Equivariant cohomology and syzygies

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(mostly joint work with Chris Allday and Volker Puppe)

Equivariant cohomology

$T = (S^1)^r$ compact torus

X "nice" T-space, e.g. $T_{\mathbb{C}}$-variety or T-manifold of finite type

$ET \to BT$ universal T-bundle

$X_T = (ET \times X)/T$ **Borel construction** (homotopy quotient)

Equivariant cohomology: $H^*_T(X) = H^*(X_T; \mathbb{Q})$

$H^*_T(X)$ is a f. g. module over
$A = H^*(BT) \cong \mathbb{Q}[t_1, \ldots, t_r]$ with $\deg(t_i) = 2$.

X **equivariantly formal**: $H^*_T(X)$ free/A
In this case, $H^*_T(X) \cong A \otimes_{\mathbb{Q}} H^*(X)$ as A-modules

Examples: compact Hamiltonian T-mfs (Frankel, Kirwan),
complete smooth $T_{\mathbb{C}}$-vars (Goresky–Kottwitz–MacPherson, Weber)

Two exact sequences

Chang–Skjelbred sequence (1974): $H_T^*(X)$ free $/A \implies$

$$0 \to H_T^*(X) \to H_T^*(X^T) \xrightarrow{\delta} H_T^{*+1}(X_1, X^T)$$

is exact, where $X_1 =$ union of orbits of dimension ≤ 1.

This is an efficient way to compute $H_T^*(X)$, in particular if X^T is finite and X_1 a union of 2-spheres ("GKM method" 1998).

augm. **Atiyah–Bredon sequence** (1974): $H_T^*(X)$ free $/A \implies$

$$0 \to H_T^*(X) \to H_T^*(X_0) \to H_T^{*+1}(X_1, X_0) \to H_T^{*+2}(X_2, X_1) \to \cdots \to H_T^{*+r-1}(X_{r-1}, X_{r-2}) \to H_T^{*+r}(X_r, X_{r-1}) \to 0$$

is exact, where $X_i =$ union of orbits of dimension $\leq i$.

The CS / GKM method only uses a small part of this sequence!

The cohomology of the AB sequence

$AB^*(X) =$ complex of A-modules

$$H_T^*(X_0) \to H_T^{*+1}(X_1, X_0) \to H_T^{*+2}(X_2, X_1) \to \cdots \to H_T^{*+r-1}(X_{r-1}, X_{r-2}) \to H_T^{*+r}(X_r, X_{r-1})$$

Theorem

$$H^i(AB^*(X)) = \mathrm{Ext}^i_A(H_*^T(X), A) \qquad \textit{for any } i \geq 0$$

The A-module $H_*^T(X)$ is a suitably defined **equivariant homology** of X. (It is *not* the homology of X_T or any other space.) Morally,

" $C_*^T(X) = \mathrm{Hom}_A(C^*(X_T), A)$ or $\mathrm{Hom}_{C^*(BT)}(C^*(X_T), C^*(BT))$ "

Poincaré duality (over $\mathbb{Q}$) lifts to equivariant PD iso over A:

$$H_T^*(X) \xrightarrow{\cap [X]} H^T_{n-*}(X) \qquad \text{where } n = \dim X$$

Syzygies

Let M be a f. g. A-module

M ***j*-th syzygy**: $\exists$ exact sequence

$$0 \to M \to F_{j-1} \to \cdots \to F_1 \to F_0$$

with $F_0, \ldots, F_{j-1}$ f. g. free$/A$

Syzygies interpolate between torsion-freeness and freeness:

$$\begin{aligned} \text{zeroeth syzygy} &= \text{any } M \\ \text{first syzygy} &= \text{torsion-free} \\ \text{second syzygy} &= \text{reflexive} \;\; (\text{i.e., } M \to M^{\vee\vee} \text{ iso}) \\ &\vdots \\ r\text{-th syzygy} &= \text{free} \\ (r+1)\text{-st syzygy} &= \text{free} \\ &\vdots \end{aligned}$$

Partial exactness

Theorem

Let $j \geq 0$. The augmented AB sequence is exact at all positions $i \leq j-2 \iff H_T^*(X)$ is a j-th syzygy.

This includes Atiyah–Bredon's result and its converse.

Corollary

The CS sequence is exact $\iff H_T^*(X)$ is reflexive

Example

$T = (S^1)^r$ acts on $(S^2)^r$. Set $X = (S^2)^r \setminus \{x, y\}$ where $x, y \in \{N, S\}^r$ are fixed points, differing in $k \geq 1$ coordinates

Then $H_T^*(X)$ is a syzygy of order $k-1$

The underlying algebraic result

Lemma

$H^*_T(X_i, X_{i-1})$ is zero or a Cohen–Macaulay module of dim $r - i$.

Let M be a f. g. R-module and K^* a complex of f. g. R-modules. Consider an augmented complex $\bar{K}^*$ with $\bar{K}^{-1} = M$,

$$0 \to M \to K^0 \to K^1 \to \cdots \to K^r \to 0$$

Assume the following:

- Each K^i is zero or CM of dimension $r - i$
- For each $\mathfrak{p} \lhd R$, if the localized complex $\bar{K}^*_{\mathfrak{p}}$ is exact at all but possibly two adjacent positions, then it is exact everywhere.

Theorem

Under these assumptions,

M is a j-th syzygy $\iff$ $H^i(\bar{K}^*) = 0$ for $i \le j - 2$

Poincaré duality

Let X be a compact oriented T-manifold.

Corollary

CS sequence is exact $\iff$ the equivariant Poincaré pairing

$$H_T^*(X) \times H_T^*(X) \to A, \quad (\alpha, \beta) \mapsto \langle \alpha \cup \beta, [X] \rangle$$

is perfect.

Corollary

If $H_T^*(X)$ is a syzygy of order $\geq r/2$, then it is free over A.

Proof

$H_T^*(X)$ syzygy of order $\geq r/2 \Rightarrow$ left half of AB sequence exact.

Syzygy order $\geq r/2$ also implies depth $H_T^*(X) \geq r/2$, hence

$$0 = \mathrm{Ext}_A^k(H_T^*(X), A) \overset{PD}{=} \mathrm{Ext}_A^k(H_*^T(X), A) = H^k(AB(X))$$

for $k \geq r/2$. So right half of AB sequence is exact, too.

Big polygon spaces

S^1 acts on $S^3 \subset \mathbb{C}^2$ via $g(u,z) = (u, gz)$

This gives an action of $T = (S^1)^r$ on $(S^3)^r$.

$$X = \left\{ ((u_1, z_1), \ldots, (u_r, z_r)) \in (S^3)^r \;\middle|\; u_1 + \cdots + u_r = 0 \right\}$$

Assume that $r = 2m+1$ is odd. Then X is a compact orientable T-manifold of dimension $3r-2$, and X^T is a “space of polygons”.

Theorem

$H_T^*(X)$ is a syzygy of order exactly m.

Open: minimal dimension for syzygy order m where $r = 2m+1$

$r = 1$: $\dim X = 1$ is minimal for syzygy order 0.
$r = 3$: $\dim X = 7$ is minimal for syzygy order 1
$r \geq 5$ odd: Is $3r-2$ still the smallest possible dimension?

Other versions

Analogous results hold in various settings:

- $G = T$ torus, rational (or real) coefficients (A–F–P)
- G compact connected Lie group, real coefficients (F)
  orbit filtration by rank of isotropy groups
  (builds on work of Goertsches–Mare)
- $G = (\mathbb{Z}_p)^r$, coefficients in $\mathbb{Z}_p$ (A–F–P)
  ($p = 2$ also done by Bourguiba–Lannes–Schwartz–Zarati)

In each case, the starting point is a "CM filtration" (X_i) of X such that $H^*_G(X_i, X_{i-1})$ is zero or CM of projective dimension i.

Thank you!

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