Unique Determination of Orders and Parameters in Multi-Term Time-Fractional Diffusion Equations by Inexact Data

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Joint work with Masahiro Yamamoto (The University of Tokyo).

Inverse Problems for Anomalous Diffusion Processes
— BIRS (Online), Banff, Canada —

Outline

Physical backgrounds

Models for time-fractional PDEs

- Continuous time random walk with mean square displacement $\langle\Delta x^2\rangle\sim t^\alpha\Longrightarrow\partial_t^\alpha u=\triangle u$ ($\alpha=1$: Brownian motion);
- Diffusion on fractal, e.g. Sierpiński gasket: $\alpha\leftarrow$ fractional dimensions (Barlow, Perkins '88, Kumagai);
- Earth science, esp. geothermics: $\sum_{j=1}^m q_j\partial_j^{\alpha_j}$ (linear combination);
- Viscoelasiticity: Spring-pot model ($\boldsymbol{G}$: relaxation tensor)

$$\partial_t^2\boldsymbol{u}=\mathrm{div}\int_0^t\boldsymbol{G}(\,\cdot\,t-\tau)\nabla\partial_\tau\boldsymbol{u}(\,\cdot\,,\tau)\,\mathrm{d}\tau\Longrightarrow\partial_t^\alpha\text{ with }1<\alpha<2.$$

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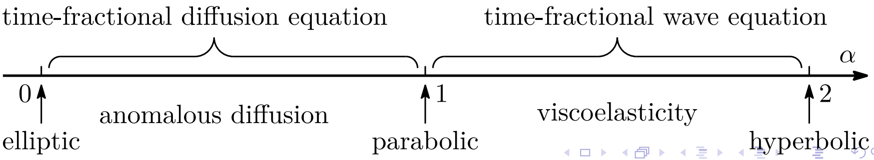
(Time-fractional) PDEs: $\partial_t^\alpha u = \triangle u + F$ $(0 < \alpha \le 2)$



Definition of Caputo derivative

Motivation Usual integral operator

$$Jf(t) := \int_0^t f(\tau)\,\mathrm{d}\tau \Longrightarrow J^n f(t) = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau)\,\mathrm{d}\tau \ (n \in \mathbb{N}).$$

$(n-1)! = \Gamma(n) \Longrightarrow$ Riemann-Liouville integral operator ($\beta > 0$):

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$\Longrightarrow$ Caputo derivative $(\alpha > 0)$:

$$\partial_t^\alpha := J^{\lceil \alpha \rceil - \alpha} \circ \frac{\mathrm{d}^{\lceil \alpha \rceil}}{\mathrm{d}t^{\lceil \alpha \rceil}}$$

($\circ$: composition, $\lceil \cdot \rceil$: ceiling function)

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Example Caputo derivative of order $\alpha \in (0,1)$:

$$\partial_t^\alpha f(t) = J^{1-\alpha}(f')(t) = \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f'(\tau)\,\mathrm{d}\tau.$$

Observation ∂_t^α accesses $\lceil \alpha \rceil$-th derivative $\Longleftarrow$ may not exist!

Redefinition of ∂_t^α in $H_\alpha(0, T)$

Idea Redefine $\partial_t^\alpha = (J^\alpha)^{-1} \Longleftarrow$ Characterization of $R(J^\alpha)$!

Basically set $0 < \alpha < 1$, $D(J^\alpha) = L^2(0, T)$,
$H^\alpha(0, T)$: Sobolev-Slobodeckij spaces.

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$$R(J^\alpha) = H_\alpha(0, T) := \overline{\{f \in C^1[0, T] \mid f(0) = 0\}}^{H^\alpha(0,T)}$$

$$= \begin{cases} H^\alpha(0, T), & 0 < \alpha < 1/2, \\ \left\{ f \in H^{1/2}(0, T) \,\middle|\, \int_0^T \frac{f^2(t)}{t} \,\mathrm{d}t < \infty \right\}, & \alpha = 1/2, \\ \{f \in H^\alpha(0, T) \mid f(0) = 0\}, & 1/2 < \alpha < 1. \end{cases}$$

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Especially, $J^\alpha : L^2(0,T)\longrightarrow H_\alpha(0,T)$ is a bijection $\Longrightarrow$ Redefine

$$\boxed{\partial_t^\alpha := (J^\alpha)^{-1}, \quad D(\partial_t^\alpha) = H_\alpha(0,T)}$$

Remark For $1/2<\alpha<1$, $H^\alpha(0,T)\subset C[0,T]$ and $\partial_t^\alpha f$ does not makes sense if $f(0)\neq 0$.

Multi-term time-fractional diffusion equation

$\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$): bounded domain, $\partial\Omega$: smooth, $\mathbb{R}_+ := (0, \infty)$.
Consider the initial-boundary value problem:

$$(\bigstar_u) \qquad \begin{cases} \displaystyle\sum_{j=1}^{m} q_j \partial_t^{\alpha_j}(u - a) + Lu = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \end{cases}$$

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$L : H^2(\Omega) \cap H_0^1(\Omega) (=: D(L)) \longrightarrow L^2(\Omega)$: elliptic operator

$$Lf(\boldsymbol{x}) := -\mathrm{div}(\boldsymbol{A}(\boldsymbol{x})\nabla f(\boldsymbol{x})) + \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla f(\boldsymbol{x}) + c(\boldsymbol{x}) f(\boldsymbol{x}) \quad (f \in D(L)),$$

$\boldsymbol{A} = (a_{ij})_{1 \le i,j \le d} \in C^1(\overline{\Omega}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$: uniformly positive-definite on $\overline{\Omega}$,
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Remark $(u - a)(\boldsymbol{x}, \cdot\,) \in D(\partial_t^{\alpha_1}) = H_{\alpha_1}(0, T)$ a.e. $\boldsymbol{x} \in \Omega \Longrightarrow$ Initial condition $u|_{t=0} = a$ if $\alpha_1 > 1/2$.

- Li, Liu, Yamamoto '15: Well-posedness.
- Jin, Lazarov, Liu, Zhou '15: Galerkin FEM.

Comparison of properties of $\partial_t^\alpha u = \triangle u$ $(0 < \alpha \le 1)$

	$\alpha = 1$	$0 < \alpha < 1$	Applications to IPs
Positivity principle	○	○	Inverse t-source problem
Time-analyticity	○	○	Coefficient inverse problem
Smoothing effect	strong	limited	Backward problem
Vanishing property	strong	weak	Inverse $\boldsymbol{x}$-source problem

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Parameter Inverse Problem

Fix $\boldsymbol{x}_0 \in \Omega, \tau > 0$ $(\tau \ll 1)$ and let u satisfy $(\bigstar_u)$, where the initial value a is unknown. Determine m, α_j, q_j $(j = 1, \ldots, m)$ simultaneously by the single point observation data of u at $\{\boldsymbol{x}_0\} \times [0, \tau]$.

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Existing literature Only uniqueness with given a and exact data!

- Hatano et al. '13: Inversion formula in the case of $m = q_1 = 1$:
$$\alpha_1 = -\lim_{t\to\infty} \frac{t\,\partial_t u(\boldsymbol{x}_0, t)}{u(\boldsymbol{x}_0, t)} = \lim_{t\to 0} \frac{t\,\partial_t u(\boldsymbol{x}_0, t)}{u(\boldsymbol{x}_0, t) - a(\boldsymbol{x}_0)}.$$
- Li, Yamamoto '15: Uniqueness with self-adjoint L (i.e., $\boldsymbol{b} \equiv \boldsymbol{0}$).
- Li, Liu, Yamamoto '19: A comprehensive survey.
- Jin, Kian '21: Uniqueness with unknown medium (i.e., L is unknown).

Outline

Parellel to $(\bigstar_u)$, introduce an auxiliary problem

$$(\bigstar_v) \qquad \begin{cases} \displaystyle\sum_{j=1}^{m'} r_j \partial_t^{\beta_j}(v-b) + Lv = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ v = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \end{cases}$$

where $m' \in \mathbb{N}$, $\beta_j, r_j > 0$: constants, $1 > \beta_1 > \cdots > \beta_{m'} > 0$.

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Theorem 1 (Uniqueness by inexact data)

Let u, v satisfy $(\bigstar_u)$ and $(\bigstar_v)$ respectively, where $a, b \in H^\gamma(\Omega)$ with $\gamma > 2 + d/2$. Pick $\boldsymbol{x}_0 \in \Omega$ such that $La(\boldsymbol{x}_0) \neq 0$, $Lb(\boldsymbol{x}_0) \neq 0$ and set $\kappa := La(\boldsymbol{x}_0)/Lb(\boldsymbol{x}_0)$.

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(i) *If $\exists\, C > 0$ and $\nu > \min\{\alpha_1, \beta_1\}$ such that*

$$(*) \qquad \textbf{Inexact data:} \quad \boxed{|u(\boldsymbol{x}_0, t) - v(\boldsymbol{x}_0, t)| \le C\, t^\nu \quad (0 \le t \le \tau),}$$

then there hold $\alpha_1 = \beta_1$, $q_1/r_1 = \kappa$.

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(ii) *If further $a, b\in H^{2+\gamma}(\Omega)$ and $(*)$ holds with $\nu>2\min\{\alpha_1,\beta_1\}$, then there hold $m=m'$, $\alpha_j=\beta_j$, $q_j/r_j=\kappa$ $(j=1,\dots,m)$.*

Comparison of existing results and Theorem 1

Results	Existing	Theorem 1
Initial value	$a \equiv b$	$a(\boldsymbol{x}_0) = b(\boldsymbol{x}_0)$, $La(\boldsymbol{x}_0) \neq 0$, $La(\boldsymbol{x}_0) \neq 0$
Data	$u \equiv v$	$u(\boldsymbol{x}_0, t) - v(\boldsymbol{x}_0, t) = O(t^\nu)$
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Example 1

In $(\bigstar_u)$ and $(\bigstar_v)$, take $\Omega = (0, \pi)$, $m = m' = 1$, $\alpha_1 = \beta_1$, $q_1 = 4$, $r_1 = 1$, $L = -\triangle$. Pick $\forall\, x_0 \in (0, \pi) \setminus \{\pi/2\}$ and select initial values

$$a(x) = \frac{\sin 2x}{2\cos x_0}, \quad b(x) = \sin x$$

$$\Longrightarrow a(x_0) = b(x_0) = \sin x_0, \quad q_1/r_1 = 4 = a''(x_0)/b''(x_0)$$

$$\Longrightarrow u(x, t) = E_{\alpha_1,1}(-t^{\alpha_1}) \frac{\sin 2x}{2\cos x_0}, \quad v(x, t) = E_{\alpha_1,1}(-t^{\alpha_1}) \sin x$$

$$\Longrightarrow u(x_0, t) = E_{\alpha_1,1}(-t^{\alpha_1}) \sin x_0 = v(x_0, t), \quad q_1 \neq r_1!$$

Assume L: self-adjoint (i.e., $\boldsymbol{b} \equiv \mathbf{0}$). By $c \geq 0$ in $\Omega \Longrightarrow$

- Distinct eigenvalues $\sigma(L) := \{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$;
- Introduce $P_n : L^2(\Omega) \longrightarrow \ker(L - \lambda_n)$ $(n \in \mathbb{N})$: eigenprojection.

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Theorem 2 (Characterization of exact data)

Define $\Sigma := \{\lambda_n/\lambda_{n'} \mid n, n' \in \mathbb{N}\}$ *and let* u, v *satisfy* $(\bigstar_u)$ *and* $(\bigstar_v)$ *respectively. Under the same assumptions in Theorem* 1*, the following claims are equivalent.*

(i) $\exists\, \tau > 0$ *such that* $u(\boldsymbol{x}_0, t) = v(\boldsymbol{x}_0, t)$ $(0 \leq t \leq \tau)$.

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(ii) *There hold* $m = m'$, $\alpha_j = \beta_j$, $q_j/r_j = La(\boldsymbol{x}_0)/Lb(\boldsymbol{x}_0) =: \kappa \in \Sigma$ $(j = 1, \ldots, m)$. *Moreover, there exist* $(\emptyset \neq) M_\kappa, M'_\kappa \subset \mathbb{N}$ *satisfying*

$$\{\lambda_n \in \sigma(L) \mid n \in M_\kappa\} = \{\kappa\lambda_n \mid \lambda_n \in \sigma(L),\ n \in M'_\kappa\}$$

and a bijection $\theta_\kappa : M_\kappa \longrightarrow M'_\kappa$ *such that*

$$\begin{cases} P_n a(\boldsymbol{x}_0) = P_{\theta_\kappa(n)} b(\boldsymbol{x}_0),\ P_n La(\boldsymbol{x}_0) = \kappa\, P_{\theta_\kappa(n)} Lb(\boldsymbol{x}_0) & (n \in M_\kappa), \\ P_n a(\boldsymbol{x}_0) = P_n La(\boldsymbol{x}_0) = 0 & (n \notin M_\kappa), \\ P_n b(\boldsymbol{x}_0) = P_n Lb(\boldsymbol{x}_0) = 0 & (n \notin M'_\kappa). \end{cases}$$

Remark Definition of $\Sigma \Longrightarrow 1 \in \Sigma$: trivial.

- $\kappa = 1$, i.e., $La(\boldsymbol{x}_0) = Lb(\boldsymbol{x}_0) \neq 0 \Longrightarrow M_1 = M_1' = \mathbb{N}$, $\theta_1 = \mathrm{Id}$,

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- $\kappa \notin \Sigma \Longrightarrow$ there must be $u(\boldsymbol{x}_0, t) \neq v(\boldsymbol{x}_0, t)$.
- $\kappa \in \Sigma \setminus \{1\} \Longrightarrow$ rather special relation between a, b.

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Example 2

$\Omega = (0, \pi)$, $L = -\triangle \Longrightarrow \lambda_n = n^2$, $\Sigma = \mathbb{Q}^2 \setminus \{0\}$, $P_n f = (f, \varphi_n)\varphi_n$, where $(\,\cdot\,,\,\cdot\,)$: $L^2(0, \pi)$ inner product, $\varphi_n(x) := \sqrt{2/\pi}\sin nx$ $(n \in \mathbb{N})$.

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Example 2

$\Omega = (0, \pi)$, $L = -\triangle \Longrightarrow \lambda_n = n^2$, $\Sigma = \mathbb{Q}^2 \setminus \{0\}$, $P_n f = (f, \varphi_n)\varphi_n$, where $(\,\cdot\,,\,\cdot\,)$: $L^2(0,\pi)$ inner product, $\varphi_n(x) := \sqrt{2/\pi}\sin nx$ $(n \in \mathbb{N})$.

Same as Example 1, take $\kappa = 4 \Longrightarrow M_4 = 2\mathbb{N}$, $M_4' = \mathbb{N}$, $\theta_4(n) = n/2$.

By Theorem 2 $\Longrightarrow$

$$(a, \varphi_{2n})\varphi_{2n}(x_0) = (b, \varphi_n)\varphi_n(x_0),\ (a, \varphi_{2n-1})\varphi_{2n-1}(x_0) = 0 \quad (n \in \mathbb{N}).$$

Especially, if $x_0 \notin \pi\mathbb{Q} \Longrightarrow a$ must take the form of

$$a(x) = \sum_{n=1}^{\infty} \frac{(b, \varphi_n)\varphi_n(x_0)}{\varphi_{2n}(x_0)}\varphi_{2n}(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{(b, \varphi_n)}{\cos nx_0} \sin 2nx.$$

Corollary 1 (Uniqueness of initial values)

Let u, v satisfy $(\bigstar_u)$ and $(\bigstar_v)$ respectively. Under the same assumptions in Theorem 2, further assume that all eigenvalues of L are simple and

$$(\#) \qquad \boldsymbol{x}_0 \notin \bigcup_{n=1}^{\infty} \{\boldsymbol{x} \in \Omega \mid \varphi_n(\boldsymbol{x}) = 0\},$$

where φ_n is the unique eigenfunction of λ_n. Then $u(\boldsymbol{x}_0, t) = v(\boldsymbol{x}_0, t)$ $(0 \le t \le \tau) \Longrightarrow$

$$a = \sum_{n \in M_\kappa} (a, \varphi_n)\varphi_n, \quad b = \sum_{n \in M'_\kappa} (b, \varphi_n)\varphi_n \quad \text{with}$$

$$(a, \varphi_n)\varphi_n(\boldsymbol{x}_0) = (b, \varphi_{\theta_\kappa(n)})\varphi_{\theta_\kappa(n)}(\boldsymbol{x}_0) \quad (n \in M_\kappa).$$

Especially if $\kappa = 1$ $(\Longleftrightarrow La(\boldsymbol{x}_0) = Lb(\boldsymbol{x}_0) \neq 0)$, then $a \equiv b$ in Ω.

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Remark (#): Special case of the rank condition (Sakawa '75: related to observability).

Parallel results for inhomogeneous equations:

$$(\bigstar'_u) \qquad \left(\sum_{j=1}^{m} q_j \partial_t^{\alpha_j} + L\right) u(\boldsymbol{x},t) = \rho(t) f(\boldsymbol{x}), \quad u|_{\partial\Omega} = 0,$$

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Theorem 1′ (Uniqueness by inexact data)

Let u, v satisfy $(\bigstar'_u)$ and $(\bigstar'_v)$ respectively, where $f, g \in H^\gamma(\Omega)$ with $\gamma > d/2$, $\rho \in L^1(\mathbb{R}_+)$ and $\exists\, \mu > -1$ such that

$$\rho(t) \sim t^\mu \quad \text{for a.e. } t \ll 1.$$

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(i) *If $\exists\, C > 0$ and $\nu > \min\{\alpha_1, \beta_1\} + \mu$ such that*

$$(*) \qquad \textbf{Inexact data:} \quad \boxed{|u(\boldsymbol{x}_0,t) - v(\boldsymbol{x}_0,t)| \le C\, t^\nu \quad (0 \le t \le \tau),}$$

then there hold $\alpha_1 = \beta_1$, $q_1/r_1 = \kappa$.

(ii) *If $f, g \in H^{2+\gamma}(\Omega)$ and $(*)$ holds with $\nu > 2\min\{\alpha_1, \beta_1\} + \mu$, then there hold $m = m'$, $\alpha_j = \beta_j$, $q_j/r_j = \kappa$ $(j = 1, \ldots, m)$.*

Remark Assumption $\rho(t) \sim t^\mu$ for a.e. $t \ll 1$: not restrictive.

Counterexample $\rho(t) = \exp(-1/t)$ near $t = 0 \Longrightarrow \rho^{(i)}(0) = 0$ $(\forall\, i = 0, 1, \ldots) \Longleftarrow$ Severe ill-posedness.

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Theorem 2′ (Characterization of exact data)

Let Σ, M_κ, M'_κ and θ_κ be the same as those in Theorem 2. Let u, v satisfy $(\bigstar'_u)$ and $(\bigstar'_v)$ respectively. Under the same assumptions in Theorem 1′, $u(\boldsymbol{x}_0, t) = v(\boldsymbol{x}_0, t)$ $(0 \le t \le \tau) \Longleftrightarrow m = m'$, $\alpha_j = \beta_j$, $q_j/r_j = f(\boldsymbol{x}_0)/g(\boldsymbol{x}_0) =: \kappa \in \Sigma$ $(j = 1, \dots, m)$ and

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Corollary 1′ (Uniqueness of source terms)

Let u, v satisfy $(\bigstar'_u)$ and $(\bigstar'_v)$ respectively. Under the same assumptions in Corollary 1,

$$\left.\begin{aligned} u(\boldsymbol{x}_0, t) = v(\boldsymbol{x}_0, t)\ (0 \le t \le \tau) \\ f(\boldsymbol{x}_0) = g(\boldsymbol{x}_0) \ne 0 \Longleftrightarrow \kappa = 1 \end{aligned}\right\} \Longleftrightarrow \begin{cases} m = m', \alpha_j = \beta_j, q_j = r_j, \\ f \equiv g \text{ in } \Omega. \end{cases}$$

Outline

Proof of Theorem 1 (1)

For simplicity, only consider $m = m' = 1 \Longrightarrow$

$$(\bigstar_u) \qquad q\,\partial_t^{\alpha}(u-a) + Lu = 0, \quad u|_{\partial\Omega} = 0,$$

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$$\left.\begin{array}{r}(*)\ \text{Inexact data} : |u(\boldsymbol{x}_0,t) - v(\boldsymbol{x}_0,t)| \le C\,t^\nu\ \ (0 \le t \le \tau)\\ \text{Asymptotic behavior} : u(\boldsymbol{x}_0,t) \sim t^{-\alpha}, v(\boldsymbol{x}_0,t) \sim t^{-\beta}\ \ (t \to \infty)\end{array}\right\}$$

$$\Longrightarrow |u(\boldsymbol{x}_0,t) - v(\boldsymbol{x}_0,t)| \le C\,t^\nu\ \ (\forall\, t \ge 0).$$

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$$\Longrightarrow |u(\boldsymbol{x}_0,t) - v(\boldsymbol{x}_0,t)| \le C\,t^\nu\ (\color{red}{\forall\, t \ge 0}).$$

Take Laplace transform $\widehat{h}(p) := \int_{\mathbb{R}_+} \mathrm{e}^{-pt} h(t)\,\mathrm{d}t$ on both sides $\Longrightarrow$

$$(*') \qquad |\widehat{u}(\boldsymbol{x}_0,p) - \widehat{v}(\boldsymbol{x}_0,p)| \le C\,\color{red}{p^{-\nu-1}} \quad (p > 0).$$

$$\left.\begin{array}{r} \text{Take Laplace transform in } (\bigstar_u) \\ \text{Formula} : \widehat{\partial_t^\alpha h}(p) = p^\alpha \widehat{h}(p) - p^{\alpha-1} h(0) \end{array}\right\} \Longrightarrow$$

$$\begin{cases} (L + q\,p^\alpha)\widehat{u}(\,\cdot\,,p) = q\,p^{\alpha-1} a & \text{in } \Omega, \\ \widehat{u}(\,\cdot\,,p) = 0 & \text{on } \partial\Omega \end{cases} \quad (p > 0).$$

Proof of Theorem 1 (2)

Assume $\sigma(L) = \{\lambda_n\} \subset \mathbb{R}_+$ and use eigenprojection $P_n \Longrightarrow$

$$p\,\widehat{u}(\boldsymbol{x}_0, p) = q\,p^\alpha (L + q\,p^\alpha)^{-1} a(\boldsymbol{x}_0) = a(\boldsymbol{x}_0) - \sum_{n=1}^\infty \frac{P_n La(\boldsymbol{x}_0)}{\lambda_n + q\,p^\alpha}.$$

Treat v similarly, employ $(*')$ and $a(\boldsymbol{x}_0) = b(\boldsymbol{x}_0) \Longrightarrow$

$$(\blacktriangle) \qquad \sum_{n=1}^\infty \frac{P_n La(\boldsymbol{x}_0)}{\lambda_n + q\,p^\alpha} - \sum_{n=1}^\infty \frac{P_n Lb(\boldsymbol{x}_0)}{\lambda_n + r\,p^\beta} = O(p^{-\nu}) \quad (p \to \infty).$$

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Contradiction argument If $\alpha < \beta$, multiply $(\blacktriangle)$ by $p^\alpha \Longrightarrow$

$$(\blacktriangledown) \quad \underbrace{\sum_{n=1}^\infty \frac{P_n La(\boldsymbol{x}_0)}{\lambda_n p^{-\alpha} + q}}_{\longrightarrow La(\boldsymbol{x}_0)/q \neq 0} - \underbrace{\sum_{n=1}^\infty \frac{P_n Lb(\boldsymbol{x}_0)}{\lambda_n p^{-\alpha} + r\,p^{\beta-\alpha}}}_{\longrightarrow 0} = \underbrace{O(p^{-\nu+\alpha})}_{\longrightarrow 0} \quad (p \to \infty)$$

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$\Longrightarrow$ contradiction. Similarly, $\alpha > \beta$ is also impossible $\Longrightarrow$ $\boxed{\alpha = \beta}$

Again by $(\blacktriangledown)$ $\Longrightarrow$ $\boxed{q/r = La(\boldsymbol{x}_0)/Lb(\boldsymbol{x}_0)}$

Now consider

$$(\bigstar'_u) \qquad (q\,\partial_t^\alpha + L)u(\boldsymbol{x},t) = \rho(t)f(\boldsymbol{x}), \quad u|_{\partial\Omega} = 0,$$

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Lemma

Let $\rho \in L^1(\mathbb{R}_+)$ and $\exists\,\mu > -1$ such that $\rho(t) \sim t^\mu$ for a.e. $t \ll 1$. Then its Laplace transform $\widehat{\rho}(p) \sim p^{-\mu-1}$ for $p \gg 1$.

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Proof of Theorem $1'$ Same argument $\Longrightarrow$ $\widehat{u}(\boldsymbol{x},p)$, $\widehat{v}(\boldsymbol{x}_0,p)$ satisfy

$$(*') \qquad |\widehat{u}(\boldsymbol{x}_0,p) - \widehat{v}(\boldsymbol{x}_0,p)| \le C\,p^{-\nu-1} \quad (p>0).$$

Take Laplace transform in $(\bigstar'_u)$ and $(\bigstar'_v)$ and by Lemma $\Longrightarrow$

$$\widehat{u}(\,\cdot\,,p) - \widehat{v}(\,\cdot\,,p) = \underbrace{\widehat{\rho}(p)}_{\sim p^{-\mu-1}} \left(\underbrace{\sum_{n=1}^\infty \frac{P_n f}{\lambda_n + q\,p^\alpha}}_{=(L+q\,p^\alpha)^{-1}f} - \underbrace{\sum_{n=1}^\infty \frac{P_n g}{\lambda_n + r\,p^\beta}}_{=(L+r\,p^\beta)^{-1}g} \right) \quad (p \gg 1)$$

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$$\Longrightarrow \sum_{n=1}^\infty \frac{P_n f(\boldsymbol{x}_0)}{\lambda_n + q\,p^\alpha} - \sum_{n=1}^\infty \frac{P_n g(\boldsymbol{x}_0)}{\lambda_n + r\,p^\beta} = O(p^{\mu-\nu}) \quad (p \to \infty).$$

$\nu - \mu > \min\{\alpha, \beta\} \Longrightarrow$ OK.

Proof of Theorem 2 (1)

Theorem 2 (Recall; Characterization of exact data)

Define $\Sigma := \{\lambda_n/\lambda_{n'} \mid n, n' \in \mathbb{N}\}$ *and let* u, v *satisfy* $(\bigstar_u)$ *and* $(\bigstar_v)$ *respectively. Under the same assumptions in Theorem* 1*, the following claims are equivalent.*

(i) $\exists\, \tau > 0$ *such that* $u(\boldsymbol{x}_0, t) = v(\boldsymbol{x}_0, t)$ $(0 \le t \le \tau)$.

(ii) *There holds* $m = m'$, $\alpha_j = \beta_j$, $q_j/r_j = La(\boldsymbol{x}_0)/Lb(\boldsymbol{x}_0) =: \kappa \in \Sigma$ $(j = 1, \dots, m)$. *Moreover, there exist* $(\emptyset \ne) M_\kappa, M'_\kappa \subset \mathbb{N}$ *satisfying*

$$\{\lambda_n \in \sigma(L) \mid n \in M_\kappa\} = \{\kappa\lambda_n \mid \lambda_n \in \sigma(L),\ n \in M'_\kappa\}$$

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Only consider $m = m' = 1$ and show (i) $\Longrightarrow$ (ii). Time-analyticity $\Longrightarrow$

$$u(\boldsymbol{x}, t) = v(\boldsymbol{x}_0, t)\ (\forall\, t \ge 0) \implies \widehat{u}(\boldsymbol{x}_0, p) = \widehat{v}(\boldsymbol{x}_0, p)\ (p > 0).$$

Proof of Theorem 2 (2)

Now $\alpha = \beta$, $q/r = \kappa$ and (▲) becomes

$$(\blacklozenge) \quad \sum_{n=1}^{\infty} \frac{P_n a(\boldsymbol{x}_0)}{z - \lambda_n} = \sum_{n=1}^{\infty} \frac{P_n b(\boldsymbol{x}_0)}{z - \kappa\lambda_n}, \quad \sum_{n=1}^{\infty} \frac{P_n La(\boldsymbol{x}_0)}{z - \lambda_n} = \kappa \sum_{n=1}^{\infty} \frac{P_n Lb(\boldsymbol{x}_0)}{z - \kappa\lambda_n},$$

where $z := -q\,p^{\alpha} < 0$. Unique continuation $\Longrightarrow$ ($\blacklozenge$) holds for $z \in \mathbb{C} \setminus \Lambda$, where $\Lambda := \sigma(L) \cup \kappa\,\sigma(L)$, $\kappa\,\sigma(L) := \{\kappa\lambda_n \mid n \in \mathbb{N}\}$.

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<u>**Proof of $\kappa \in \Sigma$**</u> If $\kappa \notin \Sigma \Longleftrightarrow \sigma(L) \cap \kappa\, \sigma(L) = \emptyset$. Apply Cauchy's integral formula to ($\blacklozenge$) $\Longrightarrow$

$$P_n La(\boldsymbol{x}_0) = 0 \ (\forall\, n \in \mathbb{N}) \implies La(\boldsymbol{x}_0) = 0 \implies \text{contradiction}$$

$\Longrightarrow$ $\boxed{\kappa \in \Sigma}$ $\Longrightarrow \sigma(L) \cap \kappa\, \sigma(L) \neq \emptyset \Longrightarrow M_\kappa, M'_\kappa, \theta_k$: well-defined.

Proof of Theorem 2 (2)

Now $\alpha = \beta$, $q/r = \kappa$ and (▲) becomes

$$(\blacklozenge) \quad \sum_{n=1}^{\infty} \frac{P_n a(\boldsymbol{x}_0)}{z - \lambda_n} = \sum_{n=1}^{\infty} \frac{P_n b(\boldsymbol{x}_0)}{z - \kappa\lambda_n}, \quad \sum_{n=1}^{\infty} \frac{P_n La(\boldsymbol{x}_0)}{z - \lambda_n} = \kappa \sum_{n=1}^{\infty} \frac{P_n Lb(\boldsymbol{x}_0)}{z - \kappa\lambda_n},$$

where $z := -q\,p^{\alpha} < 0$. Unique continuation $\Longrightarrow$ ($\blacklozenge$) holds for $z \in \mathbb{C} \setminus \Lambda$, where $\Lambda := \sigma(L) \cup \kappa\,\sigma(L)$, $\kappa\,\sigma(L) := \{\kappa\lambda_n \mid n \in \mathbb{N}\}$.

Proof of $\kappa \in \Sigma$ If $\kappa \notin \Sigma \Longleftrightarrow \sigma(L) \cap \kappa\,\sigma(L) = \emptyset$. Apply Cauchy's integral formula to ($\blacklozenge$) $\Longrightarrow$

$$P_n La(\boldsymbol{x}_0) = 0 \ (\forall\, n \in \mathbb{N}) \implies La(\boldsymbol{x}_0) = 0 \implies \text{contradiction}$$

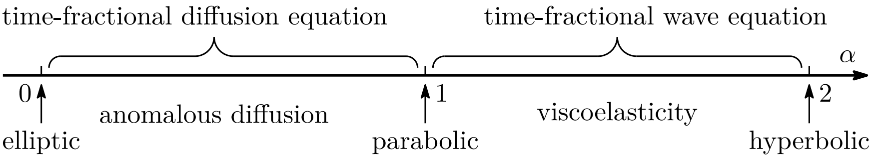
$\Longrightarrow \boxed{\kappa \in \Sigma} \Longrightarrow \sigma(L) \cap \kappa\,\sigma(L) \neq \emptyset \Longrightarrow M_\kappa, M'_\kappa, \theta_k$: well-defined.

Apply Cauchy's integral formula repeatedly to ($\blacklozenge$) $\Longrightarrow$

$$\begin{cases} n \notin M_\kappa \Longrightarrow \lambda_n \notin \kappa\,\sigma(L) \Longrightarrow P_n a(\boldsymbol{x}_0) = P_n La(\boldsymbol{x}_0) = 0, \\ n \notin M'_\kappa \Longrightarrow \kappa\lambda_n \notin \sigma(L) \Longrightarrow P_n b(\boldsymbol{x}_0) = P_n Lb(\boldsymbol{x}_0) = 0, \\ n \in M_\kappa \Longrightarrow \lambda_n = \kappa\lambda_{\theta_\kappa(n)} \Longrightarrow \begin{cases} P_n a(\boldsymbol{x}_0) = P_{\theta_\kappa(n)} b(\boldsymbol{x}_0), \\ P_n La(\boldsymbol{x}_0) = \kappa P_{\theta_\kappa(n)} Lb(\boldsymbol{x}_0). \end{cases} \end{cases}$$

Outline

(Time-fractional) PDEs: $\partial_t^\alpha u = \triangle u + F$ $(0 < \alpha \le 2)$



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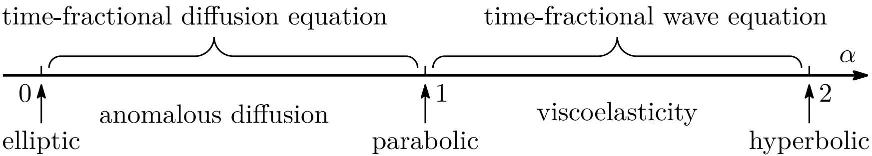

Caputo derivative : $\partial_t^\alpha f(t) = J^{1-\alpha}(f')(t) = \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f'(\tau)\,\mathrm{d}\tau$

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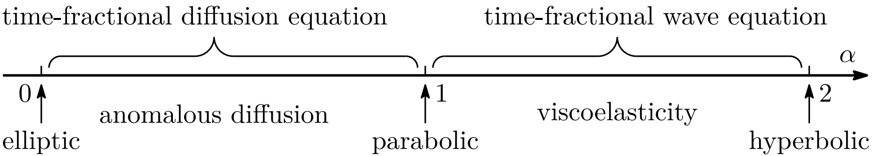

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Parameter Inverse Problem Find m, α_j, q_j $(j = 1, \dots, m)$ in

$$\sum_{j=1}^m q_j \partial_t^{\alpha_j}(u - a) + Lu = 0 \quad \text{or} \quad \left(\sum_{j=1}^m q_j \partial_t^{\alpha_j} + L\right) u = \rho(t) f(\boldsymbol{x})$$

simultaneously by single point observation of u at $\{\boldsymbol{x}_0\} \times [0, \tau]$.

Highlights Essential difference from literature:

- Only use the short-time inexact data near $t = 0$;
- The initial value a or the source term f are unknown.

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Main achievements

- **Inexact data** $\boxed{|\mathrm{err}| \le C\, t^{\nu}}$ $\Longrightarrow$ Uniqueness of m, α_j and $q_j/La(\boldsymbol{x}_0)$ or $q_j/f(\boldsymbol{x}_0)$.
- Exact data $\Longleftrightarrow$ Characterization of $a(\boldsymbol{x})$ or $f(\boldsymbol{x})$.
- In some special case, even the uniqueness of $a(\boldsymbol{x})$ or $f(\boldsymbol{x})$.

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Future topics

- Multiple point observation satisfying the rank condition $\Longrightarrow$
- Numerical reconstruction of m, α_j, q_j and $a(\boldsymbol{x})$ or $f(\boldsymbol{x})$ simultaneously.

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Thank you for your attention!

Appendix: Facts about L with $\boldsymbol{b} \neq \mathbf{0}$ (1)

Recall $Lf(\boldsymbol{x}) := -\mathrm{div}(\boldsymbol{A}(\boldsymbol{x})\nabla f(\boldsymbol{x})) + \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla f(\boldsymbol{x}) + c(\boldsymbol{x})f(\boldsymbol{x})$.

Spectrum $\sigma(L) = \{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$. $c \geq 0$ in $\overline{\Omega} \Longrightarrow \mathrm{Re}\,\lambda_n > 0$.

$$P_n := -\frac{1}{2\pi\sqrt{-1}} \int_{\gamma_n} (L - z)^{-1} \mathrm{d}z \quad \Longrightarrow \quad P_n P_{n'} = \begin{cases} P_n & (n = n'), \\ 0 & (n \neq n'). \end{cases}$$

- P_n: eigenprojection of λ_n, $d_n := \dim P_n L^2(\Omega) < \infty$;
- $\forall \varphi \in P_n L^2(\Omega) \setminus \{0\}$: generalized eigenfunction of λ_n;
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Laurant expansion for the resolvent (Kato '76):

$$(L-z)^{-1} P_n = \frac{P_n}{\lambda_n - z} + \sum_{k=1}^{d_n - 1} \frac{(-D_n)^k}{(\lambda_n - z)^{k+1}} \quad (z \notin \sigma(L)).$$

Resolvent estimate (Tanabe '75): $\exists\, z_0 > 0$ and $C > 0$ such that

$$\|(L+z)^{-1} h\|_{L^2(\Omega)} \leq \frac{C}{|z|} \|h\|_{L^2(\Omega)} \quad (\forall\, z > z_0,\ \forall\, h \in L^2(\Omega)).$$

Appendix: Facts about L with $\boldsymbol{b} \neq \mathbf{0}$ (2)

Fractional power L^γ and its domain $D(L^\gamma)$ for $\gamma > 0$:

- For $\gamma \in (0,1)$, define

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Completeness of the generalized eigenfunctions of L:

Lemma (Agmon '65)

$\forall\, h \in D(L^\gamma)$ $(\gamma \ge 0)$, $\exists \{h_N\}_{N \in \mathbb{N}} \subset D(L^\gamma)$ *such that*
$$h_N \in \sum_{n=1}^{N} P_n L^2(\Omega), \qquad \lim_{N \to \infty} \|L^\gamma (h - h_N)\|_{L^2(\Omega)} = 0.$$