Markoff triples and connectivity of Hurwitz stacks

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Markoff triples

The Markoff surface is given by the equation

$$\mathbb{X} : x^2 + y^2 + z^2 - xyz = 0$$

Let Γ denote the group of automorphisms of $\mathbb{X}$ generated by permutations of coordinates and the "Vieta involution" $R_3 : (x, y, z) \mapsto (x, y, xy - z)$.

Theorem 1 (Markoff, 1879)

The group Γ acts on $\mathbb{X}(\mathbb{Z})$ with 5 orbits, represented by $(0, 0, 0), (3, 3, 3), (-3, -3, 3), (-3, 3, -3), (3, -3, -3)$.

Conjecture 1 (Baragar 1991, Bourgain, Gamburd, Sarnak 2016)

For all primes p, Γ acts transitively on $\mathbb{X}^*(p) := \mathbb{X}(\mathbb{F}_p) - \{(0, 0, 0)\}$. In particular, $\mathbb{X}(\mathbb{Z}) \twoheadrightarrow \mathbb{X}(\mathbb{F}_p)$ is surjective. "$\mathbb{X}$ satisfies strong approximation".

Theorem 2 (Bourgain, Gamburd, Sarnak 2016)

Let $\mathbb{E}_{bgs} := \{p \text{ prime} \mid \Gamma \text{ is not transitive on } \mathbb{X}^*(p)\}$. Then
(a) $\forall \epsilon > 0$, $\#\{p \le x \mid p \in \mathbb{E}_{bgs}\} = O(x^\epsilon)$.
(b) $\forall \epsilon > 0$, $\forall p$, there is a large orbit $\mathcal{C}(p) \subset \mathbb{X}^*(p)$ s.t. $|\mathbb{X}^*(p) - \mathcal{C}(p)| \le p^\epsilon$.

Connectedness of Hurwitz stacks

Let $\mathcal{H}_{g,n}$ be the moduli stack of finite covers of genus g curves with n branch pts. For a finite group B, let $\mathcal{H}_{g,n}[B] \subset \mathcal{H}_{g,n}$ denote the substack of B-Galois covers.

Question: Classify the connected components of $\mathcal{H}_{g,n}$ using discrete invariants.

Examples: degree, monodromy group, ramification type, homological invariants...

Theorem 3

(a) (Clebsch-Hurwitz 1870's) The substack of $\mathcal{H}_{0,n}$ classifying covers with simple branching is connected.

(b) (Conway-Parker 1980's, Dunfield-Thurston 2007, Catanese, Lönne, Perroni...) For fixed B, the components of $\mathcal{H}_{g,n}[B]$ are understood for $g \gg 0$ or $n \gg 0$.

(c) (Deligne-Mumford 1969) For $B = (\mathbb{Z}/n\mathbb{Z})^{2g}$, the components of $\mathcal{H}_{g,0}(B)$ are classified by the cup product.

Conjecture 2 (McCullough-Wanderley 2013)

The connected components of $\mathcal{H}_{1,1}[\mathrm{SL}_2(\mathbb{F}_q)]$ are classified by the trace of the local monodromy around the branch point.

Note: Every noncongruence/congruence modular curve is a component of $\mathcal{H}_{1,1}$! (C., Asada, Ellenberg-McReynolds).

Relating the two problems

Let (E, O) be an elliptic curve, let $\Pi := \pi_1(E - O, x_0) = \langle a, b \rangle$. Let $\mathcal{H}_p$ be the substack of $\mathcal{H}_{1,1}[\mathrm{SL}_2(\mathbb{F}_p)]$ classifying covers whose local monodromy at $O \in E$ has trace -2. The natural forgetful map $q : \mathcal{H}_p \to \mathcal{M}_{1,1}$ is finite étale.

There are index 2 subgroups $\Gamma^+ \leq \Gamma$, $\mathrm{Out}^+(\Pi) \leq \mathrm{Out}(\Pi)$, such that

$$\begin{array}{c} \Gamma^+ \circlearrowright \mathbb{X}^*(p) \\ \downarrow \cong \quad \text{"SL}_2\text{-character variety of } \Pi\text{"} \\ \mathrm{Out}^+(\Pi) \circlearrowright \mathrm{Epi}(\Pi, \mathrm{SL}_2(\mathbb{F}_p))_{\mathrm{tr}\,\varphi([a,b])=-2} / \mathrm{GL}_2(\mathbb{F}_p) \\ \downarrow \cong \quad \text{"Galois correspondence"} \\ \pi_1(\mathcal{M}_{1,1}, E) \circlearrowright q^{-1}(E) \end{array}$$

Recall, again using the Galois correspondence:

$$\begin{array}{rcl} \{\pi_1(\mathcal{M}_{1,1}, E)\text{-orbits on } q^{-1}(E)\} & \xrightarrow{\sim} & \pi_0(\mathcal{H}_p) \\ \mathcal{O} & \mapsto & \text{The component } \mathcal{Y} \subset \mathcal{H}_p \text{ containing } \mathcal{O} \\ |\mathcal{O}| & = & \deg(\mathcal{Y} \to \mathcal{M}_{1,1}) \end{array}$$

Thus, the strong approx. conjecture is equivalent to the connectedness of $\mathcal{H}_p$.

Theorem 4 (C. 2021)

The degree of every component of $\mathcal{H}_p$ over $\mathcal{M}_{1,1}$ is divisible by p. In other words, every $\pi_1(\mathcal{M}_{1,1}, E)$-orbit on $q^{-1}(E)$, or equivalently every Γ^+-orbit on $\mathbb{X}^*(p)$, has cardinality $\equiv 0 \mod p$.

Corollary 5 (C., Fuchs, Lipman, Tran, 2022)

$\mathbb{E}_{bgs}$ is finite, and contains only primes $p < 3 \cdot 10^{27}$.

Corollary 6

(a) *For all $p \notin \mathbb{E}_{bgs}$, the reduction map $\mathbb{X}(\mathbb{Z}) \to \mathbb{X}(\mathbb{F}_p)$ is surjective.*

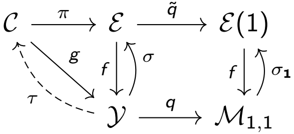
(b) *For all $p \notin \mathbb{E}_{bgs}$, the Hurwitz stack $\mathcal{H}_p$ classifying $\mathrm{SL}_2(\mathbb{F}_p)$-covers of elliptic curves only branched above the origin, with local monodromy trace -2 is connected.*

Corollary 7

Let H_p be the coarse scheme of $\mathcal{H}_p$. Then $\mathrm{genus}(H_p) \sim \frac{1}{12}p^2 + O(p^{3/2})$, and $\mathrm{genus}(H_p) \geq 2$ for all $p \geq 13$.

Proof sketch of the divisibility result

Let $\mathcal{Y} \subset \mathcal{H}_p$ be any connected component. We have a diagram (solid arrows)



Suppose there exists a section τ such that $\pi \circ \tau = \sigma$. Let e be the ramification index of π along τ. Then we have

$$q^* \underbrace{\sigma_1^* \Omega_{\mathcal{E}(1)/\mathcal{M}_{\mathbf{1},\mathbf{1}}}}_{\lambda} = \sigma^* \tilde{q}^* \Omega_{\mathcal{E}(1)/\mathcal{M}_{\mathbf{1},\mathbf{1}}} = \sigma^* \Omega_{\mathcal{E}/\mathcal{Y}} = \tau^* \Omega_{\mathcal{C}/\mathcal{Y}}^{\otimes e}$$

Taking "degrees", we get

$$\text{``}\deg(q^*\lambda) = \deg(q) \cdot \deg(\lambda) = \deg(q) \cdot \frac{1}{24} \equiv 0 \mod e\text{''}$$

To make sense of this, one needs to work over proper stacks, deal with the possible nonexistence of τ, and the possibility of fractional degrees.

Thank you!

The SL_2-character variety of Π

The SL_2-representation variety of $\Pi = \langle a, b \rangle$ is the $\mathbb{Z}$-scheme $\mathrm{Hom}(\Pi, \mathrm{SL}_2) \cong \mathrm{SL}_2 \times \mathrm{SL}_2$. The *character variety* is the GIT quotient

$$X_{\mathrm{SL}_2} := \mathrm{Hom}(\Pi, \mathrm{SL}_2) /\!\!/ \mathrm{GL}_2$$

Theorem 8 (Fricke-Vogt 1890, Brumfiel-Hilden 1995, Nakamoto 2000)

(a) *The map* $T : X_{\mathrm{SL}_2} \to \mathbb{A}^3_{\mathbb{Z}}$ *sending* $\varphi \mapsto (\operatorname{tr}\varphi(a), \operatorname{tr}\varphi(b), \operatorname{tr}\varphi(ab))$ *is an isomorphism.*

(b) *The map* $\mathrm{LM} : X_{\mathrm{SL}_2} \to \mathbb{A}^1_{\mathbb{Z}}$ *sending* $\varphi \mapsto \operatorname{tr}\varphi([a, b])$ *is given in coordinates by*

$$(x, y, z) \mapsto x^2 + y^2 + z^2 - xyz - 2$$

Thus $\mathbb{X} = \mathrm{LM}^{-1}(-2) \subset X_{\mathrm{SL}_2}$.

(c) *Away from* $\mathrm{LM}^{-1}(2)$, *the map* $\mathrm{Hom}(\Pi, \mathrm{SL}_2(\mathbb{F}_q))/\mathrm{GL}_2(\mathbb{F}_q) \to X_{\mathrm{SL}_2}(\mathbb{F}_q)$ *is a bijection.*

The action of $\mathrm{Out}^+(\Pi)$ preserves the conjugacy class of the local monodromy, and hence acts on the fibers of LM.