Monotone Lagrangians and Potentials via Toric degenerations

Yunhyung Cho

Sungkyunkwan University

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A smooth Fano variety X can be

- considered as a monotone symplectic manifold (X, ω), i.e.,

$$c_1(TX, J) = [\omega]$$

  (symplectic area of a closed surface = Chern number)

- characterized by a monotone Lagrangian submanifold $L \subset X$:

  symplectic area of a disc bounded by L = Chern number (Maslov index)

Favorite example:

- X: smooth toric Fano

- $\mu : X \to \triangle$: moment map ($\triangle$: reflexive)

- L: the central toric fiber at 0

A monotone Lagrangian torus assigns a Laurent polynomial (called a potential)

$$\Phi_L(z) := \sum_{\beta} n_\beta \, z^{\partial\beta}$$

where

- $z = (z_1, \cdots, z_n)$: coordinates on $(\mathbb{C}^*)^n$ where $n = \dim_{\mathbb{C}} X = \dim_{\mathbb{R}} L$.
- $\beta \in \pi_2(X, L)$
- $\partial\beta \in H_1(L; \mathbb{Z})$
- n_β: # Maslov index two discs bounded by L

For X: smooth toric Fano and L: monotone toric fiber, it turned out that

- $\Phi_L(z) = \sum_F z^{\nu_F}$ (F: facet of $\triangle$, ν_F: primitive inward normal) [Cho-Oh]

- $\Phi_L(z)$ = LG mirror [Fukaya-Oh-Ohta-Ono, Tonkonog]

When X is possibly non-toric Fano,

- how to find a monotone Lagrangian torus $L \subset X$ and to compute Φ_L
- how to classify monotone Lagrangian tori (up to Hamiltonian isotopy)
- how to construct a LG-mirror $W : \check{X} \to \mathbb{C}$ (by gluing $(\mathbb{C}^*)^n$'s and Φ_L's)

Main ingredients are

- Each ($\mathbb{Q}$-Gorenstein) Fano toric degeneration produces a monotone Lagrangian torus

(E.g. Gelfand-Cetlin fibers in partial flag varieties)

- If L is such a Lagrangian torus, then $\Delta_{\text{newt}}(\Phi_L) = \Delta_0$ [Galkin-Mikhalkin, Sanda]

- If $L_1 \sim L_2$, then $\Delta_{\text{newt}}(\Phi_{L_1}) \sim \Delta_{\text{newt}}(\Phi_{L_2})$

- Theorem [Vianna] There are infinitely many monotone Lagrangian tori $L_{a,b,c} \subset \mathbb{P}^2$ not Hamiltonian isotopic to each other (where $a^2 + b^2 + c^2 = 3abc$).

- Theorem [Auroux] There are infinitely many monotone Lagrangian tori in $\mathbb{R}^6$ not Hamiltonian isotopic to each other

- Theorem [C.-M.Kim-Y.Kim-Park] For $n \geq 6$, there are infinitely many monotone Lagrangian tori in the complete flag variety $\mathrm{SL}_n/\mathbb{C}$ not Hamiltonian isotopic to each other. (They are indexed by some quivers obtained from some wiring diagrams)

Proof:

- Start with the quiver Q from the standard reduced word of the longest element
- There is a sequence of mutations μ such that

$$\lim_{k\to\infty} |v_{\mathrm{fr}} \to v_{\mathrm{mut}}| \text{ in } \mu^k(Q) = \infty$$

- Each $\mu^k(Q)$ produces a Fano toric degeneration and $\Phi_{L_k}(z)$ has more than $|v_{\mathrm{fr}} \to v_{\mathrm{mut}}|$ Laurent monomials summands.

Thank you!