Effective results for Diophantine equations over finitely generated domains

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1. Introduction
   - Finitely generated domains
   - Results over arbitrary finitely generated domains

2. Some words on the proofs
   - The method of Evertse and Győry
   - Some words about the proof of the Theorem on division points

Topic of the talk

- Let $A = \mathbb{Z}[z_1, \ldots, z_r]$ be an integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}$.
- Assume that $r > 0$.
- We considered several types of Diophantine problems over A:
  - Thue equations
  - hyper- and superelliptic equations
  - the Schinzel-Tijdeman equation
  - unit points on curves
  - division points on curves

Main goal

Prove effective results for such equations, i.e. results which imply that these equations have finitely many solutions and provide a theoretical way to find all these solutions

Historical remarks

- Győry in the 1980's introduced effective specializations to prove effective results over a special type of finitely generated domain
- Using this method Győry proved effective results over special finitely generated domains for
  - unit equations
  - norm form equations
  - index form equations
  - discriminant form equations
  - polynomials and integral elements of given discriminant
- Brindza, Pintér, Végső and others used this method to prove results for several other types of equations
- In 2013 Evertse and Győry combined the method of Győry with results of Aschenbrenner and proved effective results for unit equations in two unknowns over arbitrary finitely generated domains.

Historical remarks – The new method of Evertse and Győry

- In 2013 Evertse and Győry combined the method of Győry with results of Aschenbrenner and proved effective results for unit equations in two unknowns over arbitrary finitely generated domains.
- Using this new method general effective results have been proved for several types of equations over arbitrary finitely generated domains
  - Thue equations (B., Evertse, Győry)
  - hyper- and superelliptic equations (B., Evertse, Győry)
  - the Schinzel-Tijdeman equation (B., Evertse, Győry)
  - unit points on curves (B.)
  - division points on curves (B.)
  - the Catalan equation (Koymans)
  - discriminant form and discriminant eqiations (Evertse, Győry)
  - norm form equations (Evertse, Győry)
  - decomposable form equations (Evertse, Győry)

The finitely generated domain A

- Let $A = \mathbb{Z}[z_1, \ldots, z_r]$ be as above, and put

$$I := \{f \in \mathbb{Z}[X_1, \ldots, X_r] \mid f(z_1, \ldots, z_r) = 0\}.$$

Then we have

$$A \cong \mathbb{Z}[X_1, \ldots, X_r]/I.$$

Further, the ideal I is finitely generated, say

$$I = (f_1, \ldots, f_t).$$

- We may view $f_1, \ldots, f_t$ as a representation for A.
- A is a domain of char 0 $\iff$ I is a prime ideal with $I \cap \mathbb{Z} = (0)$
- Given a set of generators $\{f_1, \ldots, f_t\}$ for I this can be checked effectively

Representing elements of A

Let A be as above and let K be its quotient field.

- For $\alpha \in A$, we call f a *representative* for α, or we say that f represents α, if $f \in \mathbb{Z}[X_1, \dots, X_r]$ and $\alpha = f(z_1, \dots, z_r)$.
- Further, for $\alpha \in K$ we call (f, g) a *representation pair* for α, or say that (f, g) represents α if $f, g \in \mathbb{Z}[X_1, \dots, X_r]$, $g \notin I$ and $\alpha = f(z_1, \dots, z_r)/g(z_1, \dots, z_r)$.
- Using an ideal membership algorithm for $\mathbb{Z}[X_1, \dots, X_r]$ one can decide effectively
  - whether two polynomials $f', f'' \in \mathbb{Z}[X_1, \dots, X_r]$ represent the same element of A, i.e., $f' - f'' \in I$
  - whether two pairs $(f', g'), (f'', g'')$ in $\mathbb{Z}[X_1, \dots, X_r]$ represent the same element of K, i.e., $g' \notin I$, $g'' \notin I$ and $f'g'' - f''g' \in I$

Effective computations in A

- Based on results of Aschenbrenner one can perform arithmetic operations on A and K by using representatives.
- For $0 \neq f \in \mathbb{Z}[X_1, \ldots, X_r]$, denote by
  - $\deg f$ the total degree of f
  - $h(f)$ the logarithmic height of f, i.e. the logarithm of the maximum of the absolute values of its coefficients.
  - $s(f)$ the *size* of f, which is defined by

$$s(f) := \max(1, \deg f, h(f)) \qquad \text{for} \quad f \neq 0$$
$$s(0) := 1$$

- It is clear that there are only finitely many polynomials in $\mathbb{Z}[X_1, \ldots, X_r]$ of size below a given bound, and these can be determined effectively.

Unit points on curves

- $A := \mathbb{Z}[z_1, \ldots, z_r]$ a domain which is finitely generated over $\mathbb{Z}$, as $\mathbb{Z}$-algebra
- K the quotient field of A
- $\overline{K}$ the algebraic closure of K
- A^*, K^*, $\overline{K}^*$ denotes the unit group of A, K, $\overline{K}$, respectively.
- Γ a finitely generated subgroup of K^*
- $\overline{\Gamma}$ the division group of Γ
- $F(X, Y) \in A[X, Y]$ a polynomial, such that F is not divisible by any polynomial of the form

$$X^m Y^n - \alpha \qquad \text{or} \qquad X^m - \alpha Y^n \tag{1}$$

for any $m, n \in \mathbb{Z}_{\geq 0}$, not both zero, and any $\alpha \in A$.

Consider the equation

$$F(x, y) = 0 \quad \text{in } x, y \in \Gamma \tag{2}$$

Historical remarks for unit points and division points on curves

Let

$$\mathcal{C} := \{(x, y) \in (\mathbb{C}^*)^2 \mid f(x, y) = 0\}$$

- Lang (1960) – finiteness of $\mathcal{C} \cap \Gamma^2$ (ineffective)
- Liardet (1974) – finiteness of $\mathcal{C} \cap \overline{\Gamma}^2$ (ineffective)
- Bombieri and Gubler (2006) – effective finiteness of $\mathcal{C} \cap \Gamma^2$ in the algebraic case
- B., Evertse and Győry (2009) – explicit effective finiteness of $\mathcal{C} \cap \overline{\Gamma}^2$ in the algebraic case

Goal:

Prove effective versions of the results of Lang and Liardet in the case of arbitrary finitely generated groups.

Recall that

- $A = \mathbb{Z}[z_1, \ldots, z_r]$ integral domain finitely generated over $\mathbb{Z}$
- We assume that $r > 0$
- $A \cong \mathbb{Z}[X_1, \ldots, X_r]/\mathcal{I}$ for $\mathcal{I} := \{f \in \mathbb{Z}[X_1, \ldots, X_r] \mid f(z_1, \ldots, z_r) = 0\}$
- we have $\mathcal{I} = (f_1, \ldots, f_t)$

Let $I \subset \mathbb{Z}^2_{\geq 0}$ be a non-empty set, and let

$$F(X,Y) = \sum_{(i,j)\in I} a_{ij} X^i Y^j \in A[X,Y]$$

be a polynomial which fulfils the following condition:

F **is not divisible by** any non-constant polynomial of the form

$$X^m Y^n - \alpha \quad \text{or} \quad X^m - \alpha Y^n, \text{where } m, n \in \mathbb{Z}_{\geq 0} \text{ and } \alpha \in \overline{K}^*. \tag{3}$$

Unit points on curves over finitely generated domains

- F is given by representatives $\tilde{a}_{ij} \in \mathbb{Z}[X_1, \ldots, X_r]$ of its coefficients $a_{ij} \in A$
- We assume that $d > 1$ and $h > 1$ are real numbers with

$$\begin{cases} \deg f_1, \ldots, \deg f_t, \deg \tilde{a}_{ij} \le d \quad \text{for every} \quad (i,j) \in I \\ h(f_1), \ldots, h(f_t), h(\tilde{a}_{ij}) \le h \quad \text{for every} \quad (i,j) \in I. \end{cases} \tag{4}$$

Theorem (Bérczes, 2015)

If A is a finitely generated domain as above, and F fulfils the condition (3) then for all elements (x, y) of the set

$$\mathcal{C} := \{(x, y) \in (A^*)^2 | F(x, y) = 0\} \tag{5}$$

there exist representatives $\tilde{x}$, $\tilde{y}$, $\tilde{x}'$ and $\tilde{y}'$ of x, y, x^{-1} and y^{-1}, respectively, with their sizes bounded by

$$\exp\left\{(2d)^{\exp O(r)}(2N)^{(\log^* N)\cdot \exp O(r)} \cdot (h+1)^3\right\}.$$

Effectiveness of the above Theorem

The above result is effective, i.e. it provides an algorithm to determine, at least in principle, all elements of the set $\mathcal{C}$.

- there are only finitely many polynomials of $\mathbb{Z}[X_1,\dots,X_r]$ below our bound in the theorem
- $(x,y)\in\mathcal{C}$ is clearly fulfilled if and only if there are polynomials $\tilde{x},\tilde{y},\tilde{x}',\tilde{y}'\in\mathbb{Z}[X_1,\dots,X_r]$ with their sizes below the bound (1), which fulfil

$$\tilde{x}\cdot\tilde{x}'-1,\ \tilde{y}\cdot\tilde{y}'-1,\ \tilde{F}(\tilde{x},\tilde{y})\in\mathcal{I}. \tag{6}$$

- so we can enlist all 4-tuples $(\tilde{x},\tilde{y},\tilde{x}',\tilde{y}')$ with $s(\tilde{x}),s(\tilde{y}),s(\tilde{x}'),s(\tilde{y}')$ being smaller than our bound
- using an ideal membership algorithm check if (6) is fulfilled
- finally, group all the tuples in which $(\tilde{x},\ \tilde{y})$ represent the same pair $(x,y)\in(A^*)^2$ and pick out one pair from each group
- so we get a list consisting of one representative for each element of the set $\mathcal{C}$.

Assumptions for the results on division points

- $F(X, Y) \in A[X, Y]$ is a polynomial as above
- $\gamma_1, \ldots, \gamma_s \in K^*$ are arbitrary non-zero elements of K
- they are given by corresponding representation pairs $(g_1, h_1), \ldots, (g_s, h_s)$
- $\Gamma := \left\{ \gamma_1^{l_1} \ldots \gamma_s^{l_s} \mid l_1, \ldots, l_s \in \mathbb{Z} \right\}$
- $\overline{\Gamma} := \{ \delta \in \overline{K} \mid \exists\ m \in \mathbb{Z}_{>0} : \ \delta^m \in \Gamma \}$

Further, we assume that

$$\deg f_1, \ldots, \deg f_t, \deg g_1, \ldots, \deg g_s, \deg h_1, \ldots, \deg h_s, \deg \tilde{a}_{ij} \leq d$$
$$h(f_1), \ldots, h(f_t), h(g_1), \ldots, h(g_s), h(h_1), \ldots, h(h_s), h(\tilde{a}_{ij}) \leq h,$$

where $(i, j) \in I$ and d, h are real numbers with $d > 1$ and $h > 1$.

Division points on curves I.

Theorem (Theorem for division points on curves – part (i))

(i) *Let* A, $\overline{\Gamma}$, *and* F *be as specified above. Define the set*

$$\mathcal{C} := \{(x,y) \in (\overline{\Gamma})^2 | F(x,y) = 0\}. \tag{7}$$

Then there exists a suitably large effectively computable constant C_1 *such that for*

$$M_0 := \left[N^6 (2d)^{\exp\{C_1(r+s)\}} (h+1)^{4s}\right]$$

and $m := \text{lcm}(1, \ldots, M_0)$ *we have*

$$x^m \in \Gamma \qquad \text{and} \qquad y^m \in \Gamma,$$

for every $(x,y) \in \mathcal{C}$.

Division points on curves II.

Theorem (Theorem for division points on curves – part (ii))

(ii) *Let m be the exponent fixed in part (i) and recall that*

$$\mathcal{C} := \{(x, y) \in (\overline{\Gamma})^2 | F(x, y) = 0\}. \tag{8}$$

Then there exists an effectively computable constant C_2 and integers $t_{1,x}, \ldots, t_{s,x}, t_{1,y}, \ldots, t_{s,y}$ with

$$|t_{i,x}|, |t_{i,y}| \leq \exp\left\{\exp\left\{N^{12}(2d)^{\exp\{C_2(r+s)\}}(h+1)^{8s}\right\}\right\} \tag{9}$$

for $i = 1, \ldots, s$, such that

$$x^m = \gamma_1^{t_{1,x}} \ldots \gamma_s^{t_{s,x}}, \qquad y^m = \gamma_1^{t_{1,y}} \ldots \gamma_s^{t_{s,y}}. \tag{10}$$

Reduction to a larger domain B

- $z_1, \dots, z_q$ – maximal alg. independent subset of $z_1, \dots, z_r$
- $A_0 := \mathbb{Z}[z_1, \dots, z_q]$, $K_0 := \mathbb{Q}(z_1, \dots, z_q)$
- The field K is a finite extension of K_0, i.e. $K = K_0(w)$
- We shall construct an integral extension B of A in K such that

$$A \subseteq B := A_0[w, f^{-1}], \tag{11}$$

  where $f \in A_0$ and w is a primitive element of K over K_0 which is integral over A_0, with minimal polynomial $\mathcal{F}(X) = X^D + \mathcal{F}_1 X^{D-1} + \dots + \mathcal{F}_D \in A_0[X]$, and with

$$D, \deg f, \deg \mathcal{F}_k, h(f), h(\mathcal{F}_k) \leq C(d, h, r)$$

- Further, we choose f in such a way that some "important" elements are units in B.
- We bound the size of the solutions of our equation in $x \in B$, which yields the same bound for the solutions $x \in A$.

Measuring in the domain B

- To $\alpha \in K$ we associate the up to sign unique tuple $(P_{\alpha,0}, \dots, P_{\alpha,D-1}, Q_\alpha) \in A_0^{D+1}$ such that

$$\alpha = Q_\alpha^{-1} \sum_{j=0}^{D-1} P_{\alpha,j} w^j \quad \text{with} \qquad (12)$$

$$Q_\alpha \neq 0, \quad \gcd(P_{\alpha,0}, \dots, P_{\alpha,D-1}, Q_\alpha) = 1.$$

- We put

$$\begin{cases} \overline{\deg}\, \alpha := \max(\deg P_{\alpha,0}, \dots, \deg P_{\alpha,D-1}, \deg Q_\alpha) \\ \overline{h}(\alpha) := \max(h(P_{\alpha,0}), \dots, h(P_{\alpha,D-1}), h(Q_\alpha)), \end{cases} \qquad (13)$$

where $\deg P$, $h(P)$ denote the total degree and logarithmic height of a polynomial P with rational integer coefficients.

- For $\alpha \in A$ $\overline{\deg}\, \alpha$, $\overline{h}(\alpha)$ and $\deg \tilde{\alpha}$, $h(\tilde{\alpha})$ can be bounded by each other. (The bounds also contain some parameter of A.)

Bounding the $\overline{\deg}$ of elements of B using function field results

- We look at K (more precisely at an extension of K) as a function field in one variable, over an algebraically closed field
- We do this for all variables $z_1, \dots, z_q$, where this is a maximal algebraically independent subset of $z_1, \dots, z_r$
- Using results (mainly of Mason) we bound the function field heights of the element in question in each such function field
- Next we use a result of Evertse and Győry, to estimate the $\overline{\deg}$ of the element by a bound depending on their function field heights and parameters of the domain A.

Kronecker-Győry Specializations – Bounding heights $\overline{h}(x)$

- Let $A = Z[z_1, \ldots, z_r] = Z[X_1, \ldots, X_r]/(f_1, \ldots, f_m)$ and let
$$\varphi : A \to \mathbb{Q} : z_i \mapsto \xi_i \in \overline{\mathbb{Q}} \quad (i = 1, \ldots, r)$$
be a specialization homomorphism. Then
$$\varphi(A) \subseteq \varphi(B) \subseteq \mathcal{O}_S$$
where $\mathcal{O}_S$ is a suitable S-integer ring in some number field K.
- Thus, φ maps the solutions of the equation under investigation to the solutions of a similar equation over $\mathcal{O}_S$.
- We apply 'many' specializations to A and apply our effective results to the resulting equations over $\mathcal{O}_S$
- This gives, for each solution x and specialization φ, effective upper bounds for the number field heights of $\varphi(x)$ and its field conjugates.
- Using these and a result of Evertse and Győry we deduce upper bounds for $\overline{h}(x)$.

Main steps of the proof of the Theorem on division points

Recall part (i) of the Theorem for division points on curves

(i) Let A, $\overline{\Gamma}$, and F be as specified above. Define the set

$$\mathcal{C} := \{(x,y) \in (\overline{\Gamma})^2 | F(x,y) = 0\}. \tag{14}$$

Then there exists a suitably large effectively computable constant C_1 such that for

$$M_0 := \left[N^6 (2d)^{\exp\{C_1(r+s)\}} (h+1)^{4s} \right]$$

and $m := \mathrm{lcm}(1, \dots, M_0)$ we have

$$x^m \in \Gamma \qquad \text{and} \qquad y^m \in \Gamma,$$

for every $(x,y) \in \mathcal{C}$.

Main steps of the proof of part (i) of the Theorem on division points

- for $(x, y) \in \mathcal{C}$ we bound the degree of the field $K(x, y)$
- we estimate the smallest positive integer exponent M such that for $(x, y) \in \mathcal{C}$ we have $x^M, y^M \in \Gamma_K$, where Γ_K denotes the K closure of Γ, i.e. the largest subgroup of $\overline{\Gamma}$ which belongs to K^*
- for $\gamma \in \Gamma_K$ we estimate the smallest positive integer exponent $m(\gamma)$ such that $\gamma^{m(\gamma)} \in \Gamma$
- The number $m_0 := M \cdot m(x^M) \cdot m(y^M)$ will have the property $x^{m_0}, y^{m_0} \in \Gamma$, however it depends on (x, y).
- Since we have the estimate

$$m_0 \leq N^6 (2d)^{\exp(O(r+s))} (h+1)^{4s} := M_0.$$

  the number $m := \operatorname{lcm}(1, \ldots, M_0)$ will be a uniform exponent with $x^m, y^m \in \Gamma$.

Recall part (ii) of the Theorem on division points

Theorem (Theorem for division points on curves – part (ii))

(ii) *Let m be the exponent fixed in part (i) and recall that*

$$\mathcal{C} := \{(x,y) \in (\overline{\Gamma})^2 | F(x,y) = 0\}. \tag{15}$$

Then there exists an effectively computable constant C_2 and integers $t_{1,x}, \dots, t_{s,x}, t_{1,y}, \dots, t_{s,y}$ with

$$|t_{i,x}|, |t_{i,y}| \le \exp\left\{\exp\left\{N^{12}(2d)^{\exp\{C_2(r+s)\}}(h+1)^{8s}\right\}\right\} \tag{16}$$

for $i = 1, \dots, s$, such that

$$x^m = \gamma_1^{t_{1,x}} \dots \gamma_s^{t_{s,x}}, \qquad y^m = \gamma_1^{t_{1,y}} \dots \gamma_s^{t_{s,y}}. \tag{17}$$

Reformulation of part (ii) of the Theorem on division points

Let us fix m to be the integer specified in part (i) of our Theorem and consider the set

$$\mathcal{C}_1 := \left\{(x_0, y_0) \in \Gamma^2 \mid \exists\, x, y \in \overline{\Gamma} : x^m = x_0, y^m = y_0, F(x,y) = 0\right\}. \tag{18}$$

Proposition

Let $(x_0, y_0) \in \mathcal{C}_1$. Then there exist representatives $\tilde{x}_0$ and $\tilde{y}_0$ for x_0 and y_0, respectively, with the property

$$\begin{aligned} \deg \tilde{x}_0, \deg \tilde{y}_0 &\leq \exp\left\{N^6 (2d)^{\exp O(r+s)} (h+1)^{4s}\right\} \\ h(\tilde{x}_0), h(\tilde{y}_0) &\leq \exp\left\{\exp\left\{N^{12} (2d)^{\exp O(r+s)} (h+1)^{8s}\right\}\right\} \end{aligned} \tag{19}$$

Reducing our equation to an equation over Γ

- Let ρ be a primitive m^{th} root of unity. There exists $G(U,V) = \sum_{(i,j)\in\mathcal{J}} b_{ij} U^i V^j \in A[U,V]$ with $b_{ij} \neq 0$ and

$$G(X^m, Y^m) = \prod_{k=0}^{m-1} \prod_{l=0}^{m-1} F(\rho^k X, \rho^l Y) \qquad (20)$$

  and such that b_{ij} have representatives $\tilde{b}_{ij}$ with bounded size.
- $G(X,Y)$ is divisible by a non-constant polynomial of the form $X^a Y^b - \alpha$ or $X^a - \alpha Y^b$ with $\alpha \in \overline{K}^*$, $a, b \in \mathbb{Z}_{\geq 0}$ if and only if $F(X,Y)$ is divisible by a non-constant polynomial of the form $X^u Y^v - \beta$ or $X^u - \beta Y^v$ with $\beta \in \overline{K}^*$, $u, v \in \mathbb{Z}_{\geq 0}$.
- The set

$$\mathcal{C}_1 := \left\{(x_0, y_0) \in \Gamma^2 \mid \exists x, y \in \overline{\Gamma} : x^m = x_0, y^m = y_0, F(x,y) = 0\right\}$$

  is equal to the set

$$\mathcal{C}_2 := \left\{(x_0, y_0) \in \Gamma^2 \mid G(x_0, y_0) = 0\right\}.$$

Effectiveness of the Theorem on division points

- Consider the above defined polynomial $G(X, Y)$
- For all values of the exponents t_{ix}, t_{iy} below the bound specified in part (ii) of our Theorem we check

$$G(\gamma_1^{t_{1x}} \dots \gamma_s^{t_{sx}}, \gamma_1^{t_{1y}} \dots \gamma_s^{t_{sy}}) = 0.$$

- If this is true then the elements

$$x_0 = \gamma_1^{t_{1x}} \dots \gamma_s^{t_{sx}}, \qquad y_0 = \gamma_1^{t_{1y}} \dots \gamma_s^{t_{sy}}$$

have at least one m^{th} root x and y, respectively, such that

$$F(x, y) = 0.$$

Further, each element of $\mathcal{C}$ can be obtained in such a way.

Thank you for your attention!