Bounds for the solutions of S-unit equations in two unknowns over number fields

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K number field, *S finite set* of *places* on *K* containing the *set* S_∞ of *infinite places*, $\mathcal{O}_S, \mathcal{O}_S^*$ *ring of S-integers, group of S-units*

Many Diophantine problems $\Longrightarrow$

S-**unit equations** *of the form*

$$\alpha x + \beta y = 1 \text{ in } x, y \in \mathcal{O}_S^* \tag{1}$$

(or *their* equivalent *homogeneous versions*), *where* $\alpha, \beta \in K^*$.

Extremely rich *literature*, a great number of *applications*, many *survey papers* and *books*, including

- Evertse, Gy, Stewart and Tijdeman (1988), *S-unit equations and their applications*, in: New Advances in Transcendence Theory (A. Baker, ed.), pp. 110–174, CUP;
- Evertse and Gy (2015), *Unit equations in Diophantine number theory*, CUP.

Ineffective finiteness results

- Siegel (1921): $S = S_\infty$, implicite;
- Mahler (1933): $K = \mathbb{Q}$, S arbitrary;
- Lang (1960): over arbitrary finitely generated domains of characteristic 0.

  ⋮

Upper bounds for the number of solutions

- Evertse (1984): at most $3 \cdot 7^{d+2s}$ solutions, $d = [K : \mathbb{Q}]$, $s = |S|$;
- Evertse, Gy, Stewart and Tijdeman (1988): apart from finitely many so-called S-equivalence classes of equations (1) at most 2 solutions, *sharp*.

  ⋮

The *S-unit equations* are very *important* in the *solutions* of many *other families of Diophantine equations.* For their *applications* to obtaining the *complete solution* of *Diophantine equations*, an *upper bound* on the (height of) solutions of associated *S-unit equations* is required.

General explicit bounds for the solutions of equation (1)

Gy (1974,79): *for* $S = S_\infty$, *and later for arbitrary* S (and for slightly more general solutions of the gomogeneous version of (1))

d, h, R *degree, class number, regulator* of K, $s = |S|$, P *greatest norm* of prime ideals in S ($P = 1$ if $S = S_\infty$), $H(\quad)$ *absolute height*, $h(\quad) = \log H(\quad)$ *absolute logarithmic height*

$$H := \max(h(\alpha), h(\beta), 1)$$

In slightly different form

Theorem A (Gy, 1979)

For every solution x, y *of* (1), $\max(h(x), h(y))$ *does not exceed*

$$(c_1 s)^{c_2 s} P^{d+1} H, \tag{2}$$

where $c_1 = c_1(d, h, R), c_2 = c_2(d)$ *explicitly given; sharp* in terms of H.

Main tools: best available estimates at that time from *Baker's theory of linear forms in logarithms* (complex and p-adic versions) + quantitative results on *fundamental units*

Many applications: discriminant and index equations, power integral bases, decomposable form equations, irreducible polynomials,...

Later several authors, including Sprindžuk (1982), Evertse, Gy, Stewart and Tijdeman (1988), Bombieri (1993), Bugeaud and Gy (1996), Bugeaud (1988), Yu and Gy (2006), Evertse and Gy (2015), Le Fourn (2020) and Gy (2020) improved upon or modified the previous bounds.

We now present in simplified form the bounds of Bugeaud and Gy, Yu and Gy, Le Fourn, and Gy and compare them.

$$R_S \ : \ S\text{-regulator} \ (R_S = R \text{ for } S = S_\infty)$$

$$\log^* a := \max(\log a, 1)$$

Theorem B (Bugeaud and Gy, 1996)

Improvement of bound (2) *to*

$$(c_3 s)^{c_4 s} P^d R_S (\log^* R_S) H. \qquad (3)$$

where $c_3 = c_3(d)$, $c_4 > 0$ *absolute, explicit constants.*

Considerable *improvement*: $c_1 \to c_3, c_2 \to c_4, P^{d+1} \to P^d R_S (\log^* R_S)$

Main tools in Bugeaud–Gy and later: estimates of Waldschmidt (1993), Matveev (2000) (complex case) and Yu (1994,2007) (p-adic case) from the *theory of linear forms in logarithms* + Lemmas 1–3 below on *S-regulators* and *S-units*.

$\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ prime ideals corresponding to the finite places in S

Lemma 1

If $t > 0$, then

$$R \prod_{i=1}^{t} \log N(\mathfrak{p}_i) \leq R_s \leq hR \prod_{i=1}^{t} \log N(\mathfrak{p}_i).$$

Improved version of some estimates of Hajdu (1993). $\mathcal{O}_S^*$ finitely generated of rank $s-1$; $s = |S|$

Lemma 2

There exists a fundamental system $\{\varepsilon_1, \ldots, \varepsilon_{s-1}\}$ of S-units such that

$$\prod_{i=1}^{s-1} h(\varepsilon_i) \leq c_5 R_s,$$

where $c_5 = s^{2s}$.

In fact due to Hajdu (1993).

The following lemma has several more general variants, e.g. in Bugeaud and Gy (1996) and Yu and Gy (2006).

$\mathcal{O}_K$ *ring of integers*, $\mathcal{O}_K^*$ *unit group* of K, r *rank* of $\mathcal{O}_K^*$ and

$$\mathcal{R} = \max(h, R).$$

Lemma 3

For $\alpha \in \mathcal{O}_K \setminus \{0\}$ *there exists* $\varepsilon \in \mathcal{O}_K^*$ *such that*

$$h(\varepsilon\alpha) \leq c_6 \mathcal{R} \log N(\alpha)$$

where $c_6 = 80d^r$.

In Yu and Gy (2006) two different bounds:

Theorem C (Yu and Gy, 2006)

The bound in (3) *can be replaced by*

$$c_7 P(1 + \log^* R_S / \log^* P) R_S H, \tag{4}$$

where $c_7 = (16ds)^{2(s+3)}$.

considerable *improvement* of (3): $P^d \to P, \log^* R_S \to \frac{\log^* R_S}{\log^* P}$

Remark

Combining **Theorem C** with **Mason's result** (1983) *on unit equations over function fields* and using their **effective specialization method**, Evertse and Gy (2013) obtained effective finiteness results for *unit equations over finitely generated domains*. Some *generalizations* were established by Bérczes, Evertse, Gy and Pontreau (2009) and Bérczes (2015).

Theorem D (Yu and Gy, 2006)

The bound in (4) *can be replaced by*

$$c_8 \mathcal{R}^{t+5} \frac{P}{\log^* P} R_S H, \tag{5}$$

where $c_8 = 16^{5(r+t+1)}$.

The first bound not containing factor s^s or t^t $\rightarrow$ important in some applications

Le Fourn (2020): the first replacement of P by a smaller factor

$$P' := \begin{cases} \text{the } \textit{third largest norm} \text{ of the prime ideals in } S, \text{ if } t \geq 3, \\ 1 \text{ if } t \leq 2. \end{cases}$$

Theorem E (Le Fourn, 2020)

The bound in (4) *can be replaced by*

$$2c_7 P'(1 + \log^* R_S / \log^* P') R_S H \tag{6}$$

with c_7 *occurring in* (4) *of Theorem C.*

Particularly good bound if $t \leq 2$ or P' small with respect to P. However, in (6) still occurs s^s (in c_7).

Le Fourn combined the *proof of Theorem C* of Yu and Gy (2006) *with his variant of Runge's method*, namely with his Proposition 4 below.

For a place v on K, d_v *local degree* of K at v and

$$h_v(\gamma) := \log^*(1/|\gamma|_v) \text{ for } \gamma \in K^*.$$

For a *solution* x, y of (1), put

$$A := \{\alpha x, \beta y, \frac{1}{\alpha x}\}.$$

Let S' *subset* of S, deprived S of the two prime ideals with largest norm. For $t \leq 2$, let $S' = S_\infty$.

Proposition 4 (Le Fourn, 2020)

Let x, y be a solution of (1). Then for $P \in A$ and some $v \in S'$,

$$\frac{dv}{d} h_v(P) \geq \frac{1}{|S|}(\max(h(x), h(y)) - 3H).$$

In terms of S, the following theorem gives the **currently best bound** *for the solutions of equation* (1).

Theorem 1 (Gy, 2020)

Let $t > 0$, *and* x, y *a solution of* (1). *Then* $\max(h(x), h(y))$ *is at most*

$$c_9 \mathcal{R}^{t+4} \frac{P'}{\log^* P'} \left(1 + \frac{\log^* \log P}{\log^* P'}\right) R_S H, \tag{7}$$

where $c_9 = (16ed)^{4(r+t+1)}$.

Improvement of (5) and (6) in (7):

- $P/\log P$ in (5) and $P'\left(1 + \frac{\log^* R_S}{\log^* P'}\right)$ in (6)

$$\longrightarrow \frac{P'}{\log^* P'}, \text{ resp } \left(1 + \frac{\log^* \log P}{\log^* P'}\right),$$

  particularly *significant* if P'/P small
- in (6) s^{2s} *still occurs*, in contrast with (7)
- but, because of $\mathcal{R}$, *in general* (6) *better than* (7) *in terms of* K

Proof of Theorem 1 combines Lemmas 1 to 3 and Proposition 4 and Proposition 5 below.

For a place v on K, put

$$N(v) = \begin{cases} 2 \text{ if } v \text{ infinite} \\ N(\mathfrak{p}) \text{ if } v \text{ finite and corresponds to the prime ideal } \mathfrak{p} \end{cases}$$

Proposition 5 (Evertse and Gy, 2015)

Let Γ be a finitely generated multiplicative subgroup of K^* of positive rank with system of generators $\{\xi_1, \dots, \xi_m\}$ for Γ/Γ_{tors}, $\theta = h(\xi_1) \cdots h(\xi_m)$, $\delta \in K^*$, and $\Delta = \max(h(\delta), 1)$. Then for every place v on K and any $\xi \in \Gamma$ with $\delta\xi \neq 1$, we have

$$\log |1 - \delta\xi|_v > -c_{10} \frac{N(v)}{\log N(v)} \theta\Delta \log^* \left(\frac{N(v)h(\xi)}{\Delta} \right)$$

where $c_{10} = (16ed)^{4(m+1)}$.

Proof: *combination* of estimates of Matveev (2000) and Yu (2007) concerning *logarithmic forms* with some *new results*, due to Evertse and Gy (2015), from the *geometry of numbers*.

Proof of (7) from Theorem 1 of Gy (2020); utilizes also some ideas from the proofs of *Theorem A* of Gy (1979) and *Theorem D* of Yu and Gy (2006).

Basic idea, outline of the *main steps:* x, y solution of (1), $\mathcal{H} := \max(h(x), h(y))$. For $t \geq 3, S' \subseteq S$ depriving S of its two prime ideals with largest norm, for $t \leq 2$, $S' = S_\infty$. Prop. 4⇒for $P \in A = \{\alpha x, \beta y, \frac{1}{\alpha x}\}$ and some $v \in S'$

$$\frac{d_v}{d} h_v(P) \geq \frac{1}{|S|}(\mathcal{H} - 3H). \tag{8}$$

Assuming $\mathcal{H} > 3H \Rightarrow h_v(P) > 0$.

First consider $P = \alpha x$. One can prove

$$h_v(P) \leq -\log|1 - (\beta y)^h|_v + h \log 4. \tag{9}$$

Here an upper bound is needed for the right hand side.

$y \in \mathcal{O}_S^* \Rightarrow (y) = \mathfrak{p}^{u_1} \cdots \mathfrak{p}^{u_t}$; by Lemma 3 there are

$\pi_i \in \mathcal{O}_K$ with bounded height s.t $(\pi_i) = \mathfrak{p}^h$ for $i = 1, \ldots, t$.

Lemma 2$\Rightarrow$there is a fundamental system $\{\varepsilon_1, \ldots, \varepsilon_r\}$ of units s.t. $h(\varepsilon_1) \cdots h(\varepsilon_r)$ bounded. Now

$$y^h = \zeta \varepsilon_1^{a_1} \cdots \varepsilon_r^{a_r} \pi_1^{u_1} \cdots \pi_t^{u_t},$$

ζ root of unity, $a_1, \ldots, a_r$ integers.

Let Γ multiplicative subgroup of K^* generated by $\varepsilon_1, \ldots, \varepsilon_r, \pi_1, \ldots, \pi_t$ and the roots of unity. Proposition 5$\Rightarrow$upper bound for

$$-\log|1 - (\beta y)^h|_v. \tag{10}$$

Distinguish two cases according as v finite or infinite. Using (8), (9) and (10), after a relatively long and careful computation one deduces in both cases an upper bound for $\mathcal{H}$

For $P = \beta y$ or $\frac{1}{\alpha x}$ one can proceed similarly to get (7). $\square$

Theorems A to E and Theorem 1 have many various applications.

Two recent applications of our Theorem 1

I. A classical application: Thue equations

Keeping the above *notation*

$$F(x, y) = \delta \text{ in } x, y \in \mathcal{O}_S, \tag{11}$$

where $\delta \in \mathcal{O}_S \setminus \{0\}$, $F(X, Y) \in \mathcal{O}_S[X, Y]$ binary form of degree $n \geq 3$.

Thue (1909): $K = \mathbb{Q}$, $\mathcal{O}_S = \mathbb{Z}$, F irred $\Rightarrow$ finitely many solutions, many generalizations, quantitative version, applications

Baker (1968): $-||- \Rightarrow$ explicit bound for the solutions, many improvements, generalizations, applications

Suppose in (11) F has *splitting field* K and ≥ 3 *distinct linear factors*, H :=*upper bound* for the heights of the coefficients of F, and

$$Q = N(\mathfrak{p}_1, \ldots, \mathfrak{p}_t) \text{ if } t > 0.$$

Theorem D of Yu amd Gy (2006) on *S-unit equations* $\Rightarrow$

Theorem F (Yu and Gy, 2006)

Let $t > 0$. For all solutions x, y of equation (11), $\max(h(x), h(y))$ *is bounded above by*

$$c_{10}^{s} \frac{P}{\log^* P} R_S (\log^* R_S)(\log Q), \tag{12}$$

$c_{10} > 0$ *effective, depending on $n, h(\delta), H$ and the above parameters of K.*

Improved upon several earlier bounds.

As a consequence of **Theorem 1** above of Gy (2020)$\Rightarrow$

Theorem 2 (Gy, 2020)

Under the above conditions, the bound (12) *can be replaced by*

$$c_{11}^{s}\frac{P'}{\log^{*} P'}(\log^{*}\log P)R_S(\log Q), \tag{13}$$

$c_{11} > 0$ *effective, depending on the same parameters as* c_{10}.

Improvement

$$\frac{P}{\log^{*} P}\log^{*} R_S \longrightarrow \frac{P'}{\log^{*} P'}\log^{*}\log P.$$

In terms of S, (13) *best known bound to date* for the solutions of Thue equation (11).

Remark. In Gy (2020), using our Theorem 1 above a *more general result* is deduced for a large class of *decomposable form equations* in an arbitrary number of unknowns.

Application towards Masser's ABC conjecture over number fields

For a *place* v on K, choose the *absolute value* $|\ |_v$ normalized in the usual way. The *height* of $(a, b, c) \in (K^*)^3$ is defined as

$$H_K(a, b, c) = \prod_v \max(|a|_v, |b|_v, |c|_v) \tag{14}$$

and the *radical* as

$$N_K(a, b, c) = \prod_v N(\mathfrak{p})^{\mathrm{ord}_{\mathfrak{p}} p}, \tag{15}$$

where $\mathfrak{p}$ prime ideal corresponding to v if v is finite, p rational prime $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$, and the *product* in (15) is taken over all *finite* v s.t. $|a|_v, |b|_v, |c|_c$ are *not all equal*.

Number field versions of the ABC **conjecture** of Oesterlé and Masser: Vojta (1987), Elkies (1991), Broberg (2000), Granville and Stark (2000), Browkin (2000) and Masser (2002).

Masser's ABC conjecture in number fields: *K number field of degree d, Δ_K the absolute value of its discriminant. Then for every $\varepsilon > 0$ there exists $C(\varepsilon)$ s.t.*

$$H_K(a,b,c) < C(\varepsilon)^d (\Delta_K N_K(a,b,c))^{1+\varepsilon} \tag{16}$$

for all $a, b, c \in K^*$ with $a + b + c = 0$.

- (16) *best possible* in terms of ε,
- *uniform*, it has good behaviour under field extensions,
- for $K = \mathbb{Q}$, *classical ABC conjecture*

Of particular importance: **effective version** of Masser's conjecture, when $C(\varepsilon)$ **effectively computable**.

Applications of S-unit equations towards Masser's conjecture

Let

$$a+b+c=0 \text{ with } a,b,c\in K^*, \tag{17}$$

S: smallest subset of places v on K containing S_∞ s.t. $v\in S$ for every finite v for which $|a|_v, |b|_v, |c|_v$ not all equal. Then

$$x=-a/c,\ \ y=-b/c$$

solution of the S-**unit equation**

$$x+y=1 \text{ in } x,y\in \mathcal{O}_S^*.$$

From a slightly improved version of Theorem D above we deduced

Theorem G, (Gy, 2008)

If (17) *holds, then for any* $\varepsilon > 0$

$$\log H_K(a,b,c) < c_{12}(N_K(a,b,c))^{1+\varepsilon}$$

where $c_{12} = c_{12}(d, \Delta_K, \varepsilon) > 0$ *explicitly given. Further, if*

$$N > \max(\exp\exp(\max(\Delta_K, e)), \Delta_K^{2/\varepsilon}),$$

then

$$\log H_K(a,b,c) < c_{13}(\Delta_K N_K(a,b,c))^{1+\varepsilon},$$

where $c_{13} = c_{13}(d, \varepsilon) > 0$ *explicitly given.*

Considerable *improvement* of Surroca (2007), who deduced her result from Theorem B above of Bugeaud and Gy.

Further *improvement*: $N = N_K(a,b,c)$, P' the third largest norm of prime ideals in $S \Rightarrow P' \leq N^{1/3}$.

Our Theorem 1 above $\Rightarrow$

Theorem 3 (Gy, 2022)

Under the above assumptions

$$\log H_K(a,b,c) < c_{14} P' N^{c_{15} \log_3 N^* / \log_2 N^*} \tag{18}$$

and

$$\log H_K(a,b,c) < c_{16} N^{1/3 + c_{17} \log_3 N^* / \log_2 N^*}, \tag{19}$$

where $N^* = \max(N, 16)$, c_{14} *to* c_{16} *effectively computable, depending only on* d *and* Δ_K.

(19) *exponential*, and if P' small enough with respect to N, (18) *subexponential* effective bounds towards Masser's conjecture. They are the *best known* results *to date* in this direction over number fields.

Remark 1. Independently, using a different approach, Scoones (202?) derived the same bounds in a slightly weaker form, over the Hilbert class field of K and not over K.

Remark 2. In the classical case $K = \mathbb{Q}$ when $a + b = c$ with coprime positive integers a, b, c, our bound (19) is slightly weaker than that of Stewart and Yu (2001). Further, subexponential bounds similar to (18) (for P' small) are given in Stewart and Yu (2001) and Pasten (202?). Pasten proved also a result towards Vojta's (1998) generalization of the ABC conjecture with truncated counting functions in varieties of arbitrary dimension.

Thank you for your attention!