Complete quaternionic Kähler manifolds with ends of finite volume

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QK manifolds

Definition

A Riemannian manifold (M, g) of $\mathrm{scal} \neq 0$ is called quaternionic Kähler if

1. $\exists$ parallel skew-symmetric almost quaternionic structure $Q \subset \mathrm{End}\, TM$ and
2. if $\dim M = 4$, then (M, g) is half-conformally flat Einstein.

Facts

1. QK $\implies$ Einstein and $\mathrm{Hol} \subset \mathrm{Sp(n)Sp(1)}$.
2. Up to homothety, $\exists$ only finitely many complete QK manifolds of $\mathrm{scal} > 0$ in every dimension (LeBrun-Salamon finiteness theorem, Invent. 89).
3. LeBrun-Salamon conjecture: all of these are symmetric.
4. $\exists$ non locally symmetric complete QK manifolds of $\mathrm{scal} < 0$ (Alekseevsky 75).

Statement of the problem

State of the art

All known examples of complete quaternionic Kähler manifolds of finite volume are locally symmetric.

Compare

For the holonomies $\neq \mathrm{Sp}(1)\mathrm{Sp}(n)$ from Berger's list (1955) even compact non locally symmetric examples are known. In the Ricci-flat case this is due to Yau 1978, Beauville 1983, and Joyce 1996.

Problem

Do there exist complete non locally symmetric quaternionic Kähler manifolds of finite volume or, more generally, with an end of finite volume?

Idea of our construction

Outline

- Start with a QK symmetric space X of noncompact type. It has compact smooth quotients X/Γ (Borel 1963), where $\Gamma \subset \mathrm{Isom}(X)$.
- Deform X such that it is no longer homogeneous but still QK and complete.
- Do this in a controlled way such that the deformed space X_{def} does still have a large group $G \subset \mathrm{Isom}(X_{\mathrm{def}})$ admitting a lattice Γ_{def}.
- If G acts with cohomogeneity one, $M/G \cong \mathbb{R}$ and G/Γ_{def} is compact, then $X_{\mathrm{def}}/\Gamma_{\mathrm{def}}$ has precisely two ends.
- Compute the volume of the ends.

Deforming the symmetric space

1. We begin with the QK symmetric space

$$\bar{N} = \bar{N}_n = \frac{\mathrm{SU}(2,n)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(n))}.$$

   It is a c-map space and thus admits a one-loop deformation $(\bar{N}, g^c_{\bar{N}})$ (Robles-Llana-Sauressig-Vandoren, JHEP 2006; generalizing Antoniadis-Minasian-Theisen-Vanhove)

2. A geometric proof of the fact that the one-loop deformation of any c-map space is QK was given by Alekseevsky-C.-Mohaupt, CMP 2013, Alekseevsky-C.-Dyckmanns-Mohaupt, JGP 2015 based on HK/QK.

3. $(\bar{N}, g^c_{\bar{N}})$ is complete if and only if $c \geq 0$ (C.-Dyckmanns-Suhr 2017).

The case $n = 1$

The metric $g^c_{\bar N}$ for $n = 1$:

$$g^c_{\bar N} = \frac{1}{4\rho^2}\left[\frac{\rho+2c}{\rho+c}\,\mathrm{d}\rho^2 + \frac{\rho+c}{\rho+2c}(\mathrm{d}\tilde\phi + \zeta^0\mathrm{d}\tilde\zeta_0 - \tilde\zeta_0\mathrm{d}\zeta^0)^2 \right.$$
$$\left. + 2(\rho+2c)\left((\mathrm{d}\tilde\zeta_0)^2 + (\mathrm{d}\zeta^0)^2\right)\right],$$

where $\rho > 0, \tilde\phi, \zeta^0, \tilde\zeta_0$ are real coordinates on $\mathbb{R}^{>0} \times \mathbb{R}^3 \cong \mathbb{R}^4$.

- For $c = 0$ we obtain the complex hyperbolic plane $(\bar N_1, g^0_{\bar N}) = \mathbb{C}H^2$.

A characterization in four dimensions

Theorem (C.-Murcia, arXiv 2022)

The manifolds $(\bar{N}_1, g^c_{\bar{N}})$, $c \geq 0$, are the only 4-dimensional complete QK manifolds with a principal isometric action of the Heisenberg group.

Theorem (C.-Saha, Ann. Mat. Pura Appl. 2021)

The manifolds $(\bar{N}_1, g^c_{\bar{N}})$, $c \geq 0$, are the only complete Einstein four-manifolds with a principal isometric action of the Heisenberg group and an additional $\mathrm{SO}(2)$-symmetry.

The one-loop deformed QK metric on $B \times \mathbb{C}^* \times \mathbb{C}^n$

$$g^c_{\bar{N}} = \frac{\rho + c}{\rho} g_{\mathbb{C}H^{n-1}} + \frac{1}{4\rho^2}\frac{\rho + 2c}{\rho + c} d\rho^2 + \frac{1}{4\rho^2}\frac{\rho + c}{\rho + 2c}\Bigg(d\tilde{\phi}$$

$$-4\,\mathrm{Im}\left(\bar{w}_0 dw_0 - \sum_{a=1}^{n-1}\bar{w}_a dw_a\right) + \frac{2c}{1-|X|^2}\mathrm{Im}\left(\sum_{a=1}^{n-1}\bar{X}^a dX^a\right)\Bigg)^2$$

$$-\frac{2}{\rho}\left(dw_0 d\bar{w}_0 - \sum_{a=1}^{n-1} dw_a d\bar{w}_a\right) + \frac{\rho + c}{\rho^2}\frac{4}{1-|X|^2}\left|dw_0 + \sum_{a=1}^{n-1} X^a dw_a\right|^2$$

The complex hyperbolic PSK metric

$$g_{\mathbb{C}H^{n-1}} = \frac{1}{1-|X|^2}\left(\sum_{a=1}^{n-1}|dX^a|^2 + \frac{1}{1-|X|^2}\left|\sum_{a=1}^{n-1}\bar{X}^a dX^a\right|^2\right),$$

defined on the ball $B = \{|X|^2 = \sum_{a=1}^{n-1}|X^a|^2 < 1\} \subset \mathbb{C}^{n-1}$.

Isometries

Theorem (C.-Saha-Thung, Manuscr. Math. 2021)

The one-loop deformation $(\bar{N}, g^c_{\bar{N}})$, $c > 0$, has a non-constant curvature invariant. Thus it is a complete locally inhomogeneous QK manifold.

Theorem (C.-Saha-Thung, Trans. London Math. Soc. 2021)

The one-loop deformation of any homogeneous c-map space admits a group G of isometries which acts with cohomogeneity one.

Theorem (C.-Röser-Thung, arXiv 2021)

For the spaces $(\bar{N}, g^c_{\bar{N}})$, $c > 0$, the cohomogeneity one group G is closed in $\mathrm{Isom}(\bar{N}, g^c_{\bar{N}})$ and (for $n \geq 2$) has the structure

$$G = (\widetilde{\mathrm{SU}}(1, n-1) \ltimes \mathrm{Heis}_{2n+1})/\mathbb{Z},$$

where $\mathbb{Z}$ is a diagonally embedded central subgroup.

The corresponding Lie algebra of Killing fields

$$\mathfrak{g} = \mathrm{Lie}\, G = \mathfrak{su}(1, n-1) \ltimes \mathfrak{heis}_{2n+1}.$$

The extended complex hyperbolic Killing fields

Real and imaginary parts of the following vector fields generate the subalgebra $\mathfrak{su}(1, n-1) \subset \mathfrak{g}$.

$$Y_a = \frac{\partial}{\partial \bar{X}^a} - X^a \sum_{b=1}^{n-1} X^b \frac{\partial}{\partial X^b} - w_0 \frac{\partial}{\partial w_a} - \bar{w}_a \frac{\partial}{\partial \bar{w}_0} + icX^a \frac{\partial}{\partial \tilde{\phi}}.$$

The Heisenberg algebra

The vector field $\partial/\partial\tilde{\phi}$ together with the real and imaginary parts of the following vector fields span the subalgebra $\mathfrak{heis}_{2n+1} \subset \mathfrak{g}$.

$$V_a = \frac{\partial}{\partial w_a} - i\bar{w}_a \frac{\partial}{\partial \tilde{\phi}}, \qquad V_0 = \frac{\partial}{\partial w_0} + i\bar{w}_0 \frac{\partial}{\partial \tilde{\phi}}.$$

Are there more Killing fields?

Yes:

There is an additional Killing field

$$-i\sum\left(w_k\frac{\partial}{\partial w_k}-\bar{w}_k\frac{\partial}{\partial \bar{w}_k}\right)$$

extending $\mathfrak{g}=\mathfrak{su}(1,n-1)\ltimes\mathfrak{heis}_{2n+1}$ to

$$\mathfrak{u}(1,n-1)\ltimes\mathfrak{heis}_{2n+1}.$$

Theorem (C.-Saha-Thung, TLMS 2021)

For $c>0$ we have

$$\mathrm{Isom}_0(\bar{N}_1,g_{\bar{N}}^c)=\mathrm{U}(1)\ltimes\mathrm{Heis}_3\subset\mathrm{Isom}(\bar{N}_1,g_{\bar{N}}^c)=\mathrm{O}(2)\ltimes\mathrm{Heis}_3.$$

Quotients

Theorem (C.-Röser-Thung)

- For all $n \geq 1$ there exist lattices

$$\Gamma \subset G \subset \mathrm{Isom}(\bar{N}, g^c_{\bar{N}})$$

  such that $(\bar{N}, g^c_{\bar{N}})/\Gamma$ is diffeomorphic to a cylinder $\mathbb{R} \times K$ with fibers $\{t\} \times K$ of finite volume.
- Moreover, $\{t > t_0\}$ has finite volume for all t_0, while $\{t < t_0\}$ has infinite volume ($\rho = e^t$).
- When $n \leq 2$, there are co-compact lattices Γ as above.

Corollary (C.-Röser-Thung)

There exist complete locally inhomogeneous QK manifolds with an end of finite volume in dimensions 4 and 8.

The examples in dimensions 4 and 8

Remark

- It is convenient to consider the group

$$\bar{G} = \mathrm{SU}(1, n-1) \ltimes \mathrm{Heis}_{2n+1},$$

which acts on a cyclic quotient $\hat{N} = \bar{N}/\mathbb{Z}$ of $\bar{N}$.
- The image of Γ in this group is $\bar{\Gamma} = \bar{\Gamma}_1 \ltimes \Gamma_2$, where $\bar{\Gamma}_1$, Γ_2 are lattices in $\mathrm{SU}(1, n-1)$ and Heis_{2n+1}, respectively.

The 4D examples

- For $n = 1$, $\Gamma = \bar{\Gamma} = \Gamma_2$ is just a lattice in Heis_3 and the constructed QK manifolds are diffeomorphic to a cylinder over a nilmanifold $K = \mathrm{Heis}_3/\Gamma$.

The 8D examples

Co-compact lattices in $\mathrm{SU}(1,1) \ltimes \mathrm{Heis}_5$

- Let b be a prime number and $a \in \mathbb{N}$ not a square mod b. Consider $\mathcal{O}_{a,b} = \mathrm{span}_{\mathbb{Z}}\{\mathbb{1}, I, J, K\}$, where

$$I = \sqrt{a}i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J = \sqrt{b} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K = IJ = -JI.$$

  Then $\bar{\Gamma}_{1,a,b} = \mathcal{O}_{a,b} \cap \mathrm{SU}(1,1) = \{A \in \mathcal{O}_{a,b} : \det A = 1\}$ is a co-compact Fuchsian group, which preserves the lattice Γ_2 in $\mathrm{Heis}_5 \cong \mathbb{C}^2 \times \mathbb{R}$ generated by $\mathcal{O}_{a,b} \cdot \binom{1}{0} \subset \mathbb{C}^2$.
- Up to passing to a suitable finite index normal subgroup, our lattices are $\bar{\Gamma} = \bar{\Gamma}_{1,a,b} \ltimes \Gamma_2$, where the one-loop parameter c is chosen to be a rational multiple of $\frac{\sqrt{ab}}{\pi}$.

Finite volume for large ρ

Lemma (C.-Röser-Thung)

The volume density of g_N^c in the given coordinates is of the form

$$f = \frac{1}{\rho^{n+2}} \left(1 + \frac{c}{\rho}\right)^{n-1} \left(1 + \frac{2c}{\rho}\right) f_{\mathrm{inv}},$$

where f_{inv} is the ρ-independent volume density of the locally homogeneous fibers.

Corollary

The volume of the domain $\{\rho > \rho_0\}$ (modulo Γ) is finite and asymptotic to

$$\mathrm{vol}\{\rho > \rho_0\} \sim \frac{k}{\rho_0^{n+1}} \quad (\rho_0 \to \infty).$$

Infinite volume for small ρ

Recall

$$f = \frac{1}{\rho^{n+2}} \left(1 + \frac{c}{\rho}\right)^{n-1} \left(1 + \frac{2c}{\rho}\right) f_{\mathrm{inv}},$$

Corollary

The volume of the domain $\{\rho < \rho_0\}$ (modulo Γ) is infinite and we have the following asymptotics for $\rho_1 \to 0$.

1. If $c = 0$, then

$$\mathrm{vol}\{\rho_1 < \rho < \rho_0\} = \frac{k}{\rho_1^{n+1}} + \mathrm{const}.$$

2. If $c > 0$, then

$$\mathrm{vol}\{\rho_1 < \rho < \rho_0\} = \frac{k_1}{\rho_1^{2n+1}} + O\left(\frac{1}{\rho_1^{2n}}\right).$$

Open problems

Question 1

Do there exist complete quaternionic Kähler manifolds of dimension ≥ 12 with a finite volume end, that are not locally symmetric?

Question 2

Can one use non-perturbative quantum corrections in string theory to construct complete QK manifolds of finite volume?

State of the art

- Structure of these instanton corrections is extremely complicated (Alexandrov, Pioline ...).
- A well defined QK metric has only been obtained in special situations: C.-Tulli, Ann. Henri Poincaré 2022, building on Alexandrov-Banerjee, JHEP 2015.

Instanton corrections

Theorem (C.-Tulli)

The quaternionic Kähler manifolds described in this talk can be deformed in a neighborhood of $\rho = \infty$ such as to include instanton corrections associated with mutually local variations of BPS structures.

Remarks

- We do not known when such metric can be extended to a complete QK metric and when the volume of the extended metric is finite.
- It is expected that the volume becomes finite after inclusion of all instanton corrections predicted by string theory if the metric describes an effective field theory consistently coupled to gravity at the quantum level ($\rightarrow$ swampland distance conjectures).

What is a variation of BPS structures?

Definition

A variation of BPS structures over a complex manifold M is a tuple (M, Λ, Z, Ω), where:

- $\Lambda \to M$ is a local system of lattices $\Lambda_p \cong \mathbb{Z}^r$ with skew, integer-valued pairing $\langle -, - \rangle$.
- Z is a holomorphic section of $\Lambda^* \otimes \mathbb{C} \to M$.
- $\Omega : \Lambda \to \mathbb{Z}$ is a function (of sets) satisfying $\Omega(\gamma) = \Omega(-\gamma)$ and the Kontsevich-Soibelman wall-crossing formula (WCF).

It is admissible if

- $\forall K \subset M$ compact and parallel norm on $\Lambda|_K \otimes \mathbb{C}$ $\exists C > 0$:

$$\forall \gamma \in \Lambda|_K \cap \mathrm{Supp}(\Omega) : \quad |Z_\gamma| > C|\gamma|.$$

- $\forall R > 0$

$$\sum_{\gamma \in \Lambda_p} |\Omega(\gamma)| e^{-R|Z_\gamma|}$$

converges normally on compact subsets of M.

Mutual locality

Definition

An admissible variation of BPS structures (M, Λ, Z, Ω) is called mutually local if

$$\langle \gamma, \gamma' \rangle = 0$$

for all $\gamma, \gamma' \in Supp(\Omega)$.

Remarks

- Mutual locality implies that there are no walls:

  $$\{p \in M \mid \exists \gamma, \gamma' \in \mathrm{Supp}(\Omega) \cap \Lambda_p, \ \langle \gamma, \gamma' \rangle \neq 0, \ Z_\gamma / Z_{\gamma'} \in \mathbb{R}_{>0}\} = \varnothing$$

  The only remnant of the WCF in this setting is that $\Omega(\gamma)$ is constant for every local section γ of Λ.
- In our setting M is a conical affine special Kähler manifold, $\Lambda \subset TM$ is a ∇-parallel lattice and $Z : M \to \Lambda^* \otimes \mathbb{C}$ is a conical Kählerian Lagrangian section.

Form of the QK metric (for a PSK domain $\overline{M}$)

$$
\begin{aligned}
g_{\overline{N}} = & \frac{\rho + c}{\rho + f^{\text{inst}}} \left(g_{\overline{M}} - e^{\mathcal{K}} \sum_{\gamma} \Omega(\gamma) V_{\gamma}^{\text{inst}} \left| dX_{\gamma} + X_{\gamma} \left(\frac{d\rho}{2(\rho + c)} + \frac{d\mathcal{K}}{2} \right) \right|^2 \right) \\
& + \frac{1}{2(\rho + f^{\text{inst}})^2} \left(\frac{\rho + 2c - f^{\text{inst}}}{2(\rho + c)} d\rho^2 + 2 d\rho df^{\text{inst}}|_{\overline{N}} + (df^{\text{inst}})^2|_{\overline{N}} \right) \\
& + \frac{4(\rho + c + f_-^{\text{inst}})}{(\rho + f^{\text{inst}})^2 (\rho + 2c - f_1^{\text{inst}})} \left(d\sigma - \frac{1}{4\pi} \langle \theta, d\theta \rangle - \frac{c}{4} d^c \mathcal{K} + \eta_+^{\text{inst}}|_{\overline{N}} + \frac{f_+^{\text{inst}} - c}{\rho + c + f_-^{\text{inst}}} \eta_-^{\text{inst}}|_{\overline{N}} \right)^2 \\
& - \frac{1}{2\pi(\rho + f^{\text{inst}})} (W_i + W_i^{\text{inst}}|_{\overline{N}}) (N + N^{\text{inst}})^{ij} (\overline{W}_j + \overline{W}_j^{\text{inst}}|_{\overline{N}}) \\
& + \frac{(\rho + c) e^{\mathcal{K}}}{\pi(\rho + f^{\text{inst}})^2} \left| X^i (W_i + W_i^{\text{inst}}|_{\overline{N}}) + 2\pi i \sum_{\gamma} \Omega(\gamma) A_{\gamma}^{\text{inst}}(V) \left(dX_{\gamma} + X_{\gamma} \left(\frac{d\rho}{2(\rho + c)} + \frac{d\mathcal{K}}{2} \right) \right) \right|^2 \\
& + \frac{\rho + c + f_-^{\text{inst}}}{\rho + f^{\text{inst}}} \left(\frac{d^c \mathcal{K}}{2} + \frac{2}{\rho + c + f_-^{\text{inst}}} \eta_-^{\text{inst}}|_{\overline{N}} \right)^2 - \frac{\rho + c}{\rho + f^{\text{inst}}} \left(\frac{d^c \mathcal{K}}{2} \right)^2, \quad \text{where}
\end{aligned}
$$

$$
\begin{aligned}
\eta_{\pm}^{\text{inst}} &:= \frac{1}{2} \left(\left(2\pi \eta^{\text{inst}} - \frac{f_1^{\text{inst}}}{2} \widetilde{\eta} \right) \pm \left(2\pi \sum_{\gamma} \Omega(\gamma) \eta_{\gamma}^{\text{inst}} - \frac{f^{\text{inst}}}{2} \widetilde{\eta} \right) \right), \\
f^{\text{inst}} &:= 4\pi \sum_{\gamma} \Omega(\gamma) \iota_V \eta_{\gamma}^{\text{inst}}, \qquad f_1^{\text{inst}} := 4\pi \iota_V \eta^{\text{inst}}, \\
\eta_{\gamma}^{\text{inst}} &:= \frac{i}{8\pi^2} \sum_{n>0} \frac{e^{in\theta_{\gamma}}}{n} |Z_{\gamma}| K_1(2\pi n |Z_{\gamma}|) \left(\frac{dZ_{\gamma}}{Z_{\gamma}} - \frac{d\overline{Z}_{\gamma}}{\overline{Z}_{\gamma}} \right), \\
\eta^{\text{inst}} &:= \sum_{\gamma} \Omega(\gamma) \eta_{\gamma}^{\text{inst}} - \iota_V \left(\sum_{\gamma} \Omega(\gamma) V_{\gamma}^{\text{inst}} |dZ_{\gamma}|^2 + \frac{1}{4\pi^2} Y_i M^{ij} \overline{Y}_j \right),
\end{aligned}
$$

Form of the QK metric continued

$$
\begin{aligned}
V_\gamma^{\text{inst}} &:= \frac{1}{2\pi}\sum_{n>0} e^{in\theta_\gamma} K_0(2\pi n|Z_\gamma|),\\
A_\gamma^{\text{inst}} &:= -\frac{1}{4\pi}\sum_{n>0} e^{in\theta_\gamma}|Z_\gamma|K_1(2\pi n|Z_\gamma|)\Big(\frac{dZ_\gamma}{Z_\gamma}-\frac{d\overline{Z}_\gamma}{\overline{Z}_\gamma}\Big),\\
\widetilde{\eta} &:= d^c\log(r), \quad r^2=\frac{\rho+c}{2\pi},\\
W_i &:= d\theta_{\widetilde{\gamma}_i}-\tau_{ij}d\theta_{\gamma^j}, \qquad W_i^{\text{inst}} := \sum_\gamma \Omega(\gamma)n_i(\gamma)(2\pi A_\gamma^{\text{inst}}-iV_\gamma^{\text{inst}}d\theta_\gamma), \qquad Y_i := W_i + W_i^{\text{inst}},\\
N_{ij} &:= \mathrm{Im}\tau_{ij}, \qquad N_{ij}^{\text{inst}} := \sum_\gamma \Omega(\gamma)V_\gamma^{\text{inst}} n_i(\gamma)n_j(\gamma), \qquad M_{ij} := N_{ij}+N_{ij}^{\text{inst}}, \quad \text{where}\\
& dZ_{\widetilde{\gamma}_i} = \tau_{ij}dZ_{\gamma^j}, \quad \gamma = n_i(\gamma)\gamma^i \quad \text{for} \quad \gamma\in \mathrm{Supp}(\Omega)\subset \mathrm{span}\{\gamma^i\}, V=\widetilde{J\xi}^{\nabla}.
\end{aligned}
$$

Theorem (C.-Tulli)

The above is a QK metric and is obtained by applying the construction of [ACM,ACDM] to the following (indefinite) HK metric:

$$
g_N = dZ_{\gamma^i}M_{ij}d\overline{Z}_{\gamma^j} + \frac{1}{4\pi^2}Y_iM^{ij}\overline{Y}_j.
$$

Structure of the HK metric

Deforming the semi-flat metric

1. The instanton corrected HK metric

$$g_N = dZ_{\gamma^i} M_{ij} d\overline{Z}_{\gamma^j} + \frac{1}{4\pi^2} Y_i M^{ij} \overline{Y}_j$$

is a deformation (on some domain $N \subset T^*M/\Lambda^*$) of the rigid c-map metric (Cecotti-Ferrara-Girardello 1989)

$$g_{\mathrm{sf}} = dZ_{\gamma^i} N_{ij} d\overline{Z}_{\gamma^j} + \frac{1}{4\pi^2} W_i N^{ij} \overline{W}_j,$$

where $N_{ij} := \operatorname{Im} \tau_{ij}$ and $W_i := d\theta_{\widetilde{\gamma}_i} - \tau_{ij} d\theta_{\gamma^j}$ are determined by $dZ_{\widetilde{\gamma}_i} = \tau_{ij} dZ_{\gamma^j}$, and

2. the instanton terms in

$$M_{ij} := N_{ij} + N_{ij}^{\mathrm{inst}}, \quad Y_i := W_i + W_i^{\mathrm{inst}}$$

depend on the variation of BPS structure.

From HK to QK

Lemma (rotating Killing field)

1. The vector field $V = \widetilde{J\xi}^{\nabla}$ on T^*M/Λ^* is Killing, ω_3-Hamiltonian and rotating ($\mathcal{L}_V \omega_1 = -\omega_2$) on the domain $N \subset T^*M/\Lambda^*$ of the instanton corrected metric g_N.
2. The function $f = 2\pi r^2 - c + f^{\text{inst}}$ satisfies $df = -4\pi \iota_V \omega_3$.
3. The open set $N_+ := \{f > 0, f_1 < 0\} \subset N$, where $f_1 = f - 4\pi g_N(V, V)$, is non-empty. (From now on $N = N_+$.)

Lemma (hyperholomorphic connection)

$$P := (T^*M \times \mathbb{R})/\mathrm{Heis}(\Lambda^*)|_N \to N \subset T^*M/\Lambda^*$$

is a principal circle bundle with hyperhol. connection

$$\eta = \eta_{\text{cl}} + \eta^{\text{inst}},$$

where η_{cl} is the hyperhol. connection from [ACM] for g_{sf}.

From HK to QK continued

The construction

- Applying [ACM,ACDM] to $(P, \eta) \to (N, g_N, f)$ we obtain the QK manifold $(\overline{N}, g_{\overline{N}})$ as

$$\overline{N} := \{\mathrm{Arg}\, z^0 = 0\} \subset P, \quad g^c_{\overline{N}} := -\frac{1}{f}\left(g_P - \frac{2}{f}\sum_{i=0}^{3}(\theta_i^P)^2\right)\Bigg|_{\bar{N}},$$

where (z^i) are special coord. on M and

$$g_P := \frac{2}{f_1}\eta^2 + 2\pi g_N, \quad \theta_0^P := -\frac{1}{2}df,$$

$$\theta_1^P := \eta + \frac{\pi}{2}\iota_V g_N, \quad \theta_2^P := \frac{\pi}{2}\iota_V\omega_2, \quad \theta_3^P := -\frac{\pi}{2}\iota_V\omega_1.$$

- A direct calculation shows that $g_{\overline{N}}$ takes the explicit form on p. 22.