The local structure of a toric degeneration

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Outline

1. The construction of a toric degeneration (Ph.D. thesis).
2. A map to a toric variety. A joint work with Lara Bossinger.

Example: Bott–Samelson variety

By definition, the *Bott–Samelson variety (also called Demazure variety) associated to* $\underline{w}$ is:

$$Z_{\underline{w}} = P_{i_1} \times^B P_{i_2} \times^B \cdots \times^B P_{i_l}/B.$$

Key: $Z_{\underline{w}}$ is a $\mathbb{P}^1$-bundle over $Z_{\underline{w}_1}, \underline{w}_1 = (s_{i_1}, \ldots, s_{i_{l-1}})$; hence, $Z_{\underline{w}}$ is obtained by successive $\mathbb{P}^1$-fibrations starting from a point.

It is a smooth projective variety of dimension l.

A $\mathbb{P}^1$-bundle Z is locally isomorphic to $U \times \mathbb{P}^1$. Thus, if the base U has a toric degeneration, then Z has an abstract toric degeneration $Z \rightsquigarrow Z'$ (lifting of a toric degeneration).

On the other hand, we have a degeneration of Z to a reduced scheme Z'' by Knutson that preserves the polarization up to Veronese.

Z'' can be chosen to coincide with Z'.

Notation: given an ideal $I \subset R$, let

$$\wp(I) = \overline{R[t, t^{-1}I]} \cap R[t, t^{-1}], \quad \bar{\cdot} = \text{int closure.}$$

Then $t\wp(I)^{[1/m]}$, some $m > 0$ is radical and defines Knutson's *balanced normal cone* to I.

For example, for $R = k[x, y, z]/(y^2z - x^3 + xz^2)$, $V(t\wp(z)^{[1/3]})$ is a not-necessarily-normal toric variety.

For the valuation ν for $R_{\sqrt{zR}}$,
Rees formula:
$\wp(z) = R[t] \bigoplus (\bigoplus_1^n t^{-n}\{\nu/3 \geq n\})$.

Then
$\wp(z)^{[1/3]} = R[t] \bigoplus (\bigoplus_1^n t^{-n}\{\nu \geq n\})$.

Example: relative elliptic curve

Let $\pi : X \to S$ be a relative elliptic curve over a variety S. Assume $X \subset \mathbb{P}^2 \times S$ and π the projection.

If S admits a toric degeneration, then we *lift a toric degeneration* under the finite morphism $i : X \hookrightarrow \mathbb{P}^2 \times S$.

(Namely, use $i^{-1}(H) = X \cap H$.)

D. Anderson "Okounkov bodies and toric degenerations" has a ruled surface $\mathbb{P}(E) \to C$ example.

The above construction should recover that example.

Abstract case

Let X be a variety.

If $Y \subset X$ is a closed subvariety that admits an abstract toric degeneration and if $Y \hookrightarrow X$ is regular, then the normal cone C_Y to $Y \subset X$ is a vector bundle; thus, has an abstract toric degeneration.

Since $X \rightsquigarrow C_Y$, by induction, we conclude that X has an abstract toric degeneration. But such an abstract toric degeneration is not proper; i.e., the morphism $\pi : \mathfrak{X} \to \mathbb{A}^1$ is not proper.

Projective case

Let $X = \operatorname{Proj} R$ be a projective variety and $\widetilde{X} = \mathrm{Spec} R$.

My Ph.D. thesis: let $Y \subset X$ be a codimension-one closed subvariety that is a *good divisor* with respect to R (i.e., as a set, given by at most $\mathrm{codim} Y$ number of homogeneous equations.) Then we get

$$X \rightsquigarrow \text{a projective variety } X'$$

such that $\widetilde{X} \rightsquigarrow \widetilde{X'}$. Also, $\widetilde{X'} \to \widetilde{Y}$ is a cone.

Assume Y has a toric degeneration (that preserves polarization up to Veronese).

If $\widetilde{X'} \to \widetilde{Y}$ is in general position (so to say), then we lift a toric degeneration under it. So, we get a toric degeneration of X (that preserves ...)

In short, *reduce the problem to the relative curve case.*

Ideal filtration

By an *ideal filtration* $\mathcal{I}_*$ on a scheme X, we mean the following equivalent notions

*A filtration $\mathcal{O}_X = \mathcal{I}_0 \supset \mathcal{I}_1 \supset \cdots$ that is multiplicative.

*$\mathcal{R}$ such that $\mathcal{O}_X[t] \subset \mathcal{R} \subset \mathcal{O}_X[t, t^{-1}]$ where $\subset$ preserves the t-grading.

($\mathcal{R}$ is a *generalized extended Rees algebra.*)

If X is integral and affine, then an ideal filtration is equivalent to a non-negative quasi-valuation.

But ideal filtrations can be handled in a similar manner to ideals. In fact, this notion is a main tool to lift a degeneration.

Branchvariety

Alexeev and Knutson introduced the notion of a *branchvariety*, which, by definition, is a finite morphism from a geometrically reduced scheme.

An analog of the Hilbert scheme exists (as a stack) for branchvarieties with fixed target.

Thus, we also get an analog of the Hilbert scheme exists for ideal filtrations of finite type ("finite type" means $\mathcal{R}$ is finite over $\mathcal{O}_X[t]$).

Geometrically, ideal filtrations of finite type amount to degenerations of Rees type. Thus, we get the moduli stack of the degenerations of Rees type of a fixed variety X, which I call the *intrinsic degeneration* of X. It then contains the *intrinsic toric degeneration of* X as a substack.

(Of course, for applications, we need the intrinsic toric degeneration of projective pairs (X, H).)

A map to a toric variety

A (not-necessarily-normal) toric variety W can be thought of as a moduli space; i.e., it is determined by a map to W.

Side-remark: in this p.o.v., it is very natural to consider an infinite-dimensional toric variety.

Classical topology (Zariski later)

Assume the base field $k = \mathbb{C}$ (actually the characteristic zero is enough).

Let (M, f) be some smooth variety M together with a proper surjective morphism $f : M \to \mathbb{A}^1$; e.g., $M = \mathbb{P}^n \times \mathbb{A}^1$ and $\mathfrak{X} \subset M$ a closed subvariety such that $f|_{\mathfrak{X}}$ is flat.

Assume $W = \mathfrak{X} \cap f^{-1}(0)$ is a not-necessarily-normal toric variety. In particular, W has a stratification given by orbits.

(In fact, W can be allowed to be semi-toric.)

The Thom–Mather theory says: for each stratum (orbit) $A \subset W$ and the normal bundle $N_{A/M}$, there are embeddings $\psi_A : N_{A/W} \to M$, called tubular neighborhoods, and the projections

$$\pi_A : \operatorname{im} \psi_A \to A$$

that satisfy the compatibility condition.

Thom's isotopy lemmas say that $\operatorname{im} \psi_A$ is trivial off A. Thus, π_A, A stratum, form a *conical structure.*

The conical structure is independent of the embedding of W. So, the local structure of the degeneration is somehow dictated by W (right by deformation theory?).

By a partition of unity type argument, we can then construct $\pi : U \to W$, where U is a neighborhood of W in M such that π and π_A coincide on $\pi^{-1}(A)$ up to diffeomorphisms on strata.

Let φ be the restriction of π to a fiber X in $\mathfrak{X}$ sufficiently close to $W = \mathfrak{X} \cap f^{-1}(0)$. Then for an orbit O, $\varphi^{-1}(O) \to O$ is a finite covering in the classical topology (thus is a finite étale covering).

Integrable systems

The pull-back of the integral system on W under $\varphi : X \to W$ gives a stratum-wise integral system on X with respect to $\varphi^*(\omega_W)$, where ω_W is really a family $\{\omega_A \mid A \text{ a stratum}\}$.

If there is a sympletic structure on M, then the canonical structure is expected (by me) to be compatible with a sympletic structure; i.e., $\varphi^*(\omega_W)$ coincides with the restriction of ω_M.

We recover and extend the result of Harada and Kaveh (up to verification).

In the Zariski topology

If $W = \mathbb{P}^r$ is a projective space, then each morphism to W amounts to a f-dim subspace V of $\Gamma(X, \mathcal{L})$ that generates $\mathcal{L}$; i.e., $V \otimes_k \mathcal{O}_X \to \mathcal{L} \to 0$ or a *linear system.*

If W is a smooth toric variety, W can be interpreted by Cox as the moduli space for collections that amount to the pull-back of (D_ρ, s_{D_ρ}, ρ rays as well as trivialization data).

In my paper with Lara Bossinger, we view each morphism $X \to W$ as a partial compactification (or extension) of a monomial map. In particular, away from a base locus, any morphism $X \to W$ factors through a (local) Veronese embedding followed by a projection.

It follows in particular: given a variety X and a line bundle $\mathcal{L}$ on it, there is a morphism $X - B \to W$ along which $\mathcal{O}(m)$, some $m > 0$, pulls-back to $\mathcal{L}|_{X-B}$ (and this is essentially of the general form).

The above result gives the negative answer to the following question of Dolgachev and Kaveh:

Question: Can a toric degeneration be obtained as a degeneration by projection (or a sequence of such degenerations)?

Question: Can any variety be embedded into a smooth toric variety?

(Of course, the above question is for non-quasi-projective varieties.) The answer is probably no.

More specifically, can we embed a toric degeneration $\mathfrak{X} \to \mathbb{A}^1$ over $\mathbb{A}^1$ into a smooth toric variety (M, f)?

A homogeneous finite Khovanskii basis amounts to a choice of an embedding of a degeneration $\mathfrak{X} \subset \mathbb{P}^n \times \mathbb{A}^1$.

So, an embedding $\mathfrak{X} \subset (M, f)$ over $\mathbb{A}^1$ amounts to a generalization of such a Khovanskii basis.

Problem: Gröbner(?) degenerations inside W?