Pure and Weak Revivals in Time Evolution Problems

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Overview

- Revivals and fractalisation
- Pure revivals
- The classical setting
- Other time evolution problems
- Weak revivals
- Conclusions and further problems

Revivals and Fractalisation

Consider the free linear Schrödinger equation with periodic boundary conditions on $[0, 2\pi]$

$$\partial_t u(x,t) = i\partial_x^2 u(x,t) \quad u(x,0) = u_0(x)$$
$$\partial_x^m u(0,t) = \partial_x^m u(2\pi,t), \quad m = 0, 1.$$

- Solution representation by Fourier series

$$u(x,t) = \sum_{j\in\mathbb{Z}} \widehat{u_0}(j) e^{-ij^2 t} e_j(x),$$
$$e_j(x) = \frac{e^{ijx}}{\sqrt{2\pi}}, \quad \widehat{u_0}(j) = \langle u_0, e_j \rangle.$$

- For $u_0(x)$ with jump discontinuities there is a dichotomy in the behaviour of $u(x,t)$ at irrational times ($t/2\pi \notin \mathbb{Q}$) and rational times ($t/2\pi \in \mathbb{Q}$).

Fractalisation

Choose a step function initial condition

$$u_0(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi. \end{cases}$$

- Plot $u(x,t)$ at generic times t.
- Continuous, non-differentiable profiles.

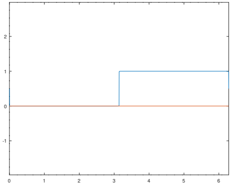


Figure: $t = 0$

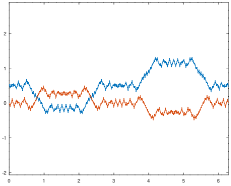


Figure: $t = 0.4$

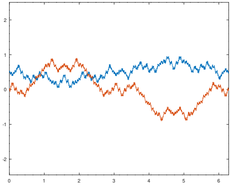


Figure: $t = 1.2$

Real (blue) and imaginary (red) parts of $u(x,t)$*.*

Revivals

- However, if t is a rational time, i.e. $\exists\ \frac{p}{q} \in \mathbb{Q}$ such that $t = 2\pi\frac{p}{q}$, then $u(x, 2\pi\frac{p}{q})$ has piece-wise constant profiles.

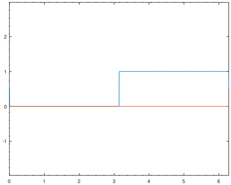


Figure: $t = 0$

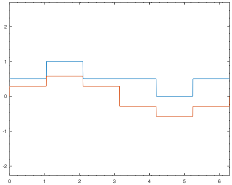


Figure: $p = 1,\ q = 3$

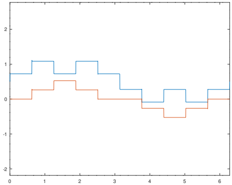


Figure: $p = 2,\ q = 5$

Real (blue) and imaginary (red) parts of $u(x, 2\pi\frac{p}{q})$.

Dichotomy in regularity:

- rational/irrational $\longleftrightarrow$ revival (discontinuous)/fractalisation (continuous+non-differentiable)

Pure revivals

For any $u_0 \in L^2(0, 2\pi)$, the following identity holds

$$u\left(x, 2\pi\frac{p}{q}\right) = \frac{1}{q}\sum_{k=0}^{q-1}\sum_{m=0}^{q-1} e^{-2\pi i m^2 \frac{p}{q}} e^{2\pi i m \frac{k}{q}} u_0^*\left(x - 2\pi\frac{k}{q}\right).$$

- u_0^* : 2π-periodic extension of u_0.
- *Proof*: Fourier coefficients of RHS are equal to

$$\widehat{u_0}(j) e^{-ij^2 2\pi \frac{p}{q}} = \widehat{u}\left(j, 2\pi\frac{p}{q}\right).$$

Pure Revivals: $u(x, 2\pi p/q)$ is a finite linear combination of translations of u_0.

Implications

(i) Finitely many jump discontinuities at rational times.
(ii) $u(x, 2\pi p/q)$ depends on finitely many values of u_0.

The classical setting

- Oskolkov'92:
  - $t/2\pi \notin \mathbb{Q} \rightarrow u(x,t)$ continuous in $x \in [0, 2\pi]$.
  - $t/2\pi \in \mathbb{Q} \rightarrow u(x,t)$ has at most countably many discontinuities in $[0, 2\pi]$.
  - u_0 continuous $\rightarrow u(x,t)$ continuous in both variables.
- Berry and Klein'96: Talbot effect = revivals/fractalisation.

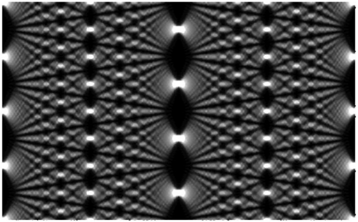


Figure: Talbot effect
(https://skullsinthestars.com/2010/03/04/rolling-out-the-optical-carpet-the-talbot-effect/)

The classical setting

- Berry'96: Quantum Revival, Conjecture: Fractal dimension is 3/2.
- Kapitanski, Rodnianksi'99 + Rodnianski'00: Fractal dimension is 3/2 indeed (for rough initial data).
- Taylor'03: Pure revival and extension to the $N-$dimensional sphere and torus.
- Olver'10 (dispersive quantisation): Pure revival for the periodic problem to

$$\partial_t u(x,t) = -iP\big(-i\partial_x\big)u(x,t),$$

where $P(\cdot)$ polynomial of order $n \geq 2$ with integer coefficients.

  - See also book by Erdoğan and Tzirakis'16 for full statement.
  - We refer to this as the classical setting.

Other time evolution problems

Question: Does the revival phenomenon survive in other time evolution problems?

- Non self-adjoint boundary conditions on $[0, 2\pi]$:

$$\partial_t u(x,t) = i\partial_x^2 u(x,t), \quad u(x,0) = u_0(x)$$
$$\beta_0 u(0,t) = u(2\pi,t), \quad \beta_1 \partial_x u(0,t) = \partial_x u(2\pi,t).$$

  - Pure revivals of u_0 and its reflection [Olver, Sheils, Smith'20, Boulton, F. , Pelloni'21].

- Self-adjoint boundary conditions on $[0, 2\pi]$:

$$\partial_t u(x,t) = \partial_x^3 u(x,t), \quad u(x,0) = u_0(x)$$
$$e^{i2\pi\theta}\partial_x^m u(0,t) = \partial_x^m u(2\pi,t), \quad m = 0,1,2.$$

  - Pure revivals if and only if $\theta \in \mathbb{Q}$ [Boulton, F., Pelloni'21].

- The revival phenomenon does not depend on the self-adjointness of the boundary conditions.
- However, the eigenstructure of the underlying differential operator seems to matter.

Weak Revivals

Separated boundary conditions on $[0, \pi]$:

$$\partial_t u(x,t) = i\partial_x^2 u(x,t), \quad u(x,0) = u_0(x)$$
$$bu(x_0,t) = (1-b)\partial_x u(x_0,t), \quad x_0 = 0, \pi \quad b \in [0,1].$$

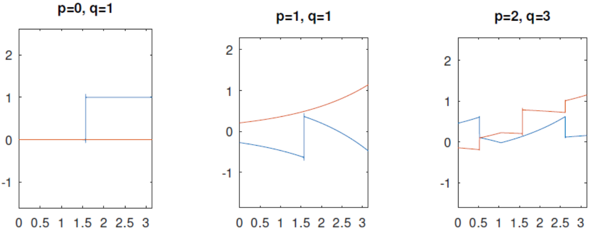
- [Boulton, F., Pelloni'21]

$$u\left(x, 2\pi\frac{p}{q}\right) = \text{pure revival of } u_0^{even}(x)$$
$$+ \left\{\text{continuous function of } x\right\}, \quad x \in [0,\pi],$$
$$:= \text{weak revival.}$$

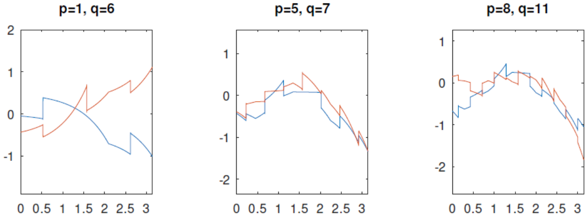
- Weak revival has a different structure from the pure revival, but it has the same implications for $u(x, 2\pi\frac{p}{q})$:
  - Finitely many jump discontinuities at rational times.
  - Dependence on finitely many values of u_0.

Weak revivals

$b = 0.35$



$b = 0.6$



Weak revivals

Consider the bi-harmonic wave equation on $[0, 2\pi]$

$$\partial_t^2 u(x,t) = -\partial_x^4 u(x,t)$$

with $u(x,0) = f(x)$ and $\partial_t u(x,0) = g(x)$ and periodic boundary conditions $\partial_x^m u(0,t) = \partial_x^m u(2\pi,t),\ m = 0,1,2,3.$

- Claim: $u(x, 2\pi\frac{p}{q})$ = pure revival + a continuous function of x.
  $:=$ weak revival.

proof. We notice that

$$\partial_t^2 u(x,t) = -\partial_x^4 u(x,t) \iff \left(\partial_t + i\partial_x^2\right)\left(\partial_t - i\partial_x^2\right) u(x,t) = 0$$

$$\iff \begin{cases} \partial_t u(x,t) = i\partial_x^2 u(x,t) + w(x,t), & u_0(x) = f(x), \\ \partial_t w(x,t) = -i\partial_x^2 w(x,t), & w_0(x) = g(x) - i\partial_x^2 f(x). \end{cases}$$

with periodic boundary conditions.

Weak revivals

By Duhamel's principle we have,

$$u(x,t) = \sum_{j\in\mathbb{Z}} \widehat{f}(j) e^{-ij^2 t} e_j(x) + \int_0^t \sum_{j\in\mathbb{Z}} \widehat{w}(j,s) e^{-ij^2(t-s)} e_j(x) ds$$

At rational times $t = 2\pi\frac{p}{q}$:

$$u\big(x, 2\pi\frac{p}{q}\big) = \text{pure revival of } f(x) + N(x, 2\pi\frac{p}{q}),$$

$$\circ\ N(x, 2\pi\frac{p}{q}) : \text{continuous function of } x \in [0, 2\pi].$$

□

Conclusions and further problems

- Weak revival is the main revival effect in the periodic even order poly-harmonic wave equation

$$\partial_t^2 u(x,t) = -(-i\partial_x)^{2r} u(x,t), \quad r \geq 2 \text{ integer}.$$

- When we go *too far outside* the classical setting, then the pure revival becomes weak revival.
- What happens for more general linear boundary conditions?
  - [Olver, Sheils, Smith'20] :

$$\partial_t u(x,t) = i\partial_x^2 u(x,t), \quad u(x,0) = u_0(x),$$
$$\beta_{11}\partial_x u(2\pi,t) + \beta_{12} u(2\pi,t) + \beta_{13}\partial_x u(0,t) + \beta_{14} u(0,t) = 0,$$
$$\beta_{21}\partial_x u(2\pi,t) + \beta_{22} u(2\pi,t) + \beta_{23}\partial_x u(0,t) + \beta_{24} u(0,t) = 0.$$

  - Third order problem with pseudo-Dirichlet boundary conditions:

$$\partial_t u(x,t) = \partial_x^3 u(x,t), \quad u(x,0) = u_0(x),$$
$$u(0,t) = u(2\pi,t) = \partial_x u(2\pi,t) = 0.$$

Conclusions and further problems

- Non-linear equations?
  [Chen and Olver'14, Erdoğan and Tzirakis'13] : Weak type revival under periodic boundary conditions on $[0, 2\pi]$:

$$\partial_t u(x,t) = i\partial_x^2 u(x,t) + i|u(x,t)|^2 u(x,t).$$

  - Galilean invariance:

$$z(x,t) = e^{i(\theta x - \theta^2 t)} u(x - 2\theta t, t)$$

    implies the weak revival effect in the case of self-adjoint quasi-periodic boundary conditions:

$$e^{i2\pi\theta}\partial_x z(0,t) = \partial_x^m z(2\pi, t), \quad m = 0, 1.$$

- Linear perturbations ?
  [Rodnianski'99, Cho, Kim, Kim, Kwon and Seo'21]: Weak type revival under periodic boundary conditions on $[0, 2\pi]$:

$$\partial_t u(x,t) = -i(-\partial_x^2 u(x,t) + V(x)u(x,t)).$$

Thank you!

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