Covolumes, units, regulator : conjectures of Bertrand and Rodriguez-Villegas

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Joint work with S. David

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Our results rest on lower bounds for the height and on (elementary) geometry of numbers.

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Unit Theorem: $\mathcal{O}_K^\star/\{\text{roots of } 1\}$ is free of rank $r = \sigma - 1$, $\mathcal{L}_K(\mathcal{O}_K^*)$ is a lattice in $H = \{\sum x_i = 0\}$

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Hölder: $\mathrm{Vol}_2(E) \le \mathrm{Vol}_1(E) \le \binom{\sigma}{m}\mathrm{Vol}_2(E)$

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Thus (*Villegas*$^+$) holds for $E = \mathcal{O}_K^*$ and when we restrict to totally real fields, by the result of Phost.

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Remark. (*Villegas*$^+$) is non trivial even if we allow $c_1 \in (0,1)$, since the trivial estimate is $\mathrm{Vol}_1(E) \gg m^{-m}$

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Remark. (recall: $\mathrm{Vol}_2(E) \leq \mathrm{Vol}_1(E) \leq \binom{\sigma}{m}\mathrm{Vol}_2(E)$)

– (*Villegas*$^+$) $\Rightarrow$ (*Villegas*$^-$) $\Rightarrow$ (*Bertrand*)

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E subgroup of $\mathcal{O}_K^*$ of rank m. Thus $m \le r-1 \in [d/2-1, d-1]$.

Conjecture (Bertrand '97)

(*Bertrand*) $\exists m \ge 2$ tel que $\mathrm{Vol}_2(E) \ge c(m)$

Conjecture (Rodriguez-Villegas)

(*Villegas*$^-$): $\mathrm{Vol}_1(E) \ge c_0$
(*Villegas*$^+$): $\mathrm{Vol}_1(E) \ge c_0 c_1^m$

Remark. (recall: $\mathrm{Vol}_2(E) \le \mathrm{Vol}_1(E) \le \binom{\sigma}{m}\mathrm{Vol}_2(E)$)

– (*Villegas*$^+$) $\Rightarrow$ (*Villegas*$^-$) $\Rightarrow$ (*Bertrand*)

– For *bounded m*, (*Villegas*$^+$) $\Leftrightarrow$ (*Villegas*$^-$)

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Remark. For $m = 2$, (*Bertrand*) is $\sim$ to (a special case of) Schinzel-Zassenhaus conjecture on the house (=max|conjugates|) of an algebraic number. S-Z is in turn a consequence of Lehmer, and for long time it was (wrongly) believed of the same difficulty. Very recently Dimitrov proved S-Z in a very beautiful way. It will be interesting to use his method to attack the $m = 2$ case of (*Bertrand*).

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Theorem (A.-David '19)

If $3 \leq m \leq d^{1-\varepsilon}$ then

$$\mathrm{Vol}_1(E) \geq \mathrm{Vol}_2(E) \geq (d/m)^{m/25}.$$

Thus (*Villegas*$^+$) holds in the above range for m.

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Theorem (A.-David '19)

(*Villegas*$^+$) holds for K (even with L_2 norm):

– totally real (Pohst): $\mathrm{Vol}_2(E) \geq 0.002 \cdot 1.4^m$

– CM: $\mathrm{Vol}_2(E) \geq 0.0002 \cdot 2.8^m$

– totally p-adique, $p \leq 7$: $\mathrm{Vol}_2(E) \geq 0.002 \cdot 1.2^m$

– totally p-adique, $p \leq 31$ and totally imaginary: $\mathrm{Vol}_2(E) \geq 0.0005 \cdot 1.06^m$

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$E_{K/K_0} \subseteq E \subseteq \mathcal{O}_K^*$

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Methods: Analytic Number Theory (as for $\mathrm{reg}(K/K_0)$): Θ series, saddle point method.

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Methods: elementary Lattices' theory + Bertrand's inequality + Friedman - Skoruppa lower bound for relative regulator

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Conjecture

$\mathcal{H}(E) \geq c(m)/[K:\mathbb{Q}]$, *i. e.* $\mathrm{Vol}_1(E) \geq c(m)[K:\mathbb{Q}]^{m-1}$

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$\mathcal{H}(E) = {\rm Vol}_{1,K}(E)/[K:\mathbb{Q}]^m$. Then:

– $\mathcal{H}(E)$ does not depend on K.

– Let $l_1, \dots, l_m \in \mathbb{N}$ generic, $E' = \langle f_1, \dots, f_m \rangle$ with $f_j^{l_j} = e_j$. Then $[\mathbb{Q}(E') : \mathbb{Q}] = l_1 \cdots l_m [\mathbb{Q}(E) : \mathbb{Q}]$ and $\mathcal{H}(E') = \mathcal{H}(E)/(l_1 \cdots l_m)$.

This suggests:

Conjecture

$\mathcal{H}(E) \geq c(m)/[K:\mathbb{Q}]$, *i. e.* ${\rm Vol}_1(E) \geq c(m)[K:\mathbb{Q}]^{m-1}$

Similarly, for the L_2 norm:

Further conjectures.

Let E be a rank m subgroup of $\mathcal{O}_K^\star$; $\epsilon_1, \dots \epsilon_m$ basis of E/E_{tors}.

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Similarly, for the L_2 norm:

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$\mathrm{Vol}_2(E) \geq c(m)[K:\mathbb{Q}]^{m/2-1}$

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What happens for m approaching $r-1 \in [d/2-1, d-1]$?

No hope to improve the dependence in m in the lower bound for the heights with present methods