Eigenvector localization of noisy non-selfadjoint Toeplitz matrices

Mathematical aspects of the Physics with non-self-adjoint Operators
BIRS – Banff, July 11 – 15, 2022

Martin Vogel
Institut de Recherche Mathématique Avancée
CNRS and Université de Strasbourg

Partly based on joint work with Anirban Basak, Johannes Sjöstrand and Ofer Zeitouni

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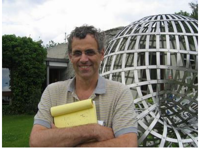
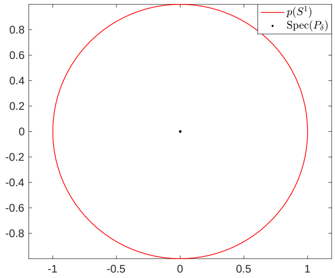


Motivational example – Left shift operator P_N on $\mathbb{C}^N$

$$P_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} : \mathbb{C}^N \to \mathbb{C}^N.$$

- **Spectrum** $\sigma(P_N) = \{0\}$, **Eigenvector** $u = (1, 0, \dots, 0)$
- **Small noise**: $P_N + N^{-\gamma} Q_N$, with $\gamma > 1/2$ and Q_N (Complex Gaussian random matrix), $\|N^{-\gamma} Q_N\| = O(N^{1/2-\gamma})$ with very high probability



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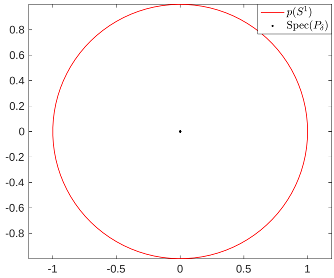

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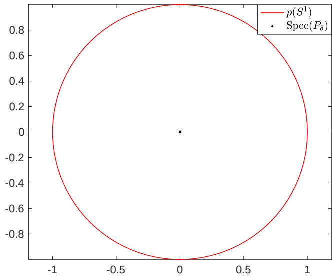

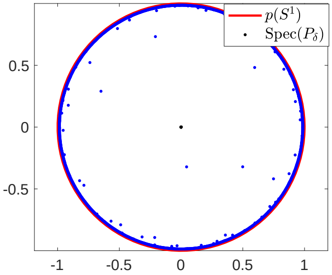

Weyl law: $\displaystyle \frac{1}{N} \sum_{\lambda \in \sigma(P_N + N^{-\gamma} Q_N)} \delta_\lambda \rightharpoonup \frac{1}{2\pi} L_{S^1}(d\lambda)$ in probability

Eigenvectors of banded noisy Toeplitz matrices

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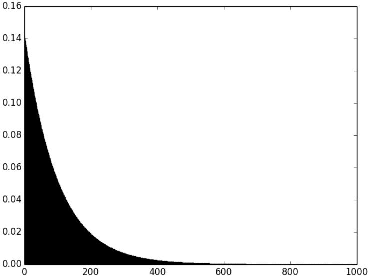
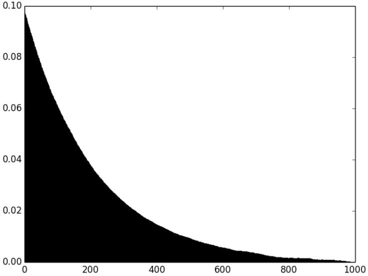


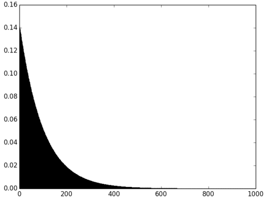
$\gamma = 2$



$\gamma = 1.5$

Eigenvectors - Phase transition?

- $P_N + N^{-\gamma}Q_N$, with $\|N^{-\gamma}Q_N\| = O(N^{1/2-\gamma})$



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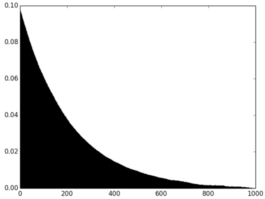

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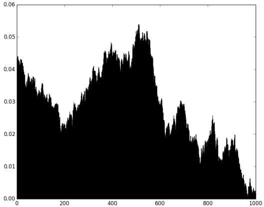
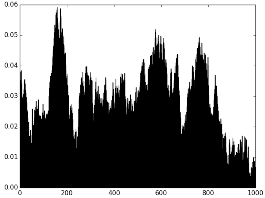
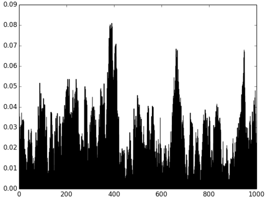


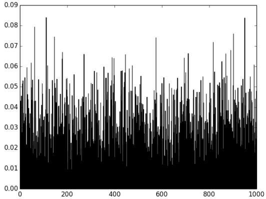
$\gamma = 1$



$\gamma = 0.9$



$\gamma = 0.75$



$\gamma = 0.5$

Non-selfadjoint operators and spectral instability

Why study the spectrum of non-selfadjoint operators?

- See all the talks of this conference!
- Scattering theory (Resonances) (see Tanya Christiansen's talk)
- Open quantum systems – Lindblad operators
- Damped systems, e.g. the damped wave equation (see Borbala Gerhat's talk)
- Stability of solutions of non-linear PDE, e.g. in hydrodynamics
- PDE Solvability theory
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Spectral instability If $P : \mathscr{H} \to \mathscr{H}$ is not self-adjoint, $(P-z)^{-1}$ may be very large far away from $\sigma(P)$:

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Pseudospectral effect Spectrum can be very unstable under small perturbations

- ε-pseudospectrum — Trefethen '97, Davies '98, Trefethen-Embree '05

$$\sigma_\varepsilon(P) \stackrel{\text{def}}{=} \sigma(P) \cup \left\{ z \in \mathbb{C};\ \|(P - z)^{-1}\| > \varepsilon^{-1} \right\}$$

$$z \in \sigma_\varepsilon(P) \Longleftrightarrow \exists Q \in \mathscr{L}(\mathscr{H}),\ \|Q\| < \varepsilon,\ z \in \sigma(P + Q)$$

(instability of spectrum w.r.t. perturbations)

Toeplitz matrices Gohberg '52, Böttcher-Silbermann '97, Böttcher-Grudsky '05

Right shift $(\tau u)(n) = u(n-1)$ with symbol $\mathrm{e}^{-i\xi} = \zeta^{-1}$

$$\mathrm{Op}(p) = \sum_{\nu=-N_-}^{N_+} p_\nu \tau^\nu : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}),$$

More generally : $\|(1+|\cdot|)p_\cdot\|_{\ell^1} < \infty$.

Symbol

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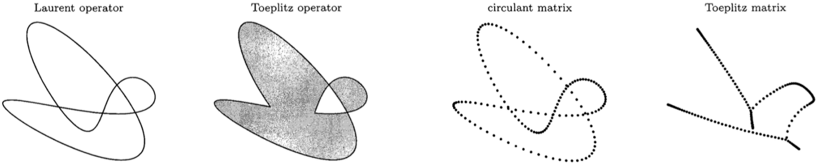
The symbol curve $p(S^1)$



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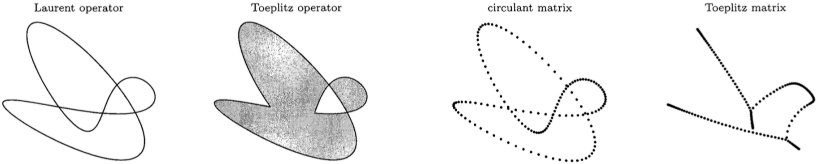


Circulant matrix: $\mathrm{Op}(p) = P_N^c : \ell^2(\mathbb{Z}/N\mathbb{Z}) \to \ell^2(\mathbb{Z}/N\mathbb{Z})$

$$\sigma(P_N^c) = p(\{\mathrm{e}^{\frac{2\pi ik}{N}}; k = 0,\dots,N-1\}), \quad u_k(n) = \frac{1}{\sqrt{N}}\exp\left(\frac{2\pi ik\cdot n}{N}\right)$$

A probabilistic Weyl law

$p(\zeta) = \sum_{\nu\in\mathbb{Z}} p_j \zeta^{-j}, \ \|(1+|\cdot|)p.\|_{\ell^1} < \infty$

Let $\Omega \Subset \mathbb{C}$ be a "nice" simply connected set with smooth boundary $\partial\Omega$

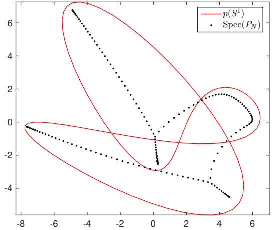
Theorem (Sjöstrand-V '16, '19, '20) Let Q_N be an iid complex Gaussian random matrix. Pick a $\delta_0 \in]0,1[$. If

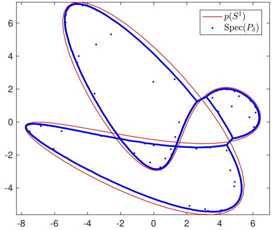
$$\mathrm{e}^{-N^{\delta_0}} \leq \delta \ll N^{-3},$$

then

$$\left| \#(\sigma(P_N + \delta Q_N) \cap \Omega) - N \int_{p(S^1)\cap\Omega} p_* \left(\frac{1}{2\pi} L_{S^1}(d\theta) \right) \right| \leq o(N),$$

with probability $\geq 1 - \mathrm{e}^{-N^{\delta_0}}$.





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- Finitely banded matrices and a general class of Q_N: $\delta = N^{-\gamma}$, $\gamma > 1/2$,

$$\mu_N = \frac{1}{N} \sum_{\lambda \in \sigma(P_N^\delta)} \delta_\lambda \rightharpoonup p_* \left(\frac{1}{2\pi} L_{S^1} \right), \quad N \to \infty, \text{ in probability},$$

Basak-Paquette-Zeitouni '18, Basak-Zeitouni '18

- For Jordan block

Śniady 02 (free probability), Bordenave-Capitaine '16
Davies-Hager '08, Sjöstrand-V '14
Guionnet-Wood-Zeitouni '14, Wood '14

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For Semiclassical Pseudo-differential operators

(RM) $P_h = hD_x + g(x)$ — Hager '06, Bordeaux-Montrieux '08

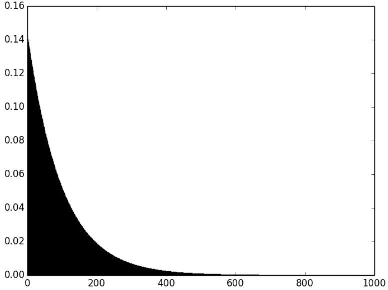
$P_h = Op_h(p)$ on $\mathbb{R}^d$ — Hager-Sjöstrand '08

Berezin-Toeplitz quantizations — Christiansen-Zworski '10, Oltman '22+, Basak-Paquette-Zeitouni '19, V '20

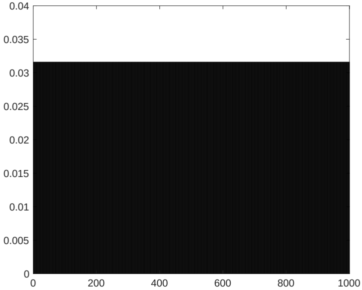
(RP) $P_h = Op_h(p)$ on $\mathbb{R}^1$ with — Hager '06

- $p(x, \xi) = p(x, -\xi)$

$P_h = Op_h(p)$ on $\mathbb{R}^d$ or compact manifold M — Sjöstrand '08, '09



Eigenvector of $P_N + N^{-\gamma} Q_N$, $\gamma > 1$



Eigenvector of P_N^c

Eigenvectors of noisy Toeplitz matrices

Singular values and singular vectors $e_1, \dots, e_M$ an ONS of eigenvectors of $(P_N - z)^*(P_N - z)$ corresp. to the eigenvalues $t_1 \leq \dots \leq t_M \leq \alpha < t_{M+1}$, with $N^{-1} \ll \alpha \ll 1$, and similarly $f_1, \dots, f_M$ for $(P_N - z)(P_N - z)^*$

Grushin Problem I

$P_N^\delta = P_N + \delta Q_N$, $\delta = N^{-\gamma}$, $\gamma > 1$

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Grushin problem Grushin '70, Sjöstrand '73, Hager-Sjöstrand '08

$$\mathcal{P}(z) = \begin{pmatrix} P_N - z & R_- \\ R_+ & 0 \end{pmatrix} : L^2(\mathbb{R}^d) \times \mathbb{C}^M \to L^2(\mathbb{R}^d) \times \mathbb{C}^M, \quad \mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix},$$

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For $\delta \|Q_\omega E\| \ll 1$ it is still invertible

$$\mathcal{P}_N^\delta(z) = \begin{pmatrix} P_N + \delta Q_N - z & R_- \\ R_+ & 0 \end{pmatrix}, \quad \mathcal{E}^\delta(z) = \begin{pmatrix} E^\delta(z) & E_+^\delta(z) \\ E_-^\delta(z) & E_{-+}^\delta(z) \end{pmatrix}, \text{ w.h.p.}$$

Grushin Problem II

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Eigenvectors as a random linear combination of singular vectors

$$E_+^{\delta}(z) : \mathcal{N}(E_{-+}^{\delta}(z)) \longrightarrow \mathcal{N}(P_N^{\delta} - z), \quad z \in \mathbb{C} \quad \text{(bijection)}.$$

Provided that all moments of the entries of Q_ω are finite,

$$E_+^{\delta}(z) = \underbrace{E_+^0(z)}_{(E_+^0 v_+)(j) = v_+(j) e_j} - \underbrace{(1 + \delta Q_\omega E^0(z))^{-1} \delta Q_\omega E^0(z) E_+^0(z)}_{= o_{N\to\infty}(1) \text{ w.h.p.}}.$$

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If $(P_N^\delta - \widehat{z})\widehat{\psi} = 0$, then w.h.p.

$$\widehat{\psi} \sim \sum_{j=1}^M \psi_j e_j(\widehat{z}), \qquad \underbrace{E_{-+}^\delta(\widehat{z})}_{\sim E_{-+}^0 - \delta E_- Q_\omega^0 E_+^0} \psi = 0.$$

Grushin Problem II

$P_N^\delta = P_N + \delta Q_N$, $\delta = N^{-\gamma}$, $\gamma > 1$

$$\mathcal{P}_N^\delta(z) = \begin{pmatrix} P_N^\delta - z & R_- \\ R_+ & 0 \end{pmatrix}, \quad \mathcal{E}^\delta(z) = \begin{pmatrix} E^\delta(z) & E_+^\delta(z) \\ E_-^\delta(z) & E_{-+}^\delta(z) \end{pmatrix}, \text{ w.h.p.}$$

with $(P_N - z)^*(P_N - z)e_j = t_j^2$, and $t_1 \leq \cdots \leq t_M \leq \alpha < t_{M+1}$, and

$$R_- u_- = \sum_1^M u_-(k) f_k, \quad (R_+ u)(k) = (u|e_k), \quad k = 1, \ldots, M$$

Eigenvectors as a random linear combination of singular vectors

$$E_+^\delta(z) : \mathcal{N}(E_{-+}^\delta(z)) \longrightarrow \mathcal{N}(P_N^\delta - z), \quad z \in \mathbb{C} \quad \text{(bijection)}.$$

Provided that all moments of the entries of Q_ω are finite,

$$E_+^\delta(z) = \underbrace{E_+^0(z)}_{(E_+^0 v_+)(j) = v_+(j) e_j} - \underbrace{(1 + \delta Q_\omega E^0(z))^{-1} \delta Q_\omega E^0(z) E_+^0(z)}_{= o_{N\to\infty}(1) \text{ w.h.p.}}.$$

If $(P_N^\delta - \widehat{z})\widehat{\psi} = 0$, then w.h.p.

$$\widehat{\psi} \sim \sum_{j=1}^M \psi_j e_j(\widehat{z}), \qquad \underbrace{E_{-+}^\delta(\widehat{z})}_{\sim E_{-+}^0 - \delta E_- Q_\omega^0 E_+^0} \psi = 0.$$

Net argument replace $\widehat{z}$ by a deterministic z close by $|z - \widehat{z}| \leq N^{-\alpha_1}$.

Grushin Problem III

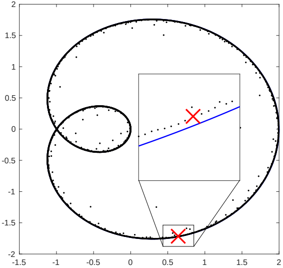
$\sup_{w\in\mathbb{C}} \mathbb{P}(|Q_{i,j}-w| \leq \varepsilon) \leq C\varepsilon^{1+\eta}$, $\eta \in]0,1]$

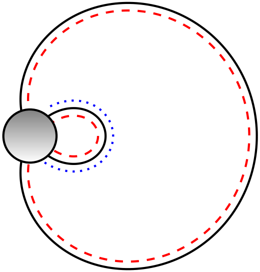
Local distribution of eigenvalues

$$\Omega(\varepsilon, N) := \{z \in \mathbb{C} : \ d(z, \mathcal{G}_{p,\varepsilon}) \asymp \log N/N, \ \mathrm{ind}_{p(S^1)}(z) \neq 0\},$$

For any $\mu > 0$ there exist $0 < \varepsilon < \infty$ (depending on γ, μ and p only) so that

$$\mathbb{P}\big(\#(\mathrm{Spec}(P_N^\delta) \cap \Omega(\varepsilon, N)) < (1-\mu)N\big) \to_{N\to\infty} 0.$$





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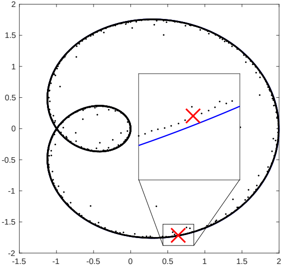

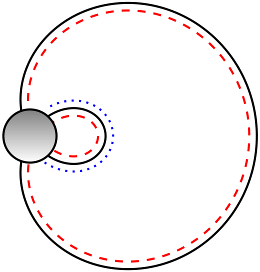

Approximate the singular vectors by quasimodes

$$\|(P_N-z)\phi\| = O(N^{-(1+C_\gamma)}\log N), \quad N\to\infty$$

For Jordan block: $\phi=(1,z,z^2,z^3,\dots,z^{N-1})$ (essentially). More generally:
Exponential states involving powers of roots of $p(\zeta)=z$

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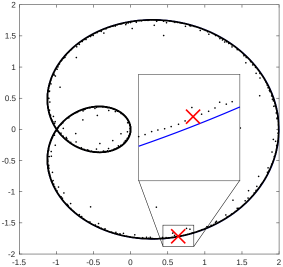

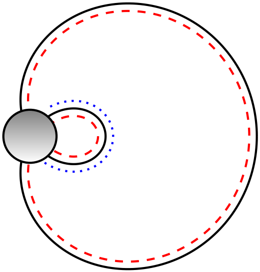

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Control of small singular values – splitting phenomenon

$$0\leq t_1\leq\cdots\leq t_d\leq N^{-(1+C_\gamma)}\log N\leq cN^{-1}\log N\leq t_{d+1},\qquad d=|\mathrm{ind}_{p(S^1)}(z)|$$

Eigenvector localization

$P_N^\delta = P_N + \delta Q_N$, $\delta = N^{-\gamma}$, $\gamma > 1$

Theorem (Basak-V-Zeitouni '21)

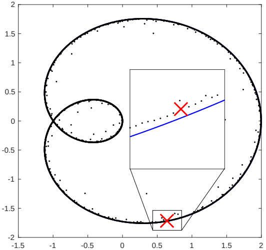
For $\widehat{z} \in \Omega(\varepsilon, C, N)$ eigenvalue of P_N^δ with eigenvector $v = v(\widehat{z})$, $\|v\|_2 = 1$.

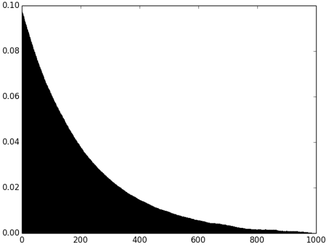
- With probability $\to 1$ as $N \to \infty$: for any $\ell \in [1, N]$,

$$\begin{aligned} \|v\|_{\ell^2([\ell,N])} &\leq c^{-1}\mathrm{e}^{-c\ell \log N/N}, & \text{if } \mathrm{ind}_{p(S^1)}(\widehat{z}) > 0, \\ \|v\|_{\ell^2([1,N-\ell])} &\leq c^{-1}\mathrm{e}^{-c\ell \log N/N}, & \text{if } \mathrm{ind}_{p(S^1)}(\widehat{z}) < 0. \end{aligned}$$

- Fix $\eta > 0$ small. Then, $\exists c_1 > 0$ so that, with probability $\geq 1 - \eta$, for every $\widehat{z} \in D(z_0, C_0 \log N/N)$, any $0 < \ell \leq \ell' \leq \widetilde{C}_0 N/\log N$ satisfying $\ell' - \ell > N^{1/2+}$ and all large N,

$$\begin{aligned} \|v\|_{\ell^2([\ell,\ell'])} &\geq c_1(\ell' - \ell)\log N/N, & \text{if } \mathrm{ind}_{p(S^1)}(\widehat{z}) > 0, \\ \|v\|_{\ell^2([N-\ell',N-\ell])} &\geq c_1(\ell' - \ell)\log N/N, & \text{if } \mathrm{ind}_{p(S^1)}(\widehat{z}) < 0. \end{aligned}$$





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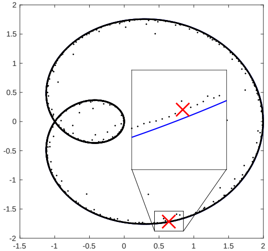

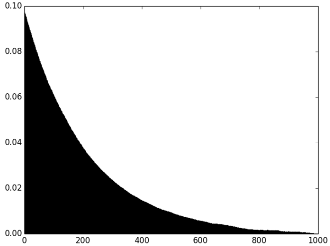

Thank you for your attention!