Cross-community collaborations in combinatorics

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1 Overview

In recent years some of the most exciting breakthroughs in Combinatorics on longstanding conjectures have resulted from innovative applications of established techniques to areas where they not necessarily used before. We would like to harness the power of collaboration and bring together open-minded participants with different areas of expertise to produce novel research in a number of globally studied areas including. We aspired to create new productive long-term bonds between members of the global community.

A large focus of the workshop was on the training and career enhancement of junior researchers. This was achieved through their fostering new collaborations with world-leading members of the global community during our focused small group work sessions. This gave junior participants opportunities to learn about and work in areas outside of their PhD/postdoctoral focus, gaining invaluable skills and knowledge. They were able to forge meaningful relationships with senior members of the community outside their home institution.

1.1 Workshop objectives

1. A primary objective of the workshop was to stimulate and foster genuinely new (and productive) collaborations amongst participants in topical areas that are not necessarily what they would usually work on and to create *new* long-term bonds between members of the global community.

2. Another key objective of the workshop was the training and career enhancement of junior participants. We have deliberately decided to make the workshop small - 21 people, to not be intimidating for more junior researchers and allow them to flourish. We aspired to a very welcoming and comfortable environment and for them to be able to develop meaningful relationships with senior members of the community.

3. We were committed to ensuring our final participant list is diverse and supports those under-represented in the mathematical sciences. Systematic barriers to inclusion are all too present in our field and we do not wish to enhance the problem.

In the sections below we will detail the scientific progress made during the workshop, and explain how we met each of these objectives.

2 Open Problems

One of the key goals of our workshop was to foster new and exciting collaborations amongst members of the combinatorics community that did not typically work together. We invited all participants to submit well thought out open problems in advance, and begun the workshop with an open problem session where these problems would be presented. In this section we summarise the problems that were suggested for the workshop.

2.1 Are trees eventually Turán-good?

For a pair of graphs G and F, say that G is *F-free* if G does not contain a subgraph isomorphic to F. Let $\mathcal{N}(H,G)$ denote the number of copies of a graph H in G, that is, the number of subgraphs of G isomorphic to H, and let

$$\text{ex}(n,H,F) = \max\{\mathcal{N}(H,G) \mid G \text{ is an } n\text{-vertex } F\text{-free graph}\}$$

be the maximum number of subgraphs isomorphic to the target graph H in an n-vertex F-free graph.

Turán's theorem states that any K_{r+1}-free graph contains at most about $(1-\frac{1}{r})\binom{n}{2}$ edges, and furthermore the unique extremal graph (up to isomorphism) is the so-called Turán graph, $T_r(n)$: the complete r-partite graph with parts as balanced as possible. In 2015 Alon and Shikhelman [15] introduced the generalized Turán problem ex(n,T,F) which is the maximum number of subgraphs isomorphic to the "target graph" T in an n-vertex F-free graph (avoiding the "forbidden" graph F). Their paper sparked broad interest and results are known for many familes of T and F.

One difficulty in precisely determining ex(n,H,F) is identifying a potential *extremal graph*. In many cases when the problem is tangible, the extremal graph turns out to be the Turán graph. Let F be a graph with chromatic number $k+1$ and say that a graph H is *F-Turán-good* if for n sufficiently large, ex$(n,H,F) = \mathcal{N}(H,T_k(n)))$. That is, the Turán graph $T_k(n)$ is an n-vertex F-free graph containing the maximum possible number of copies of H. The term *Turán-good* was recently introduced by Gerbner and Palmer [16], but the study of this phenomenon goes back much further to work of Györi, Pach and Simonovits [30]. See [16] for a comprehensive summary of what is known so far about F-Turán-good graphs.

Gerbner and Palmer [16, Conjecture 20] conjectured that for every graph H there exists $r_0 = r_0(H)$ such that H is K_{r+1}-Turán-good for every $r \geq r_0(H)$. This conjecture is known to hold for complete multipartite graphs, paths and C_5.

Problem 1. *Does the Gerbner-Palmer conjecture hold for trees? That is, if T is a tree, is there $r = r(T)$ such that for large enough n, the K_{r+1}-free graph on n vertices containing the maximum number of subgraphs isomorphic to T is the Turán graph $T_r(n)$?*

2.2 Rainbow saturation

Some standard terminology: a *rainbow* copy of a graph H is an edge-coloured graph is a copy of H whose edges are coloured with distinct colours.

Let $\text{sat}_{\text{rbw}}(n,H)$ be the minimum number of edges in an edge-coloured graph on n vertices which does not have a rainbow copy of H which is maximal with respect to this property. Girão, Lewis and Popielarz [1] consider this parameter (as well as a variant where there is another parameter t and the edges are coloured using colours from $[t]$). Among other things, they ask the following (in fact, they conjecture it is true).

Problem 2. *Is it true that for every graph H there is a constant $c = c(H)$ such that $\text{sat}_{\text{rbw}}(n,H) \leq cn$ for every $n \geq 1$?*

As background, define sat(n,H) to be the minimum number of edges in an n-vertex graph G which does not have a copy of H and is maximal with respect to this property. The above question is a natural extension of the following known fact: for every graph H there is a constant $c = c(H)$ such that sat$(n,H) \leq cn$ for every $n \geq 1$.

2.3 Oriented chromatic number

The *oriented chromatic number* of an oriented graph $\vec{G}$, denoted $\chi_o(\vec{G})$ is the least integer k so that there is a graph homomorphism from $\vec{G}$ to some tournament of order k. In other words, it is the least k so that the vertices of $\vec{G}$ can be k-coloured with no arcs within a colour class and all arcs between two colour classes oriented in the same direction. It is conjectured (see [5]) that if $\vec{G}$ is an orientation of a 3-regular graph, then $\chi_o(\vec{G}) \leq 7$ and there are examples of 3-regular oriented graphs that require 7 colours. Duffy [6] (see also [5] for preliminary results) has shown that if G is an orientation of a connected 3-regular graph, then $\chi_o(\vec{G}) \leq 8$.

Problem 3. *Let $\vec{\mathcal{G}}(n,d)$ be a model for a random orientated d-regular graph, obtained by first selecting a random (undirected) d-regular graph uniformly at random and then orienting the edges uniformly at random. For $\vec{G} \sim \vec{\mathcal{G}}(n,3)$ what is the range for $\chi_o(\vec{G})$, with high probability?*

If $\vec{G} \sim \vec{\mathcal{G}}(n,3)$, then by deterministic results, $\chi_o(\vec{G}) \leq 8$. A first moment calculation shows that with high probability, $\chi_o(\vec{G}) > 4$. Indeed, considering local structures, with positive probability, $\vec{G}$ contains a directed 5-cycle which has oriented chromatic number 5.

In the case of 2-regular graphs, an orientation of a 2-regular graph can have oriented chromatic number in $\{2,3,4,5\}$, but one can show (see [7]) that for a random oriented 2-regular graph, the oriented chromatic number is either 4 or 5 with high probability, each occurring with positive probability.

2.4 An Upper Bound on the DP-Chromatic Index

DP-colourings or correspondence colourings are a generalisation of list colourings introduced by Dvok and Postle [8]. Given a graph G, a *cover* $\mathcal{H}$ of G has the following properties: each vertex v of G has an associated set $L(v)$ of vertices in $\mathcal{H}$ where the induced graph on $L(v)$ is complete, and all the sets $L(v)$ are disjoint (this is analogous to the list of colours allowed on v). In addition, for each edge uv of G there is an matching between $L(u)$ and $L(v)$. An $\mathcal{H}$-colouring of G is an independent set in $\mathcal{H}$.

The DP-chromatic number $\chi_{DP}(G)$ of G is the smallest k such that G admits a DP-colouring for any cover $\mathcal{H}$ where all of the sets $L(v)$ have size k. One can easily see that $\chi_{DP}(G) \geq \chi_l(G)$ where $\chi_l(G)$ is the list-chromatic number of G. However, the two are not always equal.

One can similarly define the DP-chromatic index $\chi'_{DP}(G)$ of a graph G: it is the DP-chromatic number of the line graph $Line(G)$ of a G, i.e. the graph with vertex set $E(G)$ and two vertices adjacent if and only if the corresponding edges of G share an endpoint.

The famous edge-list colouring conjecture says that the list-chromatic index of a graph is equal to the chromatic index. Vizing made a weaker conjecture, that the list-chromatic index is at most $\Delta(G)+1$. Bernshteyn and Kostochka asked whether this holds for the DP-chromatic index.

Problem 4 (Problem 1.13 from [9])**.** *Is $\chi'_{DP}(G) \leq \Delta(G)+1$? Or does there exist a graph G with $\chi'_{DP}(G) \geq \Delta(G)+2$?*

This is stronger than Vizing's original conjecture, however introducting DP-colourings has proved useful in proving results about list-colourings as it is more 'local'. For example, Dvok and Postle originally introduced DP-colouring to answer a long-standing question of Borodin that every planar graph without cycles of lengths 4 to 8 is 3-list-colorable [8].

2.5 Arc-doubling in eulerian digraphs

The square of a digraph D without parallel arcs is the digraph D' that is obtained from D by adding the uw whenever $uv, vw \in A(D)$ for some $v \in V(D) - \{u, w\}$, omitting parallel arcs.

The following conjecture is due to Mody [19] where the property is proved for tournaments.

Conjecture 5. *Let D be an eulerian digraph without arcs in opposite direction and without parallel arcs. Then the square of D contains at least twice as many arcs as D.*

This conjecture would imply Seymour's second neighbourhood conjecture for eulerian digraphs. For some results on Seymour's second neighbourhood conjecture, see [20].

Update: Conjecture 5 also appears as [41, Conjecture 6.15] attributed to Seymour and/or Jackson.

2.6 A Turán Problem for Simplicial Spheres

Given a set of vertices V, say a collection S of subsets of V is a k-*complex* (short for k-dimensional homogeneous simplicial complex) if it is closed under subset inclusion and every maximal subset (called a facet) consists of $k+1$ vertices. Observe that a $(k+1)$-uniform hypergraph gives rise to a k-complex by taking the down-closure of its edges. Linial proposed a topological version of Turán's problem for k-complexes [11, 12]. The following remains open:

Problem 6. *Let $k \geq 3$ and let S^k denote the k-sphere. How many facets can an n-vertex k-complex $\mathcal{S}$ have while containing no homeomorphic copy of S^k? In other words, if such an $\mathcal{S}$ has $O(n^{k+1-\lambda})$ facets, what is the optimal value of $\lambda = \lambda(S^k)$?*

From [10], $\lambda(S^1) = 1$ and from [13], $\lambda(S^2) = \frac{1}{2}$. It is conjectured that $\lambda(S^k)$ is also the correct value of the "universal exponent" λ_k, a lower bound for all $\lambda(S)$ that depends only on the dimension of S.[14] show $\lambda_k > 0$ for all k, but another interesting problem is to determine more precisely λ_k for $k \geq 2$.

2.7 Counting tournament-free orientations of $G(n, p)$

An orientation $\vec{G}$ of a graph G is an oriented graph obtained by assigning an orientation to each edge of H. Given a fixed oriented graph $\vec{H}$ and a graph G, we can define $D(G, \vec{H})$, the number of $\vec{H}$-free orientations of G. In this problem description, we will consider the case $G = G(n, p)$, the binomial random graph.

Let $C_k^{\circlearrowright}$ denote the strongly-connected orientation of the cycle C_k. For $\vec{H} = C_k^{\circlearrowright}$, some initial estimates on $\log D(G(n,p), C_k^{\circlearrowright})$ were obtained by [21]. The correct order of growth of this function was later determined up to polylogarithmic factors in [22] and [23], who showed that $\log D(G(n,p), C_k^{\circlearrowright}) = \widetilde{\Theta}(n \,/\, p^{1/(k-2)})$ with high probability when $p \gg n^{-1+1/(k-1)}$, where $\widetilde{\Theta}(\cdot)$ is analogous to $\Theta(\cdot)$ notation but with polylogarithmic factors omitted.

Although the upper bounds in [22] and [23] only deal with forbidding cycles (or families of oriented graphs), the lower-bound construction described in Proposition 7.6 of [22] works for any oriented graph $\vec{H}$ and suggests the following problem.

Problem 7 (Question 7.7 of [22])**.** *Let $\vec{H}$ be a strongly connected tournament with $k := v(H) \geq 4$ and let $p \gg n^{-2/(r+1)}$. Prove that, with high probability as $n \to \infty$,*

$$\log D(G(n,p), \vec{H}) = \tilde{\Theta}\left(\frac{n}{p^{(k-1)/2}}\right)$$

As mentioned, the lower bound is known in all cases, but a matching upper bound is only known when the tournament is a cycle, that is, when $v(H) = 3$.

2.8 Maximum twin-width of an n-vertex graph

The *twin-width* of a graph is a new graph parameter, which was recently introduced by Bonnet, Kim, Thomassé and Watrigant [24] and already found numerous applications. Informally, twin-width measures the complexity of reducing the graph to a one vertex graph by iterative identification of vertices with similar neighborhoods. See e.g. [25, Section 2.1] for the formal definition.

Some of the very basic extremal questions about twin-width remain open. Ahn, Hendrey, Kim and Oum [25] explored the maximum twin-width of n-vertex graphs and proved that it is at most $n/2+O(\sqrt{n \ln n})$.

Conjecture 8. *Every n-vertex graph has twin-width at most $(n-1)/2$.*

The bound in Conjecture 8 is tight for conference graphs.

2.9 Extremal problems for circulant graphs

The area of 'extremal problems for regular graphs' asks questions like: 'Which d-regular graph on n vertices has the most independent sets?' (answer by Kahn and Zhao: a union of $K_{d,d}$'s). The d-regular graph with the fewest indpendent sets is a union of K_{d+1}'s (easy to show) but if we ask for the minimizer of the number of independent sets or the independence number over d-regular triangle-free graphs, the question becomes much harder (Shearer's upper bound on $R(3,k)$ comes from a lower bound on the minimum).

For background reading on this area see [26, 27, 28, 32]. The solution to some problems of this form also can be used to identify computational thresholds in approximate counting and sampling problems [29]. The type of techniques used in this area include the entropy method, inductive arguments, and the 'occupancy method' combining statistical physics tools and linear programming.

Define a *circulant graph* $G(n,S)$ on n vertices with difference set $S \subset \mathbb{Z}$ as the graph with vertex set $[n]$ and $(i,j) \in E(G)$ iff $i - j \equiv x \mod n$ for some $x \in S$. (For instance, the cycle C_n is obtained as $G(n,\{1\})$).

It is not too hard to show that for any fixed set S and any $\lambda \geq 0$, the limit

$$f(\lambda, S) := \frac{1}{n} \log Z_{G(n,S)}(\lambda)$$

exists and can be computed in terms of eigenvalues of a set of discrete recurrence relations involving the set S.

From the function $f(\lambda,S)$ you can read off some interesting graph parameters including the asymptotic growth rate of the number of independent sets of $G(n,S)$ and the limiting independence ratio of $G(n,S)$ (size of max independent set divided by n). We can associate $f(\lambda,S)$ to the infinite 'circulant' graph[1] $G(\mathbb{Z},S)$ with vertex set $\mathbb{Z}$ and edges (i,j) when $i - j \in S$.

The following open-ended problem is to explore some extremal problems for independent sets (or other homomorphism counts or for other partition functions) for different classes of circulant graphs.

Problem 9. *Pose and solve some extremal problems for (infinite) circulant graphs.*
For instance,

- *For a given d, which d-regular difference set S maximizes (minimizes) $f(\lambda,S)$?*
- *For a given d and g, which d-regular difference set S of girth at least g maximizes (minimizes) $f(\lambda,S)$?*
- *Which triangle-free difference set S minimizes the independence ratio?*
- *Which d-regular difference set maximizes (minimizes) the asymptotic growth rate of the number of q-colorings of $G(n,S)$?*

One hope is that the connection to eigenvalues will make solving some of these problems tractable and give a good understanding of the types of structures that lead to more or fewer independent sets (or other (weighted) homomorphisms) in a graph.

2.10 Monochromatic paths in multipartite hypergraphs

This is a problem on monochromatic partitions for hypergraphs. As in the classical Ramsey problem, one is given a graph G whose edges are coloured with two colours by an adversary, and one wishes to find certain monochromatic subgraphs. Instead of just one monochromatic copy, as in Ramsey's theorem, in monochromatic partitioning problems we want to find a collection of such copies that together cover the whole vertex set of the host graph.

Gerencsér and Gyárfás [34] observed that in any 2-colouring of the edges of K_n there are two disjoint monochromatic paths, of different colours, that together cover the vertex set of K_n. The same is true for 2-colourings of the edges of the complete 3-uniform hypergraph $\mathcal{K}^{(3)}$, and tight paths [35].

For complete bipartite graphs, it is known that there is always a partition into 3 monochromatic paths. In fact, for a certain class of colourings 3 monochromatic paths are needed, while for all other colourings 2

[1] Maybe there's another term for a difference graph on $\mathbb{Z}$?

paths are enough. (See [33].) It would be nice to know what happens in the complete 3-partite 3-uniform hypergraph $\mathcal{K}^{(3)}_{3\times n}$. There are 2-colourings of the edges of this hypergraph which cannot be partitioned into less than 4 monochromatic tight paths, but are these colourings 'worst possible'?

Problem 10. *Find p such that for any 2-colouring of the edges of $\mathcal{K}^{(3)}_{3\times n}$ there is a partition into p monochromatic paths.*

One could also look for a asymptotic cover, which is probably easier.

2.11 Repeated patterns

We say two copies of a graph H in an edge-colouring of K_n are *colour-isomorphic* if there is an isomorphism between these copies preserving the colours. Given $n,\ k \geq 2$ and a fixed graph H, define $f_k(n,H)$ to be the smallest integer C such that there is a proper edge-colouring of K_n with C colours containing no k vertex-disjoint colour-isomorphic copies of H.

The main question on this topic is naturally the following: given a graph H and an integer $k \geq 2$, determine $f_k(n,H)$. Among other results, Conlon and Tyomkin [36] showed that if H is a non-bipartite graph, then $f_2(n,H) = n$ for odd n, and $n-1 \leq f_2(n,H) \leq n+1$ for even n.

Problem 11. *Let H be a non-bipartite graph and let n be even. Determine $f_2(n,H)$.*

A natural starting point is to consider colour-isomorphic triangles.

Problem 12. *Determine $f_2(n,K_3)$.*

Answering a question of Conlon and Tyomkyn [36] conjectured in a stronger form way by Ge, Jing, Xu and Zhang [37], Janzer [38] proved that for any positive integers $k, r \geq 2$, we have

$$f_r(n, C_{2k}) = \Omega(n^{\frac{r}{r-1}\cdot\frac{k-1}{k}}). \tag{1}$$

Conlon and Tyomkyn showed that if H contains a cycle, then there exists r such that $f_r(n,H) = O(n)$. In view of this, one may consider the following problem.

Problem 13. *Given a graph H that contains a cycle, determine the smallest r such that $f_r(n,H) = O(n)$.*

Note that (1) shows that for an even cycle C_{2k} we have $r \geq k$. In [37], it is proved that $f_3(n,C_4) = O(n)$.

2.12 Partitioning Geometric Graphs

A geometric graph $G = G(P,E)$ is a graph drawn in the plane where the points P are in general position and edges E are straight line segments. A partition of a graph G is a set of edge-disjoint subgraphs of G such that each edge of G is in exactly one subgraph.

Problem 14. *Is there a constant $c < 1$ such that every complete geometric graph can be partitioned into at most cn plane subgraphs?*

The case $c = 1$ is easy, as we can simply take stars centered at every vertex. Bose, Hurtado, Rivera-Campo and Wood [39] showed that each geometric drawing of the complete graph can be partitioned into $n - \sqrt{\frac{n}{12}}$ plane subgraphs which is the current best result. If the pointset lies on a circle, then the graph can be partitioned into $\frac{n}{2}$ plane subgraphs but not less, since there are $\lfloor \frac{n}{2} \rfloor$ edges which pairwise cross.

If we ask about packing plane subgraphs in general graphs, edge colorings in planar graphs H are a special case. From a straight-line drawing of H, slightly extend all line segments, so that they cross at the vertices of H, and nowhere else.For maximum degree $\Delta = 4$ or $\Delta = 5$ it is conjectured to be NP-hard to decide whether a planar graph is Δ or $\Delta + 1$-edge colorable, while it was shown that planar graphs with $\Delta \geq 7$ are Δ-edge colorable.

2.13 $(F, \overline{F}^b)$-free graphs

Problem 15. *For $t \in \mathbb{N}$, are there $2^{O(n \log n)}$ bipartite graphs on n vertices that contain neither P_t nor the bipartite complement $\overline{P_t}^b$ as induced subgraph?*

This is related to the notion of adjacency labelling scheme (with labels of size $O(\log n)$). Relevant papers include the place where the conjecture (for all forests) was posed [2], the place where the conjecture was proved for star forests [4], and a theorem on $(F, \overline{F}^b$-free bipartite graphs [3].

3 Presentations

In this section we give details on the talks at the workshop. We invited a small number of senior researchers to give talks on powerful current methods or exciting recent results. Towards are goal of training younger researchers, we encouraged any non-senior researcher who wanted to speak to volunteer give a talk and had presentations from Natalie Behague, Florian Hoersch, JD Nir, Mahsa Shirazi and Corrine Yap.

3.1 Plenary talks

Speaker: Sergey Norin
Title: Burning Large Trees
Abstract: The burning number $b(G)$ of the graph G is the minimum k such that $V(G)$ can be covered by vertex sets of subgraphs $G_1, \ldots, G_k$ such that G_i has radius at most $i-1$. The burning number conjecture of Bonato, Janssen and Roshanbin states that $b(G) \leq \lceil\sqrt{n}\rceil$ for any connected n vertex graph. We will show that $b(G) \leq (1+o(1))\sqrt{n}$ by considering continuous and fractional variants of the problem.

Speaker: Richard Montgomery
Title: On the Erds-Gallai cycle decomposition conjecture
Abstract: In the 1960's, Erds and Gallai conjectured that the edges of any n-vertex graph can be decomposed into $O(n)$ cycles and edges. In 2014, Conlon, Fox and Sudakov made the first general progress on this by showing an n-vertex graph can always be decomposed into $O(n \log \log n)$ cycles and edges. I will discuss how to improve the $\log \log n$ in this bound to the iterated logarithm function, and the tools and methods involved. This is joint work with Matija Buci.

Speaker: Will Perkins
Title: The statistical physics perspective in combinatorics
Abstract: I'll introduce some objects, concepts, and questions from statistical physics, then explain how one can look at problems in extremal and enumerative combinatorics from this perspective. I'll describe methods from statistical physics for enumerating independent sets in graphs and how these can be combined with combinatorial tools like graph containers.

Speaker: Maya Stein
Title: Towards a Posa Seymour conjecture for hypergraphs
Abstract: A central problem in extremal graph theory is to study degree conditions that force a graph G to contain a copy of some large or even spanning graph F. One of the most classical results in this area is Dirac's theorem on Hamilton cycles. An extension of this theorem is the Posa-Seymour conjecture on powers of Hamilton cycles, which has been proved for large graphs by Komlos, Sarkozy and Szemeredi.

Extension of these results to hypergraphs, using codegree conditions and tight (powers of) cycles, have been studied by various authors. We give an overview of the known results, and then show a codegree condition which is sufficient for ensuring arbitrary powers of tight Hamilton cycles, for any uniformity. This could be seen as an approximate hypergraph version of the Posa-Seymour conjecture. On the way to our result, we show that the same codegree conditions are sufficient for finding a copy of every spanning hypergraph of bounded tree-width which admits a tree decomposition where every vertex is in a bounded number of bags.

This is joint work with Nicolas Sanhueza-Matamala and Matias Pavez-Signe.

4 Scientific Progress and Ongoing Collaborations

In this section we summarise some of the more tangible progress that has been made as a consequence of the workshop. We note, in addition, that a number of other collaborations and research visits been planned by participants who met at our workshop, so more progress is to come!

4.1 Turán-good graphs

During the workshop, Morrison, Norin, Rzążewski and Wesolek solved Problem 1. Moreover, they proved the conjecture of Gerbner and Palmer [16] holds for all graphs.

Theorem 16. *Let H be a graph and $r \geq 300v(H)^9$. Then H is K_{r+1}-Turán-good.*

Theorem 16 follows from a more technical result, which also implies that for $r \geq 300v(H)^9$ Turán graphs always maximize the number of copies of H among K_{r+1}-free graphs on any given number of vertices, i.e., the requirement that the number of vertices is large compared to r is unnecessary.

4.2 Twin-width

Behague, Johnston, Hörsch, Morrison, Nir, Norin, Rzążewski and Shirazi made progress in a number of directions towards proving Conjecture 8.

They were able to show that the conjecture holds for the Erdős-Rényi random graph $G(n, 1/2)$ and for all graphs on up to at most 14 vertices. They also obtained bounds on the twin-width of $G(n, p)$ for a wide range of p. This work is ongoing.

4.3 Rainbow saturation

As a consequence of discussions at this workshop, Behague, Johnston and Morrison, along with Odgen (a masters student of Morrison) initiated a collaboration which resulted in them solving Problem 2 fully. They have proved the following theorem.

Theorem 17. *For every graph H there exists a constant $c = c(H)$ such that*

$$sat_{rbw}(n, H) \leq cn.$$

This result is currently being written up.

4.4 Monochromatic paths in multipartite hypergraphs

During working sessions at the workshop, Bowtell, Skokan, Stein, Wesolek made progress towards resolving Problem 10. They can obtain two disjoint monochromatic loose paths covering all but $o(n)$ of the vertices in a 3-partite 3-graph. In addition, they can cover all vertices (or all but a small constant and for all n) by 3 such paths. They are currently working on extending their initial argument using absorbing and explore other generalisations, i.e. from loose paths in 3-partite 3-graphs to ℓ-paths in k-partite k-graphs for some more values of ℓ and k.

5 Training and career enhancement of junior participants

The workshop was carefully designed to maximise the benefit to less senior researchers (details of how we did this can be seen in our original proposal). As organisers, we designed the working groups in such a way as to ensure that all junior researchers were in some group with more senior participants, so that they could network and learn from their expertise. We also planned group social activities in the evenings to facilitate networking in a more relaxed setting. We believe we were very successful in these goals, as we have received several very positive emails from participants after the workshop thanking us for the invitation and telling us what they gained from the experience.

One PhD student wrote: *"The workshop was one of the most valuable experiences I've had in my mathematical career thus far. On the professional side of things: I did not know most of the participants beforehand but through the workshop made many new professional and personal connections. I have begun two new collaborations that we plan to continue beyond the workshop. Through the talks and the problem presentations, I learned a lot about the breadth of problems which people in combinatorics are working on.*

As a graduate student, I was not sure what to expect, but the organizers fostered a wonderfully collaborative atmosphere among all the participants. The fact that all participants, both junior and senior, were encouraged to contribute problems and give talks played a big part. I felt like I was able to make meaningful contributions to the mathematical conversations, which is not always the case at other mathematical conferences/events, and I think that was because of how welcoming the organizers and other participants were. The setting of Banff and the many informal social activities were, dare I say, equally as valuable as the research time itself. Whether during a hike, or mealtimes, or coffee breaks, I got to know most of the other participants and received great mentorship and advice about my future academic career. Overall, I am incredibly grateful that I had the opportunity to participate in this workshop."

6 Equity, Diversity and Inclusion

In order to achieve our third objective, we spent a lot of time and consideration before the workshop on ensuring our final participant list was diverse and strongly includes those from groups under-represented in the mathematical sciences.

More details about our nomination process can be found in our EDI statement, we present a summary here. We solicited nominations for PhD students and postdocs to invite, both from invited participants and other members of the community. This resulted in us being able to invite many wonderful young researchers that were not already personally known to us the organisers. This resulted in all participants meeting new colleagues and future collaborators at the workshop. We asked in particular for nominations of those under-represented in the mathematical sciences and for nominations of participants that were based at different institutions to the nominator (to try to ensure diversity and minimise nepotism).

As noted in our EDI statement, we wrote that *"We are aiming for at least 50% of our final participants to identify as female and for at least 20% to be visible minorities."* We are happy to confirm that we achieved these targets, despite COVID causing a number of participants to drop out close to the workshop and us having to issue new invites at the last minute.

We also wrote: *"We will select a diverse and representative subset of the more senior participants to give longer talks"*. We did do this, but unfortunately two of our plenary speakers dropped out at the very last minute, and several other senior participants did not feel prepared to talk at such short notice, so the final cohort of plenary speakers was not as diverse as it had been planned to be.

7 Conclusion

We believe that the workshop was a great success, both scientifically and for the career enhancement of more junior researchers. We are very happy to have been able to facilitate such a positive impact on the more junior members of our community, especially given how disruptive the previous few years have been. We very much hope to be able to repeat this success with a similar event in the future.

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