About approximation sets for properly intersecting divisors and effective techniques for local Weil and height functions

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Outline

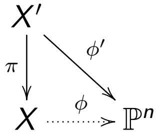
- The concept of *linear section* with respect to a linear series.
- *Subspace topology* and *linear scattering* of Diophantine arithmetic inequalities.
- A construction of *Diophantine approximation vectors and sets*.
- *Compactness of approximation sets*: outline of proof.
- Comments about *effective calculation* for *local Weil* and *logarithmic height functions*.

Notation

- Throughout $\mathbf{K}$ denotes a number field with fixed algebraic closure $\overline{\mathbf{K}}$.

The linear sections of a linear system

- Let $0 \neq V \subseteq \mathrm{H}^0(X, L)$ be a nonzero subspace for L an effective line bundle on a geometrically irreducile projective variety X.
- Fix a basis $s_0, \ldots, s_n$ for V. By *elimination of indeterminacy of rational maps*, there exists the following commutative diagram:



The linear sections of a linear system cont.

- **Defns.**
  - The *proper linear sections* $\Lambda \subsetneq X$ of X with respect to $|V|$ are described by the condition that

    $$\Lambda = \pi(\phi'^{-1}(T))$$

    for proper linear subspaces

    $$T \subsetneq \mathbb{P}^n_{\mathbf{K}}.$$

  - The *linear sections* of L are the totality of the linear sections of $|L^{\otimes m}|$, $m > 0$.
  - Let $Y(\mathbf{K}) \subseteq X(\mathbf{K})$ be a nonempty set of rational points. Then, $Y(\mathbf{K})$ is said to be *dense* with respect to the linear system $|V|$ if it is contained in no finite union of proper linear sections.

Motivation for linear sections

- *Guiding problems* for the *Subspace Theorem*, for the case of hyperplanes in $\mathbb{P}^n$:
  - Effectively describe the Diophantine exceptional set and/or determine its *linear scattering* (i.e., determine the smallest integer h so that the *Diophantine exceptional set* is contained in a finite union of h proper linear subspaces).
- *Existing viewpoints*:
  - Vojta: qualitative and effective description of exceptional set.
  - Evertse and Schlickewei: quantitative description of the exceptional set via parametric formulation of subspace theorem.
  - Schmidt: study the Diophantine exceptional set via approximation sets for rational points.

Selected recent progress

- The concept of linear section allows these existing viewpoints to extend to the context of linear systems. It allows for a way to discuss *linear scattering* of more general *height inequalities* (e.g., those of Ru and Vojta).
- As one more recent representative example, the concept of *linear section* allows for a *qualitative* understanding of the *linear scattering* of the following sequences of implications:
  - **Thm (-).** The logarithmic parametric subspace theorem for linear systems and logarithmic twisted height functions $\Rightarrow$ Logarithmic formulation of Faltings-Wüstholz approximation theorem for linear systems $\Rightarrow$ Logarithmic subspace theorem for linear systems.

Diophantine approximation sets: construction

- The starting point is:
  - **Thm (Ru-Vojta).** Let $D_1, \ldots, D_q$ be a collection of nonzero effective and properly intersecting Cartier divisors on a geometrically integral projective variety $X/\mathbf{K}$. Put $D = D_1 + \cdots + D_q$. Let L be a big line bundle on X and having (stable) base locus $\mathrm{Bs}(L)$. Let $S \subset M_{\mathbf{K}}$ be a finite set of places. Then, for all $\epsilon > 0$, there exists an *optimal constant* $\gamma = \gamma(L; D_1, \ldots, D_q; S) > 0$ and a sufficiently large integer $m > 0$, such that the collection of $\mathbf{K}$-rational points

$$x \in X \setminus \left(\mathrm{Bs}(L) \bigcup \bigcup_{i=1}^{q} \mathrm{Supp}(D_i) \right)$$

which satisfy the inequality

$$m_S(x, D) \leqslant (\gamma + \epsilon) h_L(x)$$

is *dense* with respect to the linear sections of $|L^{\otimes m}|$.

Diophantine approximation sets cont.

- Consider a collection of nonzero effective and properly intersecting Cartier divisors $D_1, \ldots, D_q$ on a geometrically integral projective variety $X/\mathbf{K}$.
- Fix a big line bundle L on X.
- Fix a finite set of places $S \subseteq M_{\mathbf{K}}$.
- Set $N := q(\#S)$.

Diophantine approximation sets cont.

- **Defns.** Inside of X, let Z be the Zariski closed subset

$$Z := \left(\mathrm{Bs}(L) \bigcup \bigcup_{i=1}^{q} \mathrm{Supp}(D_i) \right) \bigcup \{ x \in X(\mathbf{K}) : h_L(x) \leqslant 0 \}.$$

Fix $x \in X(\mathbf{K}) \setminus Z(\mathbf{K})$.

  - Fox each $v \in S$ and each $i = 1, \dots, q$, set

$$a_{iv}(x) := \frac{\lambda_{D_i}(v; x)}{h_L(x)}.$$

  - For each $v \in S$, set

$$\mathbf{a}(x; v) := (a_{1v}(x), \dots, a_{qv}(x)) \in \mathbb{R}^q.$$

  - Put

$$\mathbf{a}(x) := (\mathbf{a}(x; v))_{v \in S} \in \mathbb{R}^N.$$

Diophantine approximation sets cont.

Fix a sufficiently large integer $m > 0$ so that the conclusion of the Ru-Vojta theorem holds true with respect to the optimal constant

$$\gamma = \gamma(L; D_1, \ldots, D_q; S) \in \mathbb{R}_{>0}.$$

- **Defns.**
  - A point $\mathbf{a} \in \mathbb{R}^N$ is called an *approximation point* of (X, L), with respect to the $D_1, \ldots, D_q$ and the set of places S, if for each of its open neighbourhoods $\mathbf{a} \in B \subseteq \mathbb{R}^N$ the collection of those points $x \in X(\mathbf{K}) \setminus Z(\mathbf{K})$ which have the property that $\mathbf{a}(x) \in B$ is nonempty and *dense* with respect to the linear sections of $|L^{\otimes m}|$.
  - The *approximation set*

    $$A := \mathit{Approx}(X, L; D_1, \ldots, D_q; S) \subseteq \mathbb{R}^N$$

    is defined by the condition that

    $$A := \{\mathbf{a} \in \mathbb{R}^N : \mathbf{a} \text{ is an approximation point}\}.$$

Diophantine approximation sets: Compactness

- **Thm (-).** The approximation set

$$A := Approx(X, L; D_1, \dots, D_q; S) \subseteq \mathbb{R}^N$$

  is compact.
- Outline of proof. The idea is to show that the approximation set is closed and bounded.
  That it is closed, follows easily from the definition of A.
  That is is compact, may be deduced from the Ru-Vojta theorem.
  In fact, the approximation set A is contained in the closed and bouded region of $\mathbb{R}^N$ that consists of those $\mathbf{a} = (a_{1v}, \dots, a_{qv})_{v \in S} \in \mathbb{R}^N$ which satisfy the collection of inequalities
  - $a_{iv} \geqslant 0$ for all $v \in S$ and $i = 1, \dots, q$; and
  - $\sum_{v \in S} (\max_{i=1,\dots,q} a_{iv}) \leqslant \gamma$.

Calculation of approximation sets?

- **Question.** Defining inequalities and/or effective calculation of such approximation sets?
  - This is a difficult problem.
  - For example, to what extent is the Diophantine exceptional set that arises in the conclusion of the subspace theorem, for linear systems, effectively computable?
  - Another less ambitous (but still challenging) question is the extent to which the approximation vectors are effectively computable.
  - A first step in this direction (in full generality) involves the question of effective calculation for *presentations of Cartier divisors*.

Recall about presentations of Cartier Divisors (folowing [BG])

- Let D be a Cartier divisor on a geometrically integral projective vareity $X/\mathbf{K}$.
- Let $s_D = \mathrm{div}(D)$ be the meromorphic section of $\mathcal{O}_X(D)$ that corresponds to D.
- There are base point free line bundles L, M on X which are such that
$$\mathcal{O}_X(D) \simeq L \otimes M^{-1}.$$
- Fixing a collection of *global generating sections* $s_0, \ldots, s_k$ for L and $t_0, \ldots, t_\ell$ for M, the data
$$\mathcal{D} = (s_D, L, \mathbf{s} = (s_0, \ldots, s_k); M, \mathbf{t} = (t_0, \ldots, t_\ell))$$
is called a *presentation* of D.
- Fixing a place $v \in M_{\mathbf{K}}$, there is a *local Weil function*
$$\lambda_{\mathcal{D}}(x; v) := \max_i \max_j \left| \frac{s_i}{t_j s_D}(x) \right|_v ,$$
for $x \in X(\mathbf{K}) \setminus \mathrm{Supp}(D)(\mathbf{K})$.

Effective calculation of presentations of Cartier Divisors

- In order to do effective calculations with such local Weil functions, defined in terms of presentations of Cartier divisors, a key first step is to compute, effectively, presentations of Cartier divisors.
- To place matters into perspective, let us recall breifly some facts about *global generation* and related topics.

Recall about global generation and related topics

- To begin with, recall *Fujita's conjecture*.
- **Conjecture (Fujita).** Let L be an ample line bundle on a nonsingular projective variety X. Let K_X be the canonical line bundle. Let $n = \dim X$. Then

$$\mathrm{K}_X \otimes L^{\otimes(n+1)}$$

is globally generated and

$$\mathrm{K}_X \otimes L^{\otimes(n+2)}$$

is very ample.

Global generation and related topics cont.

- It is also helpful to recall the concept of *Castelnuovo-Mumford regularity*.
- **Defn (Mumford).** Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Let $m \in \mathbb{Z}$. Then $\mathcal{F}$ is *m-regular*, in the sense of Castelnuovo-Mumford, if it holds true that
$$\mathrm{H}^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0,$$
for all $i > 0$.
- **Mumford's m-regularity Thm. I** Let $\mathcal{F}$ be an m-regular sheaf on $\mathbb{P}^n$. Then, for all $k \geqslant 0$, it holds true that:
  (i) $\mathcal{F}(m+k)$ is generated by its global sections;
  (ii) The natural maps
$$\mathrm{H}^0(\mathbb{P}^n, \mathcal{F}(m)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \to \mathrm{H}^0(\mathbb{P}^n, \mathcal{F}(m+k))$$
  are surjective; and
  (iii) $\mathcal{F}$ is $(m+k)$-regular.

Global generation and related topics cont.

- Finally, let us recall an observation of Eisenbud which pertains to the manner in which *free resolutions* of graded modules can be used to compute cohomology groups.
- **Thm (Eisenbud).** Let $A = \overline{\mathbf{K}}[x_0, \ldots, x_n]$ and let M be a finitely generated graded A-module. Then for all $i \geqslant 0$ and all $\ell \in \mathbb{Z}$, it holds true that

$$\mathrm{H}^i(\mathbb{P}^n, \widetilde{M}(\ell)) \simeq \mathrm{Ext}^i_A(J, M)_\ell,$$

  where $J \subseteq A$ is a homogeneous ideal that is primary to the ideal $(x_0, \ldots, x_n)^a$. Here, $a = a(M)$ is the maximum of the degrees of the syzygies of M *diminishd* by $n + \ell$. In particular, if $0 \to F_m \to \cdots \to F_0 \to M \to 0$ is a graded free resolution of M and if $F_i = \bigoplus_j A(-a_{ij})$, then we may take

$$a = \left(\max_{ij} a_{ij} \right) - n - \ell.$$

Effective calculation of presentations of Cartier Divisors: Upshot

- Together, the concept of CM-regularity and the theory of graded resolutions can be used to compte presentations of line bundles on $X \subseteq \mathbb{P}^n$, assuming that such line bundles L are given, as input, in the form $L = \widetilde{M}$ for M a suitable graded $\overline{\mathbf{K}}[x_0, \dots, x_n]$-module.
- This approach also yeilds an effective description of *height functions* since they may be expressed as

$$h_L(x) = \sum_{v \in M_{\mathbf{K}}} \lambda_{\mathcal{D}}(x; v) + \mathrm{O}(1)$$

where $\mathcal{D}$ is some suitable presentation of $L = \widetilde{M}$.