BIRS Workshop on Toric degenerations

Disk Potential Functions for Polygon Spaces

(Joint with Siu-Cheong Lau. Xiao Zheng arXiv: 2211.03558)

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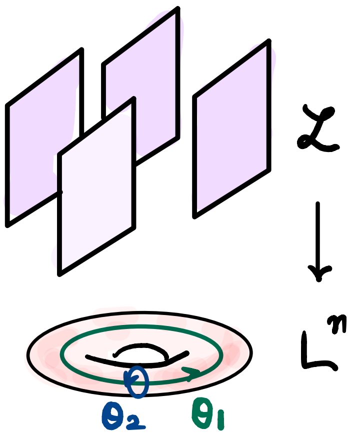
Cluster varieties via SYZ mirror symmetry

mirror

Fano symplectic manifold X^{2n}
Lagrangian torus fibrations

$\rightsquigarrow$

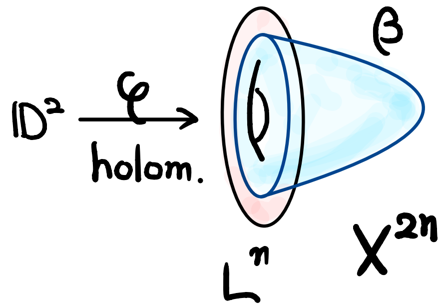
Landau–Ginzburg model
$W : \check{X} \longrightarrow \mathbb{C}$



$\check{X} = \{\text{flat } \mathbb{C}^*\text{-connections on } \mathcal{L}\}/\sim$
$\quad \cong (\mathbb{C}^*)^n$
$\quad (z_i = \mathrm{hol}_\nabla(\theta_i))$

$W = \sum_\beta n_\beta \cdot z^{\partial\beta}$

(FOOO) disk potential



Q. For a given cluster variety $\check{X}$, find a symp. mfld and Lag. torus fibrations whose SYZ mirror is $\check{X}$.

Symplectic	mirror	Complex
Q. ?		Cluster variety of Cartan A type without frozen variables

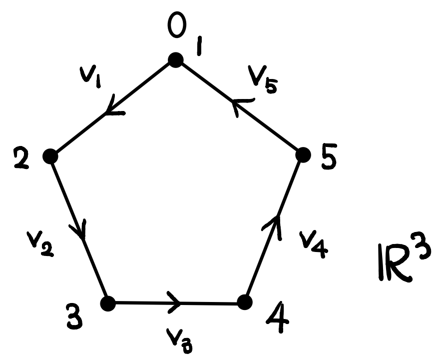
Polygon space and bending system

Symplectic	mirror	Complex
Polygon space $\mathcal{M}_{\vec{r}}$ bending system $\Phi_{\triangle}$		Cluster variety of Cartan A type without frozen variables

Fix $\vec{r} = (r_1, r_2, \cdots, r_{n+3}) \in \mathbb{R}^{n+3}_{>0}$ (the length of edges)

$SO(3) := SO(3;\mathbb{R}) \curvearrowright \mathbb{R}^3$ the linear action

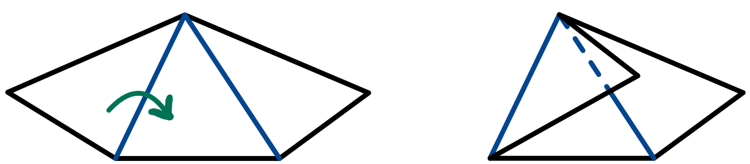
$SO(3) \curvearrowright S^2(r) := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\}$



$$\mathcal{M}_{\vec{r}} := \left\{ \vec{v} = (v_1, v_2, \cdots, v_{n+3}) \in \prod_{i=1}^{n+3} S^2(r_i) : \sum_{i=1}^{n+3} v_i = 0 \right\} \Big/ SO(3) \quad \text{(diagonal action)}$$

(Kapovich – Millson)

For $\vec{r} = (r_1, r_2, \cdots, r_{n+3}) \in \mathbb{R}^{n+3}_{>0}$, we take a triangulation of $(n+3)$-gon



(bending along the chosen diagonals.)

Have a completely integrable system $\Phi_{\triangle} : \mathcal{M}_{\vec{r}} \longrightarrow \mathbb{R}^n$

(given by the length of the chosen diagonals)

SYZ mirrors of polygon spaces

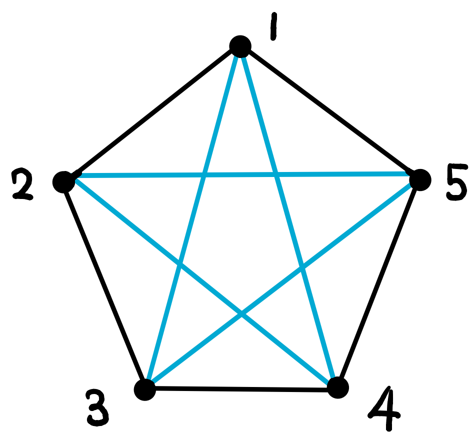
Theorem (K. – Lau – Zheng)

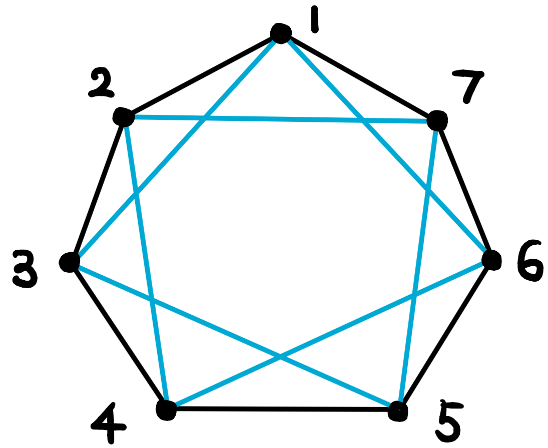
Assume that $\vec{r}$ is equilateral and generic. $X := \mathcal{M}_{\vec{r}}$ (Polygon space)

Then an SYZ mirror of X is a LG model $(\check{X}, W)$

- $\check{X} := Gr(2, \mathbb{C}^{n+3}) - D \cap \{p_{12} = p_{23} = \cdots = p_{n+2,n+3} = p_{1,n+3}\}$

  (Regard $Gr(2, \mathbb{C}^{n+3}) \subseteq \mathbb{P}(\Lambda^2 \mathbb{C}^{n+3})$ via the Plücker embedding $p_{i,j}$)
- $W : \check{X} \longrightarrow \mathbb{C}$ $\quad W(p) := \sum_{i=1}^{n+3} p_{i,i+2}$





Disk potential functions for caterpillar bending systems

Assume that $\vec{r}$ is equilateral and generic. $X := \mathcal{M}_{\vec{r}}$ (Polygon space)

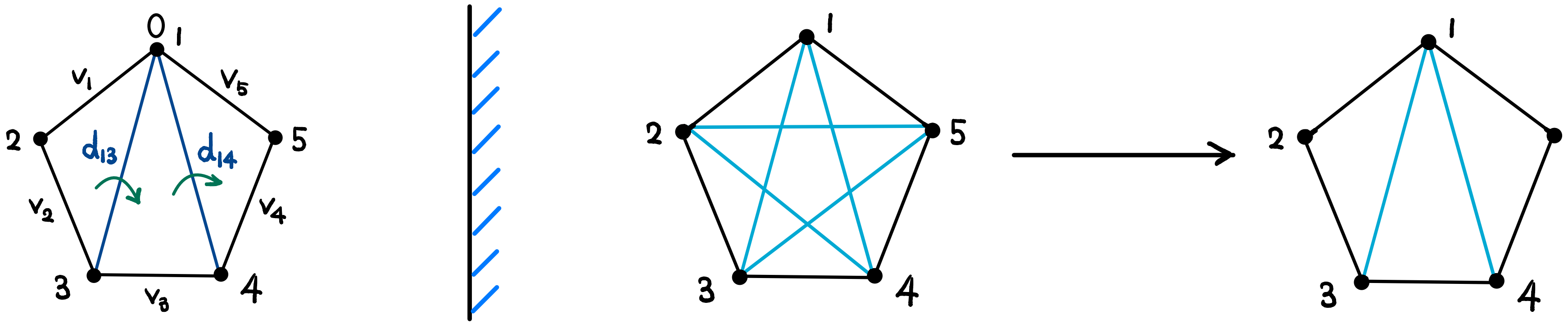
- $\Phi_{\triangle}$: the bending system $\Phi_{\triangle} = (\Phi_{13}, \Phi_{14}, \cdots, \Phi_{1.n+2})$

  $L_{\triangle} :=$ Lagrangian torus fiber of $\Phi_{\triangle}$, located at the center.

$$W_{L_{\triangle}} = \left(p_{13} + \frac{2}{p_{13}}\right) + \left(p_{1.n+2} + \frac{2}{p_{1.n+2}}\right) + \sum_{j=3}^{n+1}\left(\frac{p_{1.j+1}}{p_{1.j}} + \frac{p_{1.j}}{p_{1.j+1}} + \frac{1}{p_{1.j}\,p_{1.j+1}}\right)$$

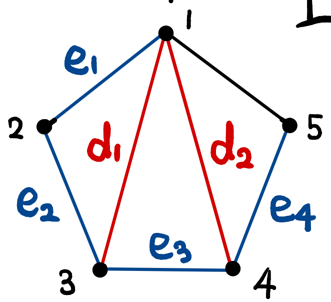
e.g> $n=2$, $\vec{r} = (1.1.1.1.1)$, $\mathcal{M}_{\vec{r}} \simeq dP_5$

$$W_{L_{\triangle}} = W\Big|_{\mathbb{C}^*_{p_{13}.p_{14}}} = \left(p_{13} + \frac{2}{p_{13}}\right) + \left(p_{1.4} + \frac{2}{p_{1.4}}\right) + \left(\frac{p_{1.4}}{p_{1.3}} + \frac{p_{1.3}}{p_{1.4}} + \frac{1}{p_{1.3}p_{1.4}}\right)$$



Frozen and cluster variables as holonomy variables

Symplectic	Complex
$Gr(2,\mathbb{C}^{n+3})$, $L_{\triangle}$: torus orbit $\Phi_{d_1},\cdots,\Phi_{d_n}$: local torus act. $\Phi_{e_1},\cdots,\Phi_{e_{n+2}}$: global torus act. Choose loops as orbits $\theta_1,\cdots,\theta_n$; $\theta_{n+1},\cdots,\theta_{2n+2}$	$\check{\mathcal{U}}_{\triangle} \simeq (\mathbb{C}^*)^{2n+2}$ $p_{1,i+2} = \mathrm{hol}_\nabla(\theta_i)$ $(i=1,\cdots,n)$ (cluster variables) $p_{i-n,i+1-n} = \mathrm{hol}_\nabla(\theta_i)$ $(i=n+1,\cdots,2n+2)$ (frozen variables)
$(L_{\triangle},\nabla) \sim (L_{\triangle'},\nabla')$	Wall-crossing = mutation
$Gr(2,\mathbb{C}^{n+3})$ $L_{\triangle}, L_{\triangle'}\cdots$ (Hausmann–Knutson) $\big\downarrow /\!/T_{U(n+3)}$ $\mathcal{M}_{\vec{r}}$, $\underline{L}_{\triangle}, \underline{L}_{\triangle'}\cdots$	$\coprod \check{\mathcal{U}}_{\triangle}$ $(\simeq(\mathbb{C}^*)^{2n+2})$, $B_{(2n+2)\times n}$ $\big\downarrow$ $p_{12}=p_{23}=\cdots=p_{n+2,n+3}=p_{1,n+3}$ $\coprod \underline{\check{\mathcal{U}}}_{\triangle}$ $(\simeq(\mathbb{C}^*)^{n})$, $\tilde{B}_{n\times n}$



$(\check{X}, W)$ $\quad W(p) := \sum_{i=1}^{n+3} p_{i,i+2}$ is a shadow of (Marsh–Rietsch)'s mirror.

Thank You!