A rigorous approach to the dynamics of self-propelled swarms via a novel central manifold approximation technique

Irina Popovici
Kostya Medynets
Carl Kolon

Mathematics Department,
United States Naval Academy
Research sponsored by ONR grant

We're hiring (4 tenure-tracks).

1. Agent-Based Models of Swarms

- Our system has n agents (oscillators); their position vectors $r_1, \ldots r_n$ (in $\mathbb{R}^2$) satisfy

$$\ddot{r}_k = (1-|\dot{r}_k|^2)\dot{r}_k - \frac{1}{n}\sum_j (r_k - r_j)$$

$$\ddot{r}_k = (1-|\dot{r}_k|^2)\dot{r}_k - (r_k - R(t))$$

$R = \frac{1}{n}(r_1 + \cdots + r_n)$ is the center of mass.

- Turing 1952, Smale 1976: cellular biology; the linear coupling represents the diffusion of enzymes past the membranes of neighboring cells

- Ebeling-Schweitzer (1998-2003): active agents $r_k \in \mathbb{R}^2$.

$$\ddot{r}_k = -f(\dot{r}_k)\dot{r}_k - \nabla U(r) + \xi$$

- $f(\dot{r}_k)$ is a non-linear dissipation function; or the self-propelling term; converts energy into motion.
- $\nabla U(r)$ describes the influences of the environment, say, the positions of other agents.

2. Paradoxical dynamics $\ddot{r}_k = (1 - |\dot{r}_k|^2)\dot{r}_k - \frac{1}{n}\sum(r_k - r_j)$

When viewed in isolation, i.e. if $\ddot{r}_k = (1 - |\dot{r}_k|^2)\dot{r}_k$, an agent's motion is rectilinear, moving in the direction of $\dot{r}_k(0)$ at speed limiting one. *In isolation, agents diverge.*

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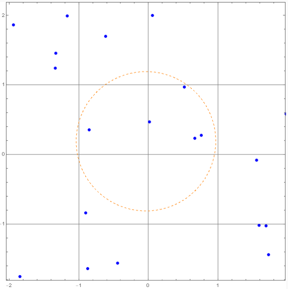
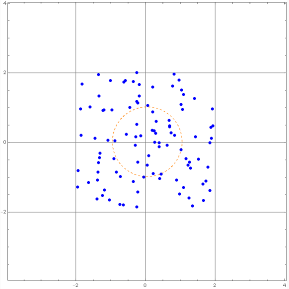
1. Regardless of the initial conditions, the agents remain in a spatially cohesive configuration around the center of mass.
2. If the center of mass escapes to infinity, then all agents *synchronize* their directions of motion.
3. If the center of mass remains bounded, then the agents exhibit *oscillatory* patterns.

3. Simulation Results: $\ddot{r}_k = (1 - |\dot{r}_k|^2)\dot{r}_k - (r_k - R)$

Left: particles with initial conditions uniformly distributed in $[-2, 2] \times [-2, 2]$, starting from rest. The frame is centered at the center of mass (eventually stationary).

Right: particles with initial velocities uniformly distributed in $[0, 1] \times [0, 1]$, starting in $[-1, 1] \times [-1, 1]$. The frame moves with the center of mass (speed approaches one).





4. $\ddot{r}_k = (1 - |\dot{r}_k|^2)\dot{r}_k - (r_k - R)$ vs other synchronizing systems

1. Van der Pol oscillators: if $r_k \in \mathbb{R}$; let by $v_k = \dot{r}_k$. We get
$$\ddot{v}_k = (1 - 3v_k^2)\dot{v}_k - \frac{1}{n}\sum_j (v_k - v_j)$$
Superimpose rectilinear oscillators in the plane: a pulsating star, periodic, unstable, made out of 1-dimensional Van der Pol oscillators, with stationary center of mass, $R(t) = R(0)$.

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2. Kuramoto oscillators: $\dot{\theta}_k = \omega_k - \frac{K}{n}\sum_j \sin(\theta_k - \theta_j)$.
Converting our r_k to polar coordinates leads to a *swarmalator*: the phase angles θ_k (and the angles for the velocity vectors) are influenced by each other and also by *spatial factors*.

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3. Bertozzi & all: $\ddot{r}_k = (1 - |\dot{r}_k|^2)\dot{r}_k - \sum_j \nabla U(r_k - r_j)$, with $U =$the Morse potential.

Heuristics and numerical simulations have been the driving factors in the study of swarms. The main challenge: reducing dimensions.

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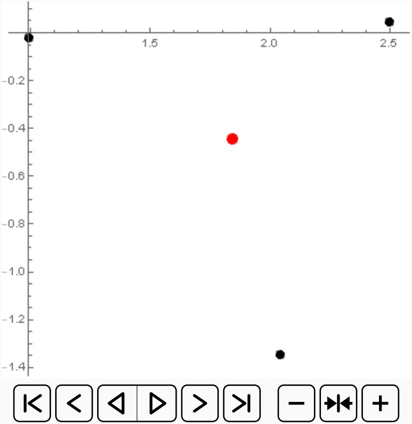
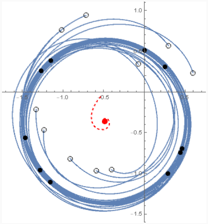
4. If the center of mass moves with constant velocity, the system must approach a translating state.

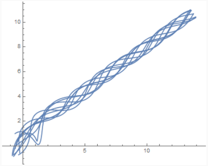
6. New Patterns: $\ddot{r}_k = (1 - |\dot{r}_k|^2)\dot{r}_k - (r_k - R)$



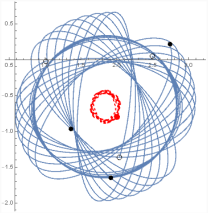
7. Limit Patterns, Absent Symmetries or Uniform Distribution



Rotating State



Translating State

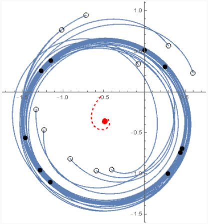


Mixed State

We work with *general* configurations of agents (no assumption of symmetry, or of uniform distribution), meaning that converting to a continuous PDE model is impossible.

Invariance under *rigid* translations and rotations: if $(r_1, \dots r_n)$ is a solution, given any translation vector R_0 in $\mathbb{R}^2$ and any rotation angle θ_0 in $[0, 2\pi]$ then $(R_0 + e^{i\theta_0} r_1, R_0 + e^{i\theta_0} r_2, \dots R_0 + e^{i\theta_0} r_n)$ is also a solution.

8. Dynamics Near Rotating States: $\ddot{r}_k = (1 - |\dot{r}_k|^2)\dot{r}_k - (r_k - R)$



- Rotating State to start from:

$$r_k = e^{i\theta_{k,0}} e^{it}$$

$$R_0 = \frac{1}{n} \sum_{k=1}^{n} e^{i\theta_{k,0}} = 0.$$

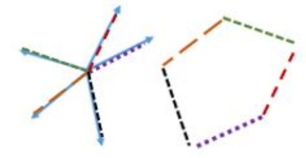
- Perturb.
- Use a rotating frame.

- Substitution (nearby):

$$r_k = e^{it} e^{i\theta_{k,0}} \left[a_k(t) + i b_k(t) \right]$$

- $4n$ unknowns: $a_k, b_k, \dot{a}_k, \dot{b}_k$.
- Rotating state/ fixed point:

$$a_k = 1, b_k = 0, \dot{a}_k = 0, \dot{b}_k = 0$$



9. Model Simplification? The Jacobian for Rotating States

$$J = \begin{array}{c} \\ \partial \dot{a} \\ \partial \dot{b} \\ \partial \dot{q} \\ \partial \dot{p} \end{array} \begin{array}{c} \begin{array}{cccc} \partial a & \partial b & \partial q & \partial p \end{array} \\ \begin{pmatrix} \mathrm{O}_n & \mathrm{O}_n & \mathrm{I}_n & \mathrm{O}_n \\ \mathrm{O}_n & \mathrm{O}_n & \mathrm{O}_n & \mathrm{I}_n \\ C & -S & \mathrm{O}_n & 2\mathrm{I}_n \\ S - 2\mathrm{I} & C & -2\mathrm{I}_n & -2\mathrm{I}_n \end{pmatrix} \end{array}$$

$$S = \left\{\tfrac{1}{n}\sin(\theta_{m,0} - \theta_{k,0})\right\}_{k,m=1}^{n}, \quad C = \left\{\tfrac{1}{n}\cos(\theta_{m,0} - \theta_{k,0})\right\}_{k,m=1}^{n}.$$

Two major issues:

- n (or $n+1$) directions are neutral directions; the other $3n$ are stable.
- the rotating states are not isolated (most theoretical tools require they be detangled).

10. Dimension-Reduction: Study the Flow on the Cental Manifolds

Consider the system

$$\begin{aligned} x' &= A_c x + f(x,y) \\ y' &= B_s y + g(x,y), \text{ with} \qquad (x,y) \in \mathbf{R}^c \times \mathbf{R}^s \end{aligned}$$

where all the eigenvalues of the matrix A_c have zero real parts and all the eigenvalues of the matrix B_s have negative real parts.

There exists a locally invariant manifold $y = h(x)$ tangent to the neutral directions, such that the stability of the original system is equivalent to that of

$$x' = A_c x + f(x, h(x)).$$

Taylor approximations for h can be computed using:

$$\nabla h(x)[A_c x + f(x, h(x))] = B_s h(x) + g(x, h(x))$$

11. Taylor Approximations Don't Capture the Dynamics in the Presence of Non-Isolated Fixed Points; An Example

A 3-dimensional system with a with a 1-dimensional set of equilibrium points within a 2-dimensional center manifold:

$$\begin{aligned}\dot{x} &= xy - x^2 \sin x \\ \dot{y} &= -y^3 + y^2 z \\ \dot{z} &= -z + x(y - x\sin x)(\sin x + x\cos x) + x\sin x.\end{aligned}$$

Rewrite the last equation: $\dfrac{d(z - x\sin x)}{dt} = (-1)(z - x\sin x)$

- Fixed points: $\{(x, x\sin x, x\sin x),\ x \in \mathbb{R}\}$.
- The origin has Jacobian matrix $\mathrm{diag}\{0, 0, -1\}$.
- The surface $\{(x, y, x\sin x)\}$ is the center manifold.
- The stable manifold is the $z-$axis.

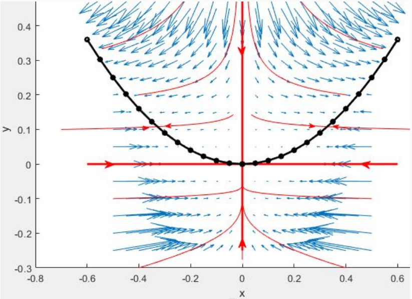
12. Taylor Approximations Failing in the Presence of Non-Isolated Fixed Points; An Example, cont

Substitute the central manifold equation $z = x \sin x$ to reduce the system to two dimensions:

$$\begin{aligned} \dot{x} &= x(y - x \sin x) \\ \dot{y} &= -y^2(y - x \sin x) \end{aligned}$$

- The origin is stable (but not asymptotically stable).
- The $\omega-$ limit points are precisely the equilibrium points $(x, x \sin x)$.
- The only points in the upper half plane whose limit point is the origin are the points on the $y-$ axis.

13. Taylor Approximations Failing in the Presence of Non-Isolated Fixed Points; An Example, cont



In black: the fixed points of the system, i.e the curve $y = x \sin x$.

In red: the trajectories near the origin, converging to the fixed points of the system.

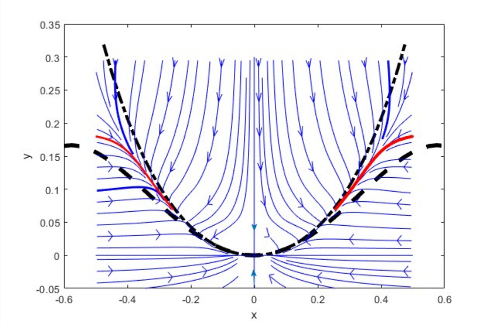
The origin is stable, but not asymptotically stable.

14. Taylor Approximations Failing in the Presence of Non-Isolated Fixed Points; An Example, cont

Approximate the central manifold flow using Taylor polynomials:

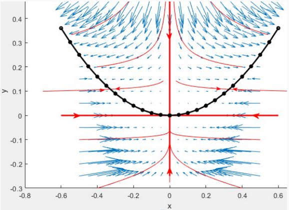
$$\begin{aligned} \dot{x} = & \quad x(y - x\sin x) \\ \dot{y} = & \quad -y^2(y - xT_{2k-1}(x)), \end{aligned}$$

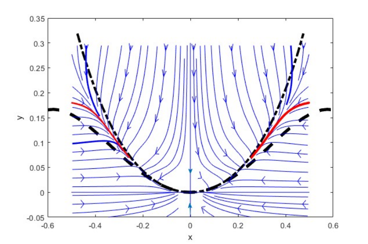
where $T_{2k-1}(x) =$ the Taylor polynomial of degree $2k-1$ for $\sin x$.
When k is even $xT_{2k-1}(x) \leq x\sin x$, and the truncated flow is:



15. Taylor Approximations Failing: Splitting the Isoclines

Left: the true flow. Right: the truncated (Taylor) flow.



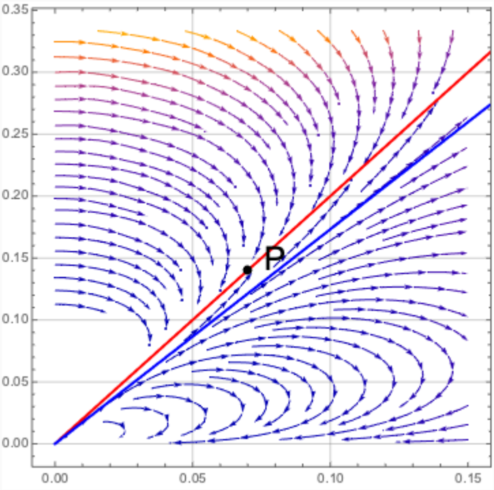


The coincidence between the isoclines $\dot{x} = 0$ and $\dot{y} = 0$ is lost.

The flow breaches the isocline $y = x \sin x$; there is transport across what was supposed to be a set of fixed points.

Once a trajectory enters the region between the isoclines $y = xT(x)$ and $y = x \sin x$, it is trapped there, and it approaches $(0, 0)$.

16. Challenges of having non-isolated fixed points: degeneracy



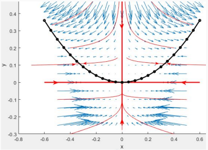
17. A New Approximation for the Central Manifold $y = h(x)$

Assume that the system is in standard form $\begin{aligned} x' &= A_c x + f(x,y) \\ y' &= B_s y + g(x,y), \end{aligned}$
where A_c has only eigenvalues $\lambda = 0$; B_s has Re $\lambda < 0$.

Assume the set E of equilibrium points is known and that there exists some curve $\gamma_E \subset E$.

Goal: approximate $h(x)$ with arbitrary precision near the fixed points, so that the reduced-dimension flow $\dot{x} = A_c x + f(x, h(x))$ is consistent with the scales of the original problem.

Use traditional Taylor approximation in sectors away from E.



18. A New Approximation for the Central Manifold $y = h(x)$

Use a contraction operator on the space of slow-growing functions to approximate $h(x)$ and the vector field $f(x, h(x))$.

If it is possible to explicitly find $h_0(x)$ such that $B_s h_0(x) + g(x, h_0(x)) = 0$, start there. Otherwise begin with an approximate solution: a function h_0 such that $B_s h_0(x) + g(x, h_0(x)) = \mathcal{O}(|x| dist(x, E))$.

Iterate, and keep track of the errors. After one step

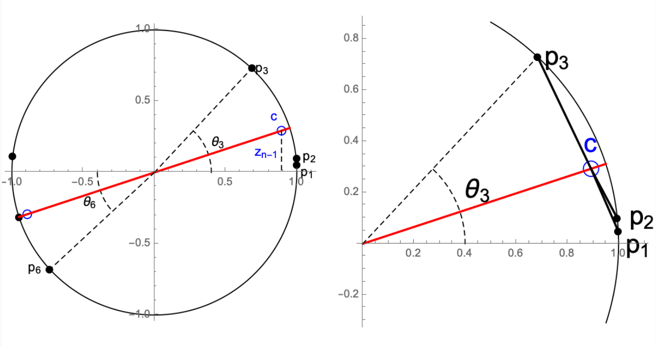
$$|f(x, h(x)) - f(x, h_{approx}(x))| = \mathcal{O}(|x|^2 dist(x, E)).$$

Using $h_{approx} = \mathbf{x} \sin x + x(y - x \sin x)(\sin x + x \cos x)$ with error $|y - x \sin x| \mathcal{O}(|(x, y)|^2)$ we get a system with an equivalent flow:

$$\dot{y} = -y^2 (y - x \sin x)[1 + \mathcal{O}(||(x, y)||)]$$

19. Stability of Rotating State Configurations

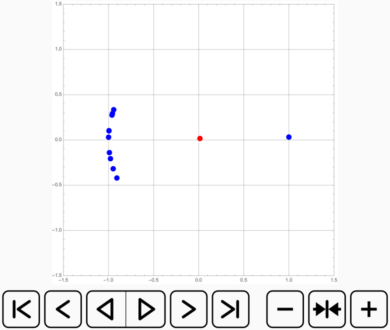
Theorem: Every rotating state solution is stable. Furthermore, every solution that starts near a rotating state converges to a nearby rotating state.



- Left/Right Group Dispersions
- Collapsing particles: very slow convergence (rate $\frac{1}{\sqrt{t}}$).

20. Rotating State Are Stable, but Very Slow Convergence Rate: $1/\sqrt{t}$

- If n is odd, the rate of convergence is exponential.
- If n is even, the rate of convergence is much slower: $\frac{1}{\sqrt{t}}$.



21. Ultimate Boundedness for Other Network Topologies

Theorem: (1) Dissipative in the $(\dot{r}_k, \ddot{r}_k)$-coordinates. (2) $r_k - R$ is ultimately bounded.

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Consider

$$\ddot{r}_k = -p_k(|\dot{r}_k|)\dot{r}_k - \sum_m a_{k,m} r_m,$$

$r_k \in \mathbb{R}^d$, where $A = \{a_{k,m}\}$ is a symmetric positive-semidefinite matrix, and $p_k(z)z \to \infty$ as $z \to \infty$.

For the parabolic potential system, $A = I - \frac{1}{n}\mathbb{I}$.

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Theorem. Dissipative in the $(\dot{r}_k, \ddot{r}_k)$-coordinates.

22. Ultimate Boundedness

Theorem.

- Consider the system

$$\ddot{r}_k = -p_k(|\dot{r}_k|)\dot{r}_k - \sum_{m=1}^{n} a_{k,m} r_m,$$

  where $A = \{a_{k,m}\}$ is symmetric and and positive semidefinite, $r_k \in \mathbb{R}^d$, and $p_k(r)r \to \infty$ as $r \to \infty$. $\exists$ C_1 and C_2 such that for any solution $r(t)$ we have that

$$|\dot{r}_k(t)| \leq C_1, \; |\ddot{r}_k(t)| \leq C_2$$

  for all t large enough.
- If A is invertible, then $\exists$ C_3 such that $|r_k(t) \leq C_3|$ for all t large enough.
- If A is non-invertible and Q is the projection onto its kernel, then $|r_k(t) - \sum_{j=1}^{n} q_{k,j} r_j(t)| \leq C_3$ for all t large enough.

The End; Full biblio references in the first listed (arXiv) paper

C. Kolon, C. Medynets and I. Popovici,*On the stability of a multi-agent system satisfying a generalized Lienard equation*, arXiv:2105.11419

C. Medynets and I. P.,*On Spatial Cohesiveness of Second Order Self-Propelled Swarming Systems*, arXiv:2110.06344v2

K. P. O'Keeffe, H. Hong, and S. Strogatz, *Oscillators that sync and swarm*, Nature Communications, 2017

M. Dorsogna,Y. Chuang, A. Bertozzi, and L. Chayes, *Self-Propelled Particles with Soft-Core Interactions: Patterns, Stability, and Collapse*, Physical review letters, 2006

A. Haraux and M. A. Jendoubi *The convergence problem for dissipative autonomous systems*, Springer Briefs, 2015

J. Hale, *Diffusive coupling, dissipation, and synchronization*, 1997

24. Recent Midshipmen Projects

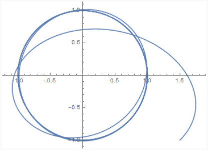
Swarming Dynamics on Riemannian Manifolds, MIDN Cami Herman, 2022.

Dynamical Systems with Delayed Response, MIDN Rachel Manhertz, 2022.

Synchronization of Coupled Nonlinear Oscillators: The Asymmetry of East-West Jet Lag, MIDN Hunter McGavran, 2020.

Stability of Nonlinear Swarms on Surfaces, MIDN Carl Kolon, 2018

Stationary R: $\ddot{r} = (1 - |\dot{r}|^2)\dot{r} - r$. Stable Rotating State.



Substitution: $u = r \cdot r$, $v = \dot{r} \cdot \dot{r}$, $w = r \cdot \dot{r}$.

$$\begin{cases} \dot{u} = 2w \\ \dot{v} = 2v(1-v) - 2w \\ \dot{w} = w(1-v) - u + v \end{cases}$$

Rotating state corresponds to $(u = 1, v = 1, w = 0)$

- Eigenvalues: $\{-0.35 \pm 1.72i, -1.2956\}$.
- **Invariant Surface**: $uv - w^2 = 0$ (1-dim, $\dot{r}(0) \| \ddot{r}(0)$)
- Lyapunov Function: $\boxed{V = u + v - \log(uv - w^2)}$.
- Full Derivative $\frac{dV}{dt} = -2(1 - v)^2$.
- **LaSalle's Invariance Principle**: (u, v, w) *converges to the rotating state).*