Finite Group Actions, Cohomology of Groups and Rank Conjectures - Part II

Alejandro Adem

University of British Columbia

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Periodic Cohomology

In Lecture I we saw that if a finite group G acts freely on a sphere, then all of its abelian subgroups are cyclic. What this means is that G does not contain any subgroup of the form $\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$, p prime.

Examples include:

• $\mathbb{Z}/n\mathbb{Z}$ acts freely on any $\mathbb{S}^{2k+1}$

• Q_8 the quaternion group of order eight, is a subgroup of $\mathbb{S}^3$ and so acts freely on it.

• More generally, if V is an orthogonal/unitary G–representation, then G acts on $S(V)$. According to Wolf, G acts freely on some $S(V)$ if and only if (i) every subgroup of order pq (where p, q are prime) in G is cyclic and (ii) G does not contain $SL_2(\mathbb{F}_p)$ with $p > 5$ as a subgroup.

We have a geometric restriction

Theorem (Milnor, 1957)

A finite group G acting freely on a sphere must have every element of order two in its centre.

This means for example that the dihedral groups D_{2p} cannot act freely on any sphere. However if we consider actions on spaces homotopy equivalent to a sphere, this condition is not relevant.

Theorem (Swan, 1961)

G acts freely on a finite complex $X \simeq \mathbb{S}^n$ for some $n > 0$ if and only if every abelian subgroup in G is cyclic.

The most complete result was obtained using surgery:

Theorem (Madsen-Thomas-Wall, 1976)

A finite group G acts freely and smoothly on some sphere $\mathbb{S}^n$ if and only if all the abelian subgroups in G are cyclic and every involution is central.

Note however that determining precisely which groups act freely on a particular sphere is more subtle. For example, consider the semidirect product $\mathbb{Z}/3\mathbb{Z} \times_T Q_{16}$ where the element of order 8 acts non–trivially on $\mathbb{Z}/3\mathbb{Z}$. Then this group has periodic cohomology of period 4 but does not act freely on any finite homotopy $\mathbb{S}^3$. If a finite group G acts freely on an n-sphere we can only conclude that G has periodic cohomology of period dividing $n+1$.

We now consider a general notion of periodicity for the cohomology of topological spaces:

Definition

The cohomology of a connected topological space W is said to be periodic if there exist integers $r, S > 0$ and a class $\alpha \in H^r(W, Z)$ such that for any coefficient module M,

$$\cup\alpha : H^t(W, M) \to H^{t+r}(W, M)$$

is an isomorphism for all $t \geq S$.

• For a finite group G, BG has periodic cohomology if and only if every abelian subgroup of G is cyclic.

• For a discrete group Γ of finite vcd, $B\Gamma$ has periodic cohomology if and only if every finite subgroup of Γ has periodic cohomology.

Question: Is abstract periodicity always induced by cup product?

Cohomological periodicity can be characterized as follows

Theorem (Adem-Smith 2001)

A connected CW–complex X has periodic cohomology if and only if there exists an orientable spherical fibration

$$\mathbb{S}^N \to E \to X$$

where E has the homotopy type of a finite dimensional complex.

Corollary

A discrete group Γ acts freely and properly on some $\mathbb{S}^n \times \mathbb{R}^k$ if and only if Γ is countable and $B\Gamma$ has periodic cohomology.

Example: The Burnside group $\Gamma = G(d, e)$ with d generators and prime exponent $e > 665$, is countable with periodic cohomology and so acts freely on some $\mathbb{S}^n \times \mathbb{R}^k$.

Homotopy Group Actions

We now consider homotopy group actions on X (following J.Grodal).

Proposition

The following sets are in natural 1-1–correspondence and all define homotopy G–actions on a complex X:

- $[BG, BAut(X)]$
- *Fiber homotopy equivalence classes of fibrations*

  $X \to E \to BG$
- *G actions on spaces Z homotopy equivalent to X, where two spaces Z', Z'' are deemed equivalent if there exists a zig-zag of G–equivariant maps $Z' \to Z_0 \leftarrow Z_1 \to \cdots \leftarrow Z_n \to Z''$ which are homotopy equivalences.*
- *G–homotopy equivalence classes of actions on free G–spaces Z, homotopy equivalent to X.*

Given a homotopy G–action on X, and any G–space Z which realizes it, then $H^*(EG \times_G Z, R)$ is an invariant of the homotopy action. We will denote this invariant by $H^*(X_{hG}, R)$. We will assume that $H_*(X, \mathbb{Z})$ is finitely generated, whence we can use the usual spectral sequence to infer that $H^*(X_{hG}, \mathbb{F}_p)$ is finitely generated over $H^*(BG, \mathbb{F}_p)$ and has Krull Dimension ranging between zero and the p–rank of the finite group G.

Definition

- Suppose given a homotopy G–action on X as above. A finite group G is said to act with h-isotropy of p-rank equal to n on X if the Krull Dimension of $H^*(X_{hG}, \mathbb{F}_p)$ is equal to n.
- The h–isotropy rank of a homotopy group action of G on X is defined as the maximum of the h-isotropy ranks over all primes p dividing $|G|$.

In this context we obtain, for a given homotopy action of G on X:

Proposition

- *The action has trivial h–isotropy (h–free) if and only if G acts freely on a finite dimensional complex equivalent to the action (Wall's criterion).*
- *The action has h–isotropy rank equal to one if and only if* $H^*(X_{hG}, \mathbb{Z}) \neq 0$ *for infinitely many values of* $i > 0$ *and* X_{hG} *has periodic cohomology.*

We shall call an action with h–isotropy rank equal to one a homotopy periodic action. We reformulate an application of our previous results that characterizes homotopy periodic actions.

Theorem

Given a G–homotopy action on a connected complex X with $H_*(X, \mathbb{Z})$ finitely generated, then the action is homotopy periodic if and only if there exists an orientable spherical fibration

$$\mathbb{S}^n \to E \to X_{hG}$$

where E is homotopy equivalent to a finite dimensional complex.

Corollary

Let X be a simply connected finite CW–complex with a homotopy periodic G action, then there exists a finite free G–CW complex $Y \simeq X \times \mathbb{S}^N$ for some $N > 1$.

Corollary

Given a periodic homotopy G–action on a sphere $\mathbb{S}^n$, then G acts freely on a finite complex $X \simeq \mathbb{S}^N \times \mathbb{S}^n$.

Rank Two Groups

If G acts on a complex X with isotropy that has periodic cohomology, then from Quillen's results it's not hard to see that the action will be homotopy periodic. Hence we see that a direct strategy for constructing free actions on a product of two spheres is to construct an action on a single sphere which has isotropy with periodic cohomology.

For a rank two p–group P we observe that by inducing up a faithful character ρ from a central subgroup C of order p we obtain an action of P on $S(V) = S(Ind_C^P(\rho))$ where P acts with rank one isotropy.

This construction together with elementary bounds on rank yields

Theorem

A p–group P does not contain $(\mathbb{Z}/p\mathbb{Z})^3$ *as a subgroup if and only if it acts freely on a finite complex* $Y \simeq \mathbb{S}^n \times \mathbb{S}^m$.

Unitary representations $G \subset U(n)$ can also be used to construct geometric examples. The **fixity** is defined as the smallest integer f such that G acts freely on $U(n)/U(n-f-1)$. We recall results due to A-Davis-Ünlü. Fixity one gives rise to a spherical fibration over a sphere with a free G–action

$$U(n-1)/U(n-2) \to U(n)/U(n-2) \to U(n)/U(n-1).$$

Using bundle theory we can show that

Proposition

If $G \subset U(n)$ has fixity one, then G acts smoothly and freely on $\mathbb{S}^{2n-1} \times \mathbb{S}^{4n-5}$.

Example: If $G \subset U(3)$ then it acts freely on $\mathbb{S}^5 \times \mathbb{S}^7$; this includes groups such as A_5, $SL_3(\mathbb{F}_2)$ and $3A_6$.

Example: For rank two p–groups P with $p > 3$ one can show that either P acts freely on a product $S(V) \times S(W)$ or P has a representation $P \subset U(p)$ of fixity one. Hence a rank two p–group for $p > 3$ acts smoothly and freely on a product of two spheres.

Proposition

If $G \subset U(n)$ is of fixity equal to two, then G acts freely on a finite complex $X \simeq \mathbb{S}^{2n-1} \times \mathbb{S}^{4n-5} \times \mathbb{S}^{M}$ for some $M > 0$.

The next result uses propagation techniques.

Theorem

Let $G \subset U(n)$ which acts freely on U(n)/U(k) (with fixity n-k-1) $k \geq 1$. If the order of G is prime to $(n-1)!$, then G acts freely, smoothly, homologically trivially on $\mathbb{S}^{2n-1} \times \mathbb{S}^{2n-3} \times \cdots \times \mathbb{S}^{2k-1}$.

Corollary

Let P denote a finite non–abelian p–group with cyclic center and having an abelian maximal subgroup. If the rank of P is $r < p$, then there exists a free, smooth and homologically trivial action of P on $M = \mathbb{S}^{2p-1} \times \mathbb{S}^{2p-3} \times \cdots \times \mathbb{S}^{2(p-r)+1}$, a product of r spheres.

Question: does every rank two group act homotopy periodically on some sphere?

The answer is no, and we in fact can give a complete description of how it goes wrong. Let $\mathrm{Qd}(p)$ denote the semi–direct product $(\mathbb{Z}/p)^2 \rtimes SL_2(\mathbb{F}_p)$, where $SL_2(\mathbb{F}_p)$ acts via the natural representation on $\mathbb{Z}/p \times \mathbb{Z}/p$. The following result tells us that this group does not give rise to a periodic action, even up to homotopy.

Proposition (Grodal)

Every homotopy action of $\mathrm{Qd}(p)$ on a sphere X has rank two h–isotropy at the prime p. In other words, the equivariant cohomology $H^*(EQd(p) \times_{Qd(p)} X, \mathbb{F}_p)$ will always have Krull dimension two.

This requires understanding homotopy fixed points via the work of Dwyer-Wilkerson and Lannes T-functor. On the other hand this is all that can go wrong if we want to construct homotopy actions on spheres with periodic cohomology.

We recall a concept from group theory:

Definition

A subquotient of a group G is a factor group H/K where $H, K \subset G$ with $K \trianglelefteq H$. A group L is said to be involved in G if L is isomorphic to a subquotient of G. In particular, for a prime p, we say that L is p'–involved in G if L is isomorphic to a subquotient H/K of G where K has order relatively prime to p.

We can obtain a complete characterization of those finite groups that can act homotopy periodically on spheres. This builds on work in M.Jackson's thesis.

Theorem (A-Grodal)

A finite group G admits a homotopy periodic action on some sphere if and only if $rk(G) \leq 2$ and $\mathrm{Qd}(p)$ is not p'–involved in G for any $p > 2$.

This provides a complete answer for the approach using periodicity. It implies for example that every odd order rank two group G acts freely on a finite complex homotopy equivalent to a product of two spheres.

It is an open question whether or not $\mathrm{Qd}(p)$ can act freely on a product of two spheres! Any such action would not allow any equivariant projection onto a sphere. It has an irreducible nature. However, using p-compact groups, we are able to construct an exotic example for $p = 3$:

Theorem (Adem-Grodal)

$\mathrm{Qd}(3)$ *acts freely on a finite dimensional simply connected complex* Z *such that* $H^*(Z,\mathbb{Z}) \cong H^*(\mathbb{S}^{11} \times \mathbb{S}^{15},\mathbb{Z})$ *as an algebra but which is not homotopy equivalent to* $\mathbb{S}^{11} \times \mathbb{S}^{15}$*.*

Question: Does every rank 2 finite group G act freely on a finite dimensional, simply–connected complex X such that $H^*(X,\mathbb{Z})$ is an exterior algebra on two odd–dimensional generators?

Higher Rank Actions

We would like to go beyond the rank two case. Recall that $r(G)$, the rank of G, is defined as the maximal rank of a p–elementary abelian subgroup of G, over all prime divisors p of $|G|$.

Definition

The homotopy rank $h(G)$ of a finite group G is the minimal k such that G acts freely on a finite complex $X \simeq \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}$.

- Every finite group G acts freely on some product of spheres, so this is well–defined.
- The rank conjecture from Lecture I is equivalent to $r(G) \leq h(G)$.

It turns out that such a question can be investigated more completely in an algebraic context.

Let k be a field of characteristic p and consider a homogeneous system of parameters $k[\zeta_1, \ldots, \zeta_r] \subset H^*(G, k)$ where $\deg \zeta_i = n_i \geq 2$ for $i = 1, \ldots, r$.

Theorem (Benson-Carlson)

There exists a finite complex C of projective kG–modules with $H^*(C) \cong \Lambda(\overline{\zeta}_1, \ldots, \overline{\zeta}_r)$ with $\deg \overline{\zeta}_i = n_i - 1$. There is a spectral sequence with

$$E_2^{*,*} = H^*(G, k) \otimes \Lambda(\overline{\zeta}_1, \ldots, \overline{\zeta}_r)$$

converging to $H^*(Hom_{kG}(C, k))$ which satisfies Poincaré Duality in formal dimension $\sum_{i=1}^{r}(n_i - 1)$. In this spectral sequence we have $d_{n_i}(\overline{\zeta}_1) = \zeta_i$, and if $H^*(G, k)$ is Cohen–Macaulay, then

$$H^*(Hom_{kG}(C, k)) \cong H^*(G, k)/(\zeta_1, \ldots, \zeta_r).$$

We provide a geometric example of this kind of phenomena. Let P be a p–group with center $Z \cong \mathbb{Z}/p^{s_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{s_t}\mathbb{Z}$. For each factor in Z we can choose a faithful linear character χ_i that extends to a representation of Z. Let $W_i = Ind_Z^P(\chi_i)$ and consider the P–complex

$$X = S(W_1) \times \cdots \times S(W_t) \cong (\mathbb{S}^{2[P:Z]-1})^t.$$

The center Z acts freely on X, and $H^*(EP \times_P X, \mathbb{F}_p)$ can be computed from the usual spectral sequence in equivariant cohomology:

Theorem (Duflot)

In the mod p spectral sequence for the P–action on X the generators transgress to a regular sequence $\{u_1, \ldots, u_t\}$ in $H^*(P, \mathbb{F}_p)$, and

$$H^*(EP \times_P X, \mathbb{F}_p) \cong H^*(P, \mathbb{F}_p)/(u_1, \ldots, u_t).$$

We make the following observations

- If every element of order p on P is central, then the ranks agree $r(P) = r(Z) = t$ and this yields a free P–action on $X = (\mathbb{S}^{2[P:Z]-1})^t$ with

$$H^*(X/P, \mathbb{F}_p) \cong H^*(P, \mathbb{F}_p)//(u_1, \dots, u_t).$$

- If $r(P) = r(Z) + 1$, then P acts on $(\mathbb{S}^{2[P:Z]-1})^t$ with periodic isotropy, whence we conclude that there exists a finite free P–complex

$$Y \simeq (S^{2[P:Z]-1})^t \times \mathbb{S}^N.$$

- This argument by Duflot shows that

$$\mathrm{depth}\ H^*(P, \mathbb{F}_p) \geq r(Z).$$

Using gluing arguments, Benson-Carlson were able to construct a projective $\mathbb{Z}G$ complex that has the cohomology of a product of $r(G)$ spheres, leading them to conjecture that every group of rank k acts freely on a product of k spheres. This leads to

Conjecture: For a finite group G, $r(G) = h(G)$.

As we have seen, $r(G) = 1$ if and only if $h(G) = 1$ and this co-rresponds to groups with periodic cohomology. We have also seen that

Proposition

If G is a finite group with $r(G) = 2$, then $h(G) = 2$ if $\mathrm{Qd}(p)$ is not p'–involved in G for any $p > 2$.

Corollary

If G is a group of odd order then $r(G) = 2$ if and only if $h(G) = 2$.

It may be more realistic to focus on finite dimensional complexes

Question: Does every finite group G of rank equal to n act h–freely on a product of n spheres?

Motivated by the periodic case and the exotic example constructed for $\mathrm{Qd}(3)$, we can also pose the following structural question

General Question: Let X be a connected CW-complex. Under what conditions on the cohomology of X does there exist a fibration

$$F \to E \to X$$

where the fiber F has the cohomology [homotopy type] of a product of r spheres and E is homotopy equivalent to a finite dimensional complex?