Scattering for 1DNLS with variable coefficients

Piero D'Ancona

Dipartimento di Matematica
Sapienza Università di Roma

Mathematical aspects of the Physics with non self-Adjoint Operators
BIRS 10–15 July 2022

References

Based in part on joint works with

- Luca Fanelli (Bilbao)
- Federico Cacciafesta (Padova)
- Biagio Cassano (Bari)
- Angelo Zanni (Dottorato, Sapienza) - work in progress

What is scattering?

Scattering theory compares the asymptotic behaviour of the solution flows $F(t)$ and $D(t)$ of two different but 'close' evolution equations, linear or nonlinear, in some Banach space of initial data X (under the assumption that the two flows are globally and uniquely defined)

Example

$F(t)\phi = e^{it\Delta}\phi = u(t,x)$ is the solution on $\mathbb{R}_t \times \mathbb{R}^n_x$ of

$$iu_t + \Delta u = 0, \qquad u(0,x) = \phi(x) \in H^1(\mathbb{R}^n) = X$$

$D(t)\psi = e^{it(\Delta - V)}\psi = v(t,x)$ is the solution of

$$iv_t + \Delta v = V(x)v, \qquad v(0,x) = \psi(x) \in H^1(\mathbb{R}^n) = X$$

What is the relation between the global dynamics of the two flows?

Writing $o_X(1)$ to mean $\|o_X(1)\|_X \to 0$, we aim at expansions like:

$$F(t)\phi = D(t)\phi_\pm + o_X(1) \quad \text{as} \quad t \to \pm\infty \tag{1}$$

More precisely, given $\phi \in X$, can we find $\phi_+, \phi_- \in X$ such that (1) holds?

Conversely, we can try to prove the expansions

$$D(t)\psi = F(t)\psi_\pm + o_X(1) \quad \text{as} \quad t \to \pm\infty \tag{2}$$

but the symmetry is only formal, indeed in typical situations:

$F(t)$ is a 'reference' flow which is known in detail. Thus (1) is easier to solve (it often reduces to a problem with small data at time infinity) and is called the problem of the existence of the wave operator. The wave operator is the map $W_+ : \phi \mapsto \phi_+$

$D(t)$ is a 'perturbed' flow. Thus (2) contains more information and is harder to prove than (1). When a solution $D(t)\psi$ satisfies (2) we say it scatters. When all solutions scatter we say that asymptotic completeness holds

Scattering theory is very extensive

In classical potential scattering: $F(t) = e^{it\Delta}$, $D(t) = e^{it(\Delta - V)}$

In modern nonlinear scattering:

- $F(t) = e^{it\Delta}$
- $D(t)$ = solution flow of NLS $iu_t + \Delta u = \pm |u|^{\gamma-1} u$ on $\mathbb{R} \times \mathbb{R}^n$
  $+$ = defocusing equation, $-$ = focusing equation

If $\phi \in H^1$ and the equation is defocusing, $D(t)\phi$ is well defined for

$$1 \leq \gamma \leq \gamma_{H^1} := 1 + \tfrac{4}{(n-2)_+} \qquad (\gamma_{H^1} = \infty \text{ for } n = 1, 2)$$

and scattering occurs in the intercritical range (Ginibre–Velo ∼1985)

$$\gamma_{L^2} < \gamma < \gamma_{H^1}, \qquad \gamma_{L^2} := 1 + \tfrac{4}{n}$$

The most difficult energy critical case $\gamma = \gamma_{H^1}$ in $n \geq 3$ was solved by Bourgain, Tao, Visan, Ryckman, CKSTT 1999–2005

Of course scattering is not restricted to linear Schrödinger/NLS

Other settings:

- Obstacle scattering (exterior domains)
- Wave, Klein–Gordon, Dirac, Maxwell and other equations
- Equations on manifolds
- Scattering–like behaviour of solutions in compact settings (cubic 1DNLS on $\mathbb{T}$: Kappeler–Schaad–Topalov 2017)
- Stationary scattering fot the Helmholtz equation

Main problem

I am interested in scattering for the flows

- $F(t)\phi = e^{-itA}$ where A is a selfadjoint operator on $L^2(\mathbb{R}^n)$ (elliptic operator). This is the solution of the linear problem

$$iu_t - Au = 0, \qquad u(0,x) = \phi$$

- $D(t)\phi$ solution of

$$iu_t - Au = \pm|u|^{\gamma-1}u, \qquad u(0,x) = \phi$$

Necessary ingredients are

1. A good understanding of the dispersive properties of e^{-itA}
2. A good well posedness theory for the nonlinear equation

Dispersion for linear flows

The model flow is $e^{it\Delta}$. In decreasing order of strength:

- Pointwise decay $p \in [2.\infty]$

$$\|e^{it\Delta}\phi\|_{L^p} \lesssim |t|^{\frac{n}{p}-\frac{n}{2}}\|\phi\|_{L^{p'}}$$

- Strichartz–Sobolev estimates $p, r \in [2, \infty]$

$$\|e^{it\Delta}\phi\|_{L^pL^r} \lesssim \|\phi\|_{\dot{H}^s}, \qquad 0 < \tfrac{n}{r} \leq \tfrac{n}{2} - \tfrac{2}{p}, \qquad s = \tfrac{n}{2} - \tfrac{2}{p} - \tfrac{n}{r}$$

  and the inhomogeneous variants for $iu_t + \Delta u = F(t, x)$

- Smoothing estimates

$$\|\langle x\rangle^{-1/2-}|D|^{1/2}e^{it\Delta}\phi\|_{L^2L^2} \lesssim \|\phi\|_{L^2}$$

  plus the inhomogeneous variants

Dispersion for $A = -\Delta + V(x)$

For the case of a potential Yajima 1995–2002 developed a very general theory based on a property of the wave operator W

Intertwining property: (P_{ac} = projection on the ac spectrum of $\Delta - V$)

$$W^* a(t, \Delta) W = P_{ac} a(t, \Delta - V)$$

Under suitable decay, smoothness and spectral assumptions on V

$$W, W^* : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \quad \text{are bounded}$$

This gives, like in the free case

$$\|P_{ac} e^{it(\Delta - V)} \phi\|_{L^p} \lesssim |t|^{\frac{n}{p} - \frac{n}{2}} \|\phi\|_{L^{p'}}$$

and Strichartz estimates follow

- In 1D Yajima's result was improved by Weder, D.–Fanelli (see below)
- Goldberg–Visan 2006: pointwise estimates fail if $V \in C^{\frac{n-3}{2}-}(\mathbb{R}^n)$
- Strichartz estimates alone hold under weaker assumptions on the potential
- Beceanu–Goldberg 2012: Strichartz estimates in 3D for potentials of Kato class and small negative part
- Burq–Planchon–Stalker-TZadeh 2004: Strichartz estimates in $n \geq 3$ for repulsive potential of critical decay $\sim |x|^{-2}$

Dispersion for $A = (i\partial_x + b(x))^2 + V(x)$

- For electromagnetic potentials, i.e. with first order terms, pointwise estimates are an open problem (some results for the 3D wave equation Cuccagna–Schirmer 1999, D.–Fanelli 2006, Cacciafesta–D. 2013)
- Strichartz estimates hold for potentials of almost critical decay and minimal regularity D.–Fanelli 2008, Erdogan–Goldberg–Schlag 2009

Dispersion for A elliptic

Fully variable coefficients

$$A\phi = -\partial_b(a(x)\partial_b\phi) + V(x), \qquad \partial_b = \partial_x + ib(x)$$

- Strichartz estimates for e^{-itA} hold under various assumptions Staffilani-Tataru 2002, Robbiano-Zuily 2005, Bouclet-Tzvetkov 2008
- Tataru 2008: sufficient conditions are ($\epsilon \ll 1$, $\delta > 0$)

$$|a - I| + \langle x\rangle(|a'| + |b|) + \langle x\rangle^2(|a''| + |b'| + |V|) \le \epsilon\langle x\rangle^{-\delta}$$

- Cassano–D. 2015: V can be taken large, repulsive, with almost critical decay
- For large $a(x)$ the estimates fail (trapped energy). Nontrapping conditions are necessary

Dispersion in 1D

D.–Fanelli 2006: L^p boundedness of the wave operator for

$$A = -\partial_x a(x)\partial_x + b(x)\partial_x + V(x), \qquad a(x) \geq c_0 > 0$$

provided

Assumption (A)

$$a(x) \geq c_0 > 0$$

$$\langle x\rangle(|a'| + |b|) \in L^2(\mathbb{R}), \qquad \langle x\rangle^2(|V| + |a''| + |b'|) \in L^1(\mathbb{R})$$

In particular, pointwise decay and Strichartz estimates hold

Burq–Planchon 2004: Strichartz estimates hold for $a \in BV$ (but they require $b = V = 0$)

Q: what about $-\partial_x a(x)\partial_x + V(x)$? I can do this for odd solutions

Well posedness of the NLS

If e^{-itA} satisfies Strichartz estimates, or a suitable subset, the nonlinear theory is essentially identical to the free case. We consider the problem

$$iu_t - Au = |u|^{\gamma-1}u, \qquad u(0,x) = \phi(x)$$

or rather its integral version

$$u(t,x) = e^{-itA}\psi - i\int_0^t e^{-i(t-s)A}|u|^{\gamma-1}u ds \tag{3}$$

We fix the following (standard) indices:

$$p = \frac{2(\gamma^2-1)}{\gamma+3} \qquad r = \gamma+1 \qquad q = \frac{2(\gamma^2-1)}{\gamma^2-2\gamma-3}$$

Note in particular the Strichartz estimates

$$\|e^{-itA}\phi\|_{L^pL^r} \lesssim \|\phi\|_{H^1}, \qquad \|\int_0^t e^{-i(t-s)A}G(s)ds\|_{L^pL^r} \lesssim \|G\|_{L^{q'}L^{r'}}$$

Since $(r'\gamma, q'\gamma) = (p,r)$ we have also $\||u|^{\gamma-1}u\|_{L^{q'}L^{r'}} = \|u\|^{\gamma}_{L^pL^r}$

Denote by $D(t)\psi$ the solution to (3), with $\psi \in H^1$

We say that $D(t)\psi$ scatters at $\pm\infty$ if for some $\psi_+, \psi_- \in H^1$ one has

$$\|D(t)\psi - e^{-itA}\psi_\pm\|_{H^1} \to 0 \quad \text{as} \quad t \to \pm\infty$$

We say that the wave operator for A exists if for any $\phi \in H^1$ there exist $\phi_+, \phi_- \in H^1$ such that

$$\|e^{-itA}\phi - D(t)\phi_\pm\|_{H^1} \to 0 \quad \text{as} \quad t \to \pm\infty$$

Theorem

Let Assumption (A) be satisfied, and $\gamma > 5$. Then for any $\psi \in H^1$ Problem (3) *has a unique global solution $u \in C(\mathbb{R}; H^1)$. Moreover,*

- *if $u \in L^p L^r$ then u scatters*
- *if $\|\psi\|_{H^1}$ is sufficiently small then u scatters*
- *the wave operator for A exists.*

An essential property for the construction of a solution with critical energy:

Theorem (Nonlinear perturbation)

Let Assumption (A) be satisfied, $\gamma > 5$*.*
For any $M > 0$ *there exist* $\epsilon, C > 0$ *such that the following holds. Let* $\|\psi\|_{H^1} < M$ *and* $\|G(t,x)\|_{L^pL^r} < \epsilon$*. Suppose* $v(t,x) \in L^pL^r$ *satisfies*

$$v(t,x) = e^{-itA}\psi - i\int_0^t e^{-i(t-s)A}|v|^{\gamma-1}v ds + G(t,x).$$

Then the solution of

$$u(t,x) = e^{-itA}\psi - i\int_0^t e^{-i(t-s)A}|u|^{\gamma-1}u ds$$

belongs to L^pL^r*, hence scatters, and* $\|u - v\|_{L^pL^r} \leq C\epsilon$

Scattering: classical approaches

In order to prove scattering of a global solution to

$$iu_t - Au = |u|^{\gamma-1}u$$

there exist several scattering criteria i.e. sufficient conditions

- Morawetz estimate: a bound of the form

  $$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{|u|^{\gamma+1}}{|x|} dxdt < \infty$$

  which can be proved by multiplier methods
  One deduces that $\|u(t)\|_{L^q} \to 0$ as $t \to +\infty$ and scattering follows
  Classical approach for $n \geq 3$ by Lin–Strauss, Ginibre–Velo

  Nakanishi modifies this method in dimension $n = 1, 2$ (time dependent Morawetz estimate)

- Bilinear smoothing, or Quadratic Morawetz estimate. A more efficient approach popularized by CKSTT (but used also by Ginibre and Velo) based on the bound

$$\int \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^3} dx dy dt < \infty$$

  This method works in dimension $n \geq 4$, and also for $n = 3$ by a suitable modification

  We used it in Cassano–D. 2015 to prove scattering for the intercritical defocusing NLS

$$iu_t - Au = |u|^{\gamma-1} u$$

  with fully variable coefficients in dimension $n \geq 3$

The Kenig–Merle approach

We follow a different approach, introduced by Kenig–Merle 2006 to study the radial, energy critical, focusing NLS

The method is flexible and has been applied and improved in a large number of works both on focusing and defocusing NLS (Holmer–Roudenko 2008, Duyckaerts–Holmer–Roudenko 2008, Fang–Xie–Cazenave 2011) and on other equations (wave, Klein–Gordon, Yang–Mills, wave maps)

For NLS with potentials translation invariance is broken. This difficulty was overcome in

- Hong 2016: cubic focusing 3DNLS with a short range potential
- Banica–Visciglia 2016: intercritical, defocusing 1DNLS with δ potential
- Lafontaine 2016: intercritical, defocusing 1DNLS with repulsive potential
- Ikeda 2021: intercritical, focusing 1DNLS with repulsive potential
- Dinh 2021: 3DNLS with potential

In particular in Banica–Visciglia 2016 an effort was done to streamline the KM technique and make it applicable to more general equations

We pursue their idea towards a black–box approach and an ‘abstract’ profile decomposition method

Main result

Recall Assumption (A) implying Strichartz estimates: $a(x) \geq c_0 > 0$, and

$$\langle x \rangle (|a'| + |b|) \in L^2(\mathbb{R}), \qquad \langle x \rangle^2 (|V| + |a''| + |b'|) \in L^1(\mathbb{R})$$

(This is work in progress and the assumptions can certainly be improved!)

Theorem (D.–Zanni)

Let $\gamma > 5$. Suppose Assumption (A) is satisfied. There exist $\epsilon > 0$ such that the following holds. If $V \geq 0$, $xV'(x) \leq 0$, $\langle x \rangle (|V| + |V'|) < \infty$ and

$$\langle x \rangle^2 (|a - 1| + |a'| + |a''| + |a'''|) < \epsilon$$

then asymptotic completeness in H^1 holds for the equation

$$iu_t - Au = |u|^{\gamma-1} u$$

where $Au = -\partial_b a(x) \partial_b u + V(x) u$.

Outline of the KM method

Recall the choice

$$p = \frac{2(\gamma^2-1)}{\gamma+3} \qquad r = \gamma + 1 \qquad q = \frac{2(\gamma^2-1)}{\gamma^2-2\gamma-3}$$

which implies $\||u|^{\gamma-1}u\|_{L^{q'}L^{r'}} = \|u\|^{\gamma}_{L^pL^r}$ and the Strichartz estimates

$$\|e^{-itA}\phi\|_{L^pL^r} \lesssim \|\phi\|_{H^1}, \qquad \|\int_0^t e^{-i(t-s)A}G(s)ds\|_{L^pL^r} \lesssim \|G\|_{L^{q'}L^{r'}}$$

Steps:

1. Linear profile decomposition
2. Construction of a critical solution
3. Rigidity and scattering

1: Linear profiles

The idea of profile decomposition was initally developed for the NLWE Gerard 1998, Bahouri–Gerard 1999 and then extended to the NLS Merle–Vega 1998, Keraani 2001. Its origin might be traced back to the concentration–compactness principle of Lions 1984

Main idea: let (v_n) be a bounded sequence in $H^1(\mathbb{R}^n)$ and let $p \in [2, 2^*)$

- If the supports of v_n are localized in a bounded set, we can extract a convergent subsequence in L^p
- But in general this does not happen and the mass of v_n may split in several bumps moving towards infinity, or may flatten out
- Can we single out one of the bumps and follow it? yes!
- The trick is to smoothen out v_n (by frequency truncation) and find a point x_n where the smoothed v_n is large. Then a significant part of the L^p norm of v_n must be localized near x_n

- The translated sequence $\tau_{x_n} v_n = v(x - x_n)$ is bounded in H^1. We extract a subsequence which converges weakly to the first profile

$$\tau_{x_n} v_n \rightharpoonup \psi^1 \quad \text{in} \quad H^1, \qquad \psi^1 \neq 0$$

- Define the remainder $R_n^1 = v_n - \psi^1(x + x_n)$
- If $\|R_n^1\|_{L^p} \to 0$ we stop. If not, we iterate taking R_n^1 as the new sequence v_n and obtaining a second sequence x_n^2 and a second profile ψ^2, and so on

Conclusion: for every J we can find $\psi^1, \dots, \psi^J \in H^1$ and sequences $(x_n^1), \dots, (x_n^J)$ such that $|x_n^j - x_n^k| \to +\infty$ if $j \neq k$ and

$$v_n = \sum_{j=1}^J \psi^j(x - x_n^j) + R_n^J,$$

$$\lim_{J \to +\infty} \lim_{n \to +\infty} \|R_n^J\|_{L^p} = 0$$

$$\|v_n\|_{H^1}^2 = \sum_{j=1}^J \|\psi^j\|_{H^1}^2 + \|R_n^J\|_{H^1}^2 + o(1) \quad \text{as} \quad n \to +\infty$$

The property $|x_n^j - x_n^k| \to +\infty$ if $j \neq k$ is crucial and is called the orthogonality of the two–index sequence x_n^j. (It is also obvious: if $x_n^j \sim x_n^k$ the two bumps are running together and form a single bump)

The idea works also for the critical embedding $H^1 \hookrightarrow L^{2^*}$, but then one must also account for concentration effects, and an additional scaling parameter is needed

Keraani's idea is to do the same for a sequence of solutions $e^{it\Delta}\phi_n(x)$ of the linear Schrödinger equation. The embedding $H^1 \hookrightarrow L^p$ is replaced by the Strichartz estimate $H^1 \hookrightarrow L^p L^r$

To state the result for our problem we use the notations

- $\tau_z u(x) = u(x - z)$ the translation operator
- $A_z = \tau_{-z} A \tau_z$. For instance, if $Au = \partial_x(a(x)\partial_x u(x))$ then

$$A_z u = \partial_x(a(x+z)\partial_x u(x))$$

  Assumptions on A_z for $z \to +\infty$ are asymtotic assumptions on the coefficients of A at spatial infinity

- A standard sequence $(x_n) \subset \mathbb{R}$ is such that either $x_n \to +\infty$, or $x_n \to -\infty$, or $x_n = 0$ for all n

Abstract assumptions

For every $\psi \in H^1$, all real sequences $(x_n), (y_n), (s_n)$ and a $p \in (2, \infty)$

1. $\|e^{itA}\psi\|_{L^\infty H^1} \lesssim \|\psi\|_{H^1}$ and $|(A_z\psi, \psi)| \lesssim \|\psi\|_{H^1}^2$
2. $(A_{x_n}\psi)$ is precompact in H^{-1}
3. $(e^{is_n A_{x_n}}\psi)$ is precompact in L^p and, for $s_n = \overline{s}$ constant, in H^1.
4. if $|s_n| \to +\infty$ then $e^{is_n A_{y_n}}\psi \rightharpoonup 0$ in H^1 up to a subsequence

Theorem (First Profile)

Assume (1)–(2)–(3). Given any bounded sequence in $H^1(\mathbb{R})$ we can find a subsequence (v_n), $\psi \in H^1$, standard sequences (x_n), $(t_n) \subset \mathbb{R}$ s.t.

1. $\tau_{-x_n} e^{-it_n A} v_n = \psi + W_n$ *with* $W_n \rightharpoonup 0$ *in* H^1
2. $\limsup \|e^{-itA} v_n\|_{L^\infty L^\infty} \lesssim \|\psi\|_{L^2}^{1/2} \sup \|v_n\|_{H^1}^{1/2}$
3. *we have the asymptotic behaviours for* $n \to +\infty$
   - $\|v_n\|_{L^2}^2 = \|\psi\|_{L^2}^2 + \|W_n\|_{L^2}^2 + o(1)$
   - $(Av_n, v_n)_{L^2} = (A_{x_n}\psi, \psi)_{L^2} + (A_{x_n} W_n, W_n)_{L^2} + o(1)$
   - $\|v_n\|_{L^p}^p = \|e^{it_n A_{x_n}} \psi\|_{L^p}^p + \|e^{it_n A_{x_n}} W_n\|_{L^p}^p + o(1)$.

Theorem (Profile decomposition)

Assume (1)–(2)–(3)–(4). Given any bounded sequence in H^1 we can find a subsequence $(u_n)_{n\geq 1}$, and $\forall j \in \mathbb{N}$ we can find $\psi_j \in H^1$ and standard sequences $(t_j^n)_{n\geq 1}$, $(x_j^n)_{n\geq 1}$ as follows. Writing $J \in \mathbb{N}$

$$u_n = \sum_{j=1}^{J} e^{it_j^n A}\tau_{x_j^n}\psi_j + R_J^n, \tag{4}$$

1. *for all $j \neq k$ we have $|t_j^n - t_k^n| + |x_j^n - x_k^n| \to +\infty$*
2. $\limsup_n \|e^{-itA}R_J^n\|_{L^\infty L^\infty} \to 0$
3. *for each J we have the asymptotic behaviours as $n \to +\infty$*
   - $\|u_n\|_{L^2}^2 = \sum_{j=1}^{J} \|\psi_j\|_{L^2}^2 + \|R_J^n\|_{L^2}^2 + o(1)$
   - $(Au_n, u_n)_{L^2} = \sum_{j=1}^{J}(A\tau_{x_j^n}\psi_j, \tau_{x_j^n}\psi_j)_{L^2} + (AR_J^n, R_J^n)_{L^2} + o(1)$
   - $\|u_n\|_{L^p}^p = \sum_{j=1}^{J} \|e^{it_j^n A}\tau_{x_j^n}\psi_j\|_{L^p}^p + \|R_J^n\|_{L^p}^p + o(1).$
4. *if $\psi_J = 0$ for some J then $\psi_j = 0$ for all $j \geq J$*

2: The critical solution

If $\phi \in H^1$ let $u(\phi)$ the unique solution of the nonlinear equation

$$iu_t + Au = |u|^{\gamma-1}u, \qquad u(0) = \phi$$

with energy

$$E(\phi) = (Au, u)_{L^2} + \frac{1}{\gamma+1}\|u\|_{L^{\gamma+1}}^{\gamma+1}$$

Define the critical energy E_{crit} as

$$E_{crit} = \sup\{E > 0 : \forall\phi \in H^1,\ E(\phi) < E \Rightarrow u(\phi) \in L^p L^r\}.$$

Out goal: prove that $E_{crit} = \infty$

Assume by contradiction $E_{crit} < \infty$ and pick $\phi_n \in H^1$ such that

$$E(\phi_n) \downarrow E_{crit} \quad \text{and} \quad u(\phi_n) \notin L^p L^r.$$

We apply the profile decomposition

$$\phi_n = \sum_{j=1}^{J} e^{it_j^n A}\tau_{x_j^n}\psi_j + R_J^n$$

In particular we have

$$E_{crit} = \sum_{j=1}^{J} E(e^{it_j^n A} \tau_{x_j^n} \psi_j) + E(R_J^n) + o(1)$$

and hence

$$\infty > E_{crit} \geq \limsup_n \sum_{j=1}^{J} E(e^{it_j^n A} \tau_{x_j^n} \psi_j). \tag{5}$$

Theorem

There is at most one profile i.e. $J = 1$, *with* $t_1^n = x_1^n = 0$ *and* $E(\psi_1) = E_{crit}$. *The corresponding solution* $u(\psi_1) \notin L^p L^r$, *and* $\{u(\psi_1)(t) : t \geq 0\}$ *is precompact in* H^1

The procedure is the following:

- Using the profiles ψ_j as data, construct an approximate solution of the nonlinear equation, which is close to $u(\phi_n)$
- Assume by contradiction $J \geq 2$; then $E(\psi_j) < E_{crit}$, hence the approximate solution is in $L^p L^r$ and scatters
- By the nonlinear perturbation property, also $u(\phi_n)$ scatters, giving a contradiction
- We deduce $J = 1$ and hence $E(\psi_1) = E_{crit}$
- Compactness of the flow follows by applying the same profile decomposition to the bounded sequence $u(\psi_1)|_{t_n}$ with $t_n \to \infty$

We try to perform also this step in an 'abstract' setting

Abstract assumptions B

The operator A satisfies

$$\|e^{-itA_z} - e^{-itA}\|_{H^1 \to H^1} \to 0 \quad \text{as} \quad z \to 0$$

and if $x_n \to +\infty$ or $-\infty$ then

$$\|e^{it\Delta} - e^{-itA_{x_n}}\|_{H^1 \to L^p L^r} \to 0$$

$$\| \int_0^t (e^{i(t-s)\Delta} - e^{-i(t-s)A_{x_n}}) \cdot ds\|_{L^{q'} L^{r'} \to L^p L^r} \to 0$$

In the following we write for brevity

$$F(u) = |u|^{\gamma-1} u$$

Recall the profile decomposition of ϕ_n

$$\phi_n = \sum_{j=1}^{J} e^{it_j^n A}\tau_{x_j^n}\psi_j + R_J^n$$

To each profile ψ_j we associate a nonlinear solution, but the construction depends on the sequences. We will use each ψ_j as initial data 'at the point' (t_j^n, x_j^n), thus we set

$$U_j^n(t,x) = U_j(t - t_j^n, x - x_j^n)$$

where each U_j is defined according to four possible cases:

Case 1: $t_j^n = x_j^n = 0$ for all n. Then U_j is simply the solution with data ψ_j:

$$U_j = u(\psi_j)$$

Note that by orthogonality this case happens at most for one profile

Case 2: $t_j^n \to \pm\infty$ and $x_j^n = 0$ for all n, e.g. $t_j^n \to +\infty$. Then we use ψ_j as 'data at infinity', i.e. as scattering data. We know that the wave operator at $-\infty$ exists for A, thus we can define U_j as the solution of

$$i\partial_t u - Au = F(u), \qquad \lim_{t\to-\infty} \|U_j(t) - e^{-itA}\psi_j\|_{H^1} = 0.$$

Case 3: $t_j^n = 0$ for all n and $x_j^n \to \pm\infty$. We rely on the Abstract assumption B: for x large, $e^{-itA} \simeq e^{it\Delta}$. Thus we set U_j as the solution of

$$iu_t + \Delta u = F(u), \qquad u(0,x) = \psi_j$$

Case 4: $t_j^n \to \pm\infty$ and $x_j^n \to \pm\infty$, e.g. $t_j^n \to +\infty$. As in Case 2, we use ψ_j as scattering data, but this time we use Δ instead of A in view of the Abstract assumption B since $|x_j^n| \to \infty$. Now U_j is the solution of

$$i\partial_t u + \Delta u = F(u), \qquad \lim_{t\to-\infty} \|U_j(t) - e^{-itA}\psi_j\|_{H^1} = 0.$$

If we plug $U_j^n(t,x) = U_j(t - t_j^n, x - x_j^n)$ in the equation we check that

$$U_j^n(t,x) = e^{-itA}\psi_j + i\int_0^t e^{-i(t-s)A}F(U_j^n(s,x))ds + r_j^n$$

and in all cases the error r_j^n satisfies

$$\|r_j^n\|_{L^pL^r} \lesssim \|\psi_j\|_{H^1} \cdot o(1)$$

The approximate solution is obtained by summing the U_j^n:

$$W_J^n = \sum_{j=1}^{J} U_j^n$$

We assume by contradiction that $J \geq 2$; then the H^1 norm of ϕ_n must split between the profiles. Hence the profiles are subcritical, $U_j^n \in L^p L^r$, and

$$W_J^n \in L^p L^r \quad \text{and scatters}$$

Plugging W_J^n in the equation we get

$$W_J^n = e^{-itA}\phi_n + i\int_0^t e^{-i(t-s)A} F(W_J^n) ds + \rho_J^n$$

where the error $\rho_J^n =$

$$\sum_{j=1}^J r_j^n - e^{-itA} R_J^n + i\int_0^t e^{-i(t-s)A} \left[\sum_{j=1}^J F(U_j^n) - F(\sum_{j=1}^J U_j^n)\right] ds.$$

If we show that

$$\|\rho_J^n\|_{L^p L^r} \to 0,$$

then by nonlinear perturbation $u(\phi_n) \in L^p L^r \implies$ contradiction

The terms r_j^n and $e^{-itA}R_J^n$ tend to 0 in L^pL^r trivially by construction

For the last term we have by Strichartz

$$\| \int_0^t e^{-i(t-s)A} \left[\sum_{j=1}^J F(U_j^n) - F(\sum_{j=1}^J U_j^n) \right] ds \|_{L^pL^r}$$

$$\lesssim \| \sum_{j=1}^J F(U_j^n) - F(\sum_{j=1}^J U_j^n) \|_{L^{q'}L^{r'}} \lesssim \sum_{j\neq k} \| |U_j^n|^{\beta-1} U_k^n \|_{L^{q'}L^{r'}}$$

Also this term tends to 0, using the orthogonality

$$|t_j^n - t_k^n| + |x_j^n - x_k^n| \to +\infty$$

This is an exercise: if $f, g \in L^{2a}, a < \infty$, and $|x_n - y_n| \to \infty$, then

$$\|f(x-x_n)g(x-y_n)\|_{L^a} \to 0$$

by density of C_c in L^{2a}

3: Rigidity and scattering

Assuming $E_{crit} < \infty$, we have constructed a solution u_{crit} of critical energy with precompact flow in H^1

Compactness implies easily localization: $\forall \epsilon > 0 \, \exists R$ such that for all t

$$\|u_{crit}(t)\|_{H^1(|x| \geq R)} + \|u_{crit}(t)\|_{L^{\gamma+1}(|x| \geq R)} \leq \epsilon$$

To reach the final contradiction we rely on the explicit form of the equation, via a virial inequality of the form

$$\partial_t^2 \int \chi_R(x) |u_{crit}(t,x)|^2 dx \geq C E_{crit}$$

where $\chi_R = x^2$ for $|x| \leq R$ and vanishes for $|x| \geq 2R$, which is absurd

Only in this final step the smallness assumptions on a', a'', a''' and the repulsivity condition $xV' \leq 0$ are used