# Finite Group Actions, Cohomology of Groups and Rank Conjectures – Part I

Alejandro Adem

University of British Columbia

September 2022

# Quick Trip through Group Actions and Cohomology

We begin by recalling basic notions about group actions and cohomology of groups. Unless stated otherwise our groups $G$ will be assumed to be finite.

Recall that there exists a principal G-bundle $EG \to BG$ where $EG$ is a contractible space with a free action of $G$, which we can assume to be a $G$–CW complex. The orbit space $BG = EG/G$ is called the classifying space of $G$ given its role in the classification of principal $G$–bundles; from the point of view of homotopy theory it is a $K(G, 1)$ i.e. a connected space whose only non trivial homotopy group is $\pi_1(BG) = G$. Using the fact that $C_*(EG)$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ we can see that for any $\mathbb{Z}G$–module $M$,

$$H^*(BG, M) \cong Ext^*_{\mathbb{Z}G}(\mathbb{Z}, M) = H^*(G, M),$$

the cohomology of $G$ with coefficients in $M$.

One of the nicer properties of finite group cohomology is that it can be determined locally. Let $S_p(G)$ denote the lattice of all $p$–subgroups of $G$, which admits a natural action of $G$ by conjugation. Then we have a classical computation:

## Theorem (Cartan-Eilenberg)

*The restriction maps $H^*(G,\mathbb{F}_p) \to H^*(P,\mathbb{F}_p)$ where $P$ is a $p$–subgroup of $G$, induce an isomorphism*

$$H^*(G,\mathbb{F}_p) \cong \lim_{P\in S_p(G)} H^*(P,\mathbb{F}_p)$$

This limit term can be described as sequences of cohomology classes compatible with respect to maps induced by inclusion and conjugation. More explicitly, we have that $H^*(G,\mathbb{F}_p) \to H^*(Syl_p(G),\mathbb{F}_p)$ is injective, with image determined by the stability conditions arising from conjugation and inclusion.

Given a $G$–CW complex $X$ we can construct the homotopy orbit space (also known as the Borel construction)

$$EG \times_G X = EG \times X/G$$

where $G$ acts diagonally on $EG \times X$. We can then define the equivariant or Borel cohomology of $X$ as

$$H^*_G(X) = H^*(EG \times_G X, \mathbb{Z}) = Ext^*_{\mathbb{Z}G}(\mathbb{Z}, C^*(X))$$

where $C^*(X)$ denotes the cellular $G$–cochain complex of $X$. For an algebraist this is the $G$–hypercohomology of $C^*(X)$. We will assume that $X$ is a finite dimensional $G$–CW complex, with finitely generated homology.

**Example:** If $G$ acts smoothly on a compact manifold $M$, then this space has a compatible finite $G$–CW complex structure.

The first projection gives rise to a fibration

$$X \to EG \times_G X \to BG$$

yielding a Serre spectral sequence converging to $H^*_G(X)$ with $E_2$-term $E_2^{p,q} = H^p(G, H^q(X, \mathbb{Z}))$. The second projection map

$$EG \times_G X \to X/G$$

gives rise to a Leray spectral sequence with $E_1^{p,q} = H^q(G, C^p(X))$ also converging to $H^*_G(X)$ (this is related to Bredon cohomology). Note that if $G$ acts freely on $X$, then $EG \times_G X \simeq X/G$. Algebraically this corresponds to $C^*(X)$ being a free $\mathbb{Z}G$ chain complex, and the equivariant cohomology will be isomorphic to the cohomology of the invariants $C^*(X)^G$. More generally, $C^*(X)$ is a complex of permutation modules, so the $E_1$–term can be computed using the cohomology of the isotropy subgroups.

Using the fact that the homology of $X$ is finitely generated, the first spectral sequence shows that $H^*_G(X)$ is a finitely generated module over $H^*(BG)$. Using a unitary representation of $G$ we obtain a fibration with compact fibre

$$U((n)/G \to BG \to BU(n)$$

which can be used to show that $H^*(BG)$ is a finitely generated module over $H^*(BU(n)) \cong \mathbb{Z}[c_1, \ldots, c_n]$ where $c_1, \ldots, c_n$ denote the Chern classes, in even degrees. Thus from the point of view of commutative algebra the objects aren't all bad. If $p_G(t)$ denotes the Poincaré series for $H^*(EG \times_G X, \mathbb{F}_p)$, then as shown by Venkov,

$$p_G(t) = \sum_{i \geq 0} dim_{\mathbb{F}_p} \; H^i(EG \times_G X, \mathbb{F}_p) = \frac{r(t)}{\prod_{i=1}^{n}(1 - t^{2i})}$$

where $r(t) \in \mathbb{Z}[t]$. The order of the pole at $t = 1$ is the Krull Dimension of the equivariant cohomology ring.

The main results of Smith theory can be recovered using the cohomological methods first introduced by Borel. For a finite $p$–group $P$, we have:

- If $P$ acts on a space $X$ with the mod $p$ homology of a point, then $X^P \neq \emptyset$ and it has the mod $p$ homology of a point.
- If $X$ has the mod $p$ homology of a sphere then $X^P$ also has the mod $p$ homology of a sphere.

The basic ingredient here is to use a central subgroup of order $p$ and apply the fact that for $P = \mathbb{Z}/p\mathbb{Z}$, we have isomorphisms

$$H^i(EP \times_P X, \mathbb{F}_p) \cong H^i(BP \times X^P, \mathbb{F}_p)$$

for $i > dim\ X$. Also key to this is the cohomology of the group of prime order:

$$H^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = H^*(L_p^\infty, \mathbb{Z}) \cong \mathbb{Z}[u]/(pu) \text{ where deg u =2.}$$

In fact elementary abelian p–groups $V$ play a key role in transformation groups. Their cohomology mod p can be computed using the Kunneth formula. Let $V = (\mathbb{Z}/p\mathbb{Z})^m$, then for $p = 2$,

$$H^*(V, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \ldots, x_m] \text{ where deg } x_i = 1.$$

For $p$ odd

$$H^*(V, \mathbb{F}_p) \cong \Lambda^*(e_1, \ldots, e_m) \otimes \mathbb{F}_p[y_1, \ldots, y_m]$$

where deg $e_i$=1, deg $y_i = 2$.

### Theorem (Localization Theorem)

*For $V = (\mathbb{Z}/p\mathbb{Z})^m$ there exists a class $z \in H^2(BV, \mathbb{F}_p)$ such that the inclusion $X^V \to X$ induces an isomorphism*

$$H^*(EV \times_V X, \mathbb{F}_p) \cong H^*(BV \times X^V, \mathbb{F}_p)$$

*of $H^*(BV, \mathbb{F}_p)$–modules after inverting the powers of $z$.*

Note that there are important elaborations on this cohomological approach using homotopy fixed-points $Map_V(EV, X)$, due to Lannes and Dwyer-Wilkerson.

Another classical result due to Quillen is that for any finite group $G$, the Krull dimension of $H^*(EG \times_G X, \mathbb{F}_p)$ is equal to $r(X)$, the rank of the largest elementary abelian $p$–subgroup that fixes a point in $X$. In terms of group cohomology we have

Theorem (Quillen-Venkov)

*The restriction maps $H^*(G, \mathbb{F}_p) \to H^*(V, \mathbb{F}_p)$ induce an F–isomorphism*

$$H^*(G, \mathbb{F}_p) \to \lim_{V \in A_p(G)} H^*(V, \mathbb{F}_p)$$

**Example:** If $\Sigma_n$ denotes the symmetric group then in fact we have

$$H^*(\Sigma_n, \mathbb{F}_2) \cong \lim_{V \in A_2(G)} H^*(V, \mathbb{F}_2)$$

Recall that a complete resolution $\widehat{F}_*$ can be obtained by splicing a free resolution of $\mathbb{Z}$ with its dual. Following Swan, we can define the $G$–hypercohomology of $C^*(X)$ using a complete resolution, yielding the equivariant Tate cohomology of $X$, denoted $\widehat{H}^*_G(X)$. We list some properties that we will use later:

- If the $G$–action on $X$ is free, then $\widehat{H}^*_G(X) \equiv 0$, i.e. the cohomology of the orbit space no longer plays a role.
- Multiplication by $|G|$ always annihilates $\widehat{H}^*_G(X)$ i.e. it has a *finite exponent* that divides $|G|$.
- For $i > \dim X$, $\widehat{H}^i_G(X) \cong H^i(EG \times_G X, \mathbb{Z})$.
- As before there are two spectral sequences converging to $\widehat{H}^*_G(X)$, obtained from the two filtrations of the double complex $Hom_{\mathbb{Z}G}(\widehat{F}_*, C^*(X))$:

$$E_2^{p,q} = \widehat{H}^p(G, H^q(X)) \quad \text{and} \quad E_1^{p,q} = \widehat{H}^q(G, C^p(X)).$$

# Restrictions on Free Group Actions

We now focus on restrictions for free group actions.

**Question:** Given a finite complex $X$, can we describe the finite groups that act freely on $X$?

One basic restriction is given by the Euler characteristic: if $G$ acts freely on $X$, then $|G|$ must divide $\chi(X)$. So for example it's easy to see that the only non-trivial group acting freely on an even-dimensional sphere is $\mathbb{Z}/2\mathbb{Z}$. For odd dimensional spheres the situation is much more complicated.

## Theorem (Smith)

*If $G$ acts freely on $X = \mathbb{S}^n$ then it cannot contain $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ as a subgroup for any prime $p$.*

We observe that by the Lefschetz fixed-point theorem, the action must be trivial in homology if $n$ is odd.

Now for any subgroup $Q \subset G$, $\widehat{H}^*_Q(X) \equiv 0$ so the differential induces an isomorphism for all $p$:

$$d_{n+1} : \widehat{H}^p(Q, H^n(X, \mathbb{Z})) = \widehat{H}^p(Q, \mathbb{Z}) \to \widehat{H}^{p+n+1}(Q, \mathbb{Z}).$$

In particular this implies that $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ cannot be a subgroup of $G$, as its cohomology contains a polynomial algebra on two generators and so cannot have this periodic behaviour.

This condition on the cohomology of a group is called periodicity, and what this result shows is that in fact $G$ has periodic cohomology of period dividing $n+1$. Later Artin and Tate showed that this condition is in fact equivalent to every abelian subgroup of $G$ being cyclic. Such groups have been classified, in particular characterized by the condition that their $p$–Sylow subgroups are all cyclic or generalized quaternion.

We now describe a very general restriction on free group actions:

## Theorem (Browder)

*Let $X$ be a connected free $G$–CW complex. Then $|G|$ divides the product $\prod_{r=1}^{dimX} \; exp \; \widehat{H}^{-r-1}(G, H^r(X,\mathbb{Z}))$.*

The proof of this result is very simple. We will use the spectral sequence with $E_2$–term converging to $\widehat{H}^*_G(X) \equiv 0$, which has $E_2^{p,q} = \widehat{H}^p(G, H^q(X,\mathbb{Z}))$. Consider the term $E_2^{0,0} = \widehat{H}^0(G,\mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}$, which must be killed in the spectral sequence. The differentials involved here are

$$d_{r+1} : E_{r+1}^{-r-1,r} \to E_{r+1}^{0,0}$$

The terms are subquotients of the groups $\widehat{H}^{-r-1}(G, H^r(X,\mathbb{Z}))$ from which we obtain the desired result.

## Corollary

*If $G$ acts trivially on the cohomology of $X$, then $|G|$ divides $\prod_{r=1}^{dimX}$ $exp\ \widehat{H}^{-r-1}(G,\mathbb{Z})$.*

This relationship tells us that a certain amount of cohomological torsion must be present to allow for a free action on a connected complex.

We now specialize to the case when $V$ is a $p$–elementary abelian group. In this case note that for $k \neq 0$, $p \cdot \widehat{H}^k(V,\mathbb{Z}) = 0$. Recall that if $V = (\mathbb{Z}/p\mathbb{Z})^k$ then $k$ is referred to as the rank of $V$. For a finite connected complex $X$ let $d_p(X) = \#\{i > 0 \mid H^i(X,\mathbb{Z}_{(p)}) \neq 0\}$.

## Proposition

*If $V$ acts freely and homologically trivially on a connected finite complex $X$, then $rank(V) \leq d_p(X)$.*

Indeed, the previous result implies that $p^{rank(V)}$ must divide $p^{d_p(X)}$.

Some of these results extend to actions which aren't free by applying cohomological varieties and ideas due to J. Carlson. A key result is the following

## Theorem (AA)

*Let $X$ denote a connected $G$–CW complex. Let $r(X)$ denote the maximal rank among all isotropy subgroups of the action. Then there exist cohomology classes $\zeta_1, \ldots, \zeta_{r(X)} \in H^*(G, \mathbb{Z})$ such that $\exp \widehat{H}^*_G(X) | \prod_{i=1}^{r(X)} \exp \zeta_i$.*

This result says that the torsion in $\widehat{H}^*_G(X)$ has to be accounted for by at most $r(X)$ classes in $H^*(G, \mathbb{Z})$. For elementary abelian groups this yields

## Corollary

*If $V$ is a $p$–elementary abelian group acting on a finite connected complex $X$, then the exponent of $\widehat{H}^*_V(X)$ is equal to the order of an isotropy subgroup $V_x$ of maximal rank.*

We can use this approach to obtain analogous restrictions for non–free actions of elementary abelian $p$–groups. Now the top exponent in $\widehat{H}^0(V) \cong \mathbb{Z}/|V|\mathbb{Z}$ has to be reduced by at least $[V : V_x]$ in the spectral sequence, where $V_x \subset V$ is of maximal rank.

## Proposition

*Let $X$ be a connected $V$–CW complex. Then $[V : V_x]$ divides the product $\prod_{r=1}^{dimX} exp\ \widehat{H}^{-r-1}(V, H^r(X, \mathbb{Z}))$, where $V_x$ is an isotropy subgroup of maximal rank.*

## Corollary

*If $V$ acts freely and homologically trivially on a connected finite complex $X$, then $rank(V) - rank(V_x) \leq d_p(X)$ where $V_x \subset V$ is an isotropy subgroup of maximal rank.*

# Products of Spheres

We now consider the celebrated case of a product of spheres.

## Conjecture

If an elementary abelian p–group $V$ acts freely on a product of spheres $X = \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}$, then $rank(V) \leq k$.

This question has been considered by several authors. We focus on the equidimensional case.

## Theorem (Carlsson)

*If $V$ acts freely and homologically trivially on $X = (\mathbb{S}^n)^k$, then $rank(V) \leq k$*

For $X = (\mathbb{S}^n)^k$, $d_p(X) = k$, whence the result follows. This proof is very different from Carlsson's. The non-homogically trivial case requires some representation theory.

## Theorem (Adem-Browder)

*If $p$ is odd and $V$ acts freely on $X = (\mathbb{S}^n)^k$, then*

$$rank(V) \leq dim_{\mathbb{F}_p} H^n(X, \mathbb{F}_p)^G + \frac{1}{p-2}[k - dim_{\mathbb{F}_p} H^n(X, \mathbb{F}_p)^G]$$

For p=2, $n \neq 1, 3, 7$ we used a modified approach to establish the bound, applying the fact that for those values of $n$, $H^n(X, \mathbb{F}_2)$ is a permutation module by Hopf invariant one considerations. The case $n = 1$ was settled by Yalcin using Bieberbach groups. Therefore we have

## Theorem

*Let $V$ be an elementary abelian p–group acting freely on $(\mathbb{S}^n)^k$. Then $rank(V) \leq k$ if $p$ is odd, or if $p = 2$ and $k \neq 3, 7$.*

More generally the case of actions on equidimensional spheres that permute the basis in homology gives rise to a stronger bound.

### Theorem (Adem-Benson)

*Let $V$ be an elementary abelian p-group of rank r acting freely on a finite dimensional CW complex $X \simeq (\mathbb{S}^n)^t$ in such a way that the basis $u_1, u_2, \ldots, u_t$ of $H_n(X, \mathbb{F}_p)$ corresponding to the t spheres is permuted by $V$. Then the number of orbits of $V$ on $\{u_1, \ldots, u_t\}$ is at least r, i.e. $rank(V) \leq dim\ H_n(X, \mathbb{F}_p)^V$.*

More recently Hanke settled the non-equidimensional case provided $p$ is large relative to dim(X). He will tell us about this in his lectures.

It is conjectured that free actions of elementary abelian groups on finite complexes must be supported by large enough mod p cohomology. Specifically we have the following much more general conjecture:

### Conjecture (Carlsson)

If $V$ is a $p$–elementary abelian group acting freely on a finite connected complex $X$, then

$$2^{rank(V)} \leq \sum_{i=0}^{dimX} dim_{\mathbb{F}_p} H_i(X, \mathbb{F}_p)$$

This has been settled for $p = 2$ and $k \leq 4$. Of course an analogous algebraic question can be asked for free $\mathbb{F}_p V$–chain complexes, an interesting topic in its own right...

We can apply the exponent techniques from non-free actions to obtain

## Corollary

*If $V$ acts on $X = (\mathbb{S}^n)^k$, then $rank(V) - max\{rank(V_x)\} \leq k$ provided $p$ is odd or $p = 2$ and $n \neq 3, 7$.*

## Conjecture

If $X$ is a finite connected $V$–CW complex with a maximal rank isotropy subgroup $V_x$, then

$$2^{[rank(V) - rank(V_x)]} \leq \sum_{i=0}^{dimX} dim_{\mathbb{F}_p} H_i(X, \mathbb{F}_p)$$

An interesting approach is to take an extension $K$ of the field $\mathbb{F}_p$ for which there will exist a shifted subgroup of order $[V : V_x]$ acting freely on $C_*(X) \otimes K$.

# Some background book/survey references

Places where you can find the references to the original papers.

- **Cohomology of Finite Groups** (AA & R.J. Milgram), Springer-Verlag Grundlehren 309 (2nd Ed.2004).
- **Topics in transformation groups** (AA & J.F. Davis). Handbook of geometric topology, pp. 1–54, North-Holland, Amsterdam (2002).
- **Lectures on the Cohomology of Finite Groups** (AA), Contemporary Mathematics 436 (2007), 317-334.