A direct product theorem for quantum communication with applications to device-independent QKD

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$\Rightarrow$ success probability $p^{\Omega(n)}$ for n instances

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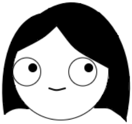
Known distribution μ on $\mathcal{X} \times \mathcal{Y}$



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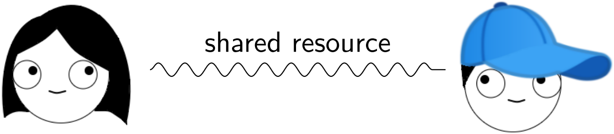

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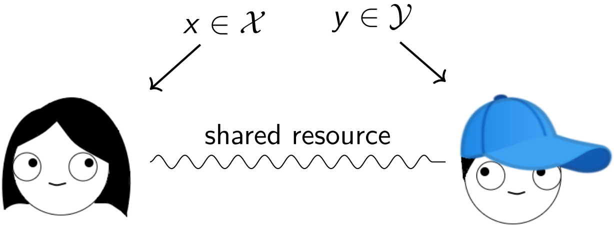

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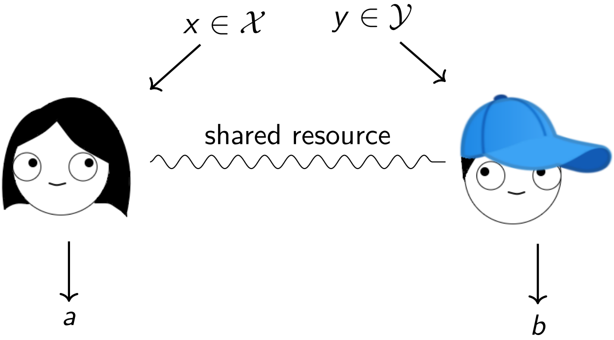

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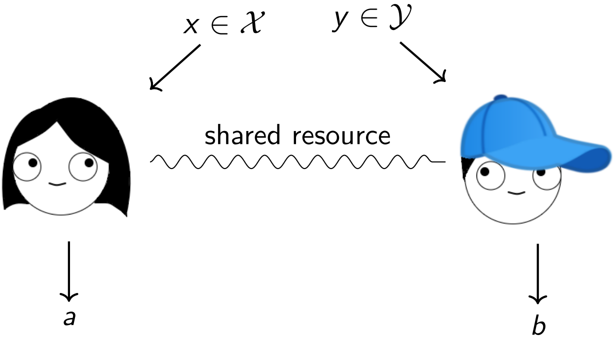

$$\Pr[\text{Success}] = \Pr_{\mu} \Pr_{\mathcal{S}}[\mathsf{V}(x, y, a, b) = 1]$$

$$\omega^*(G) = \sup_{\mathcal{S}} \Pr[\text{Success on } \mathcal{S}] \qquad (\text{resp. } \omega(G))$$

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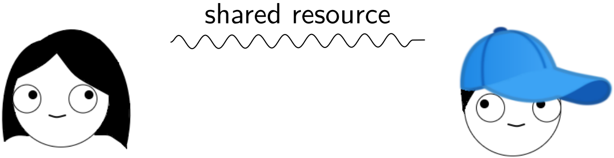

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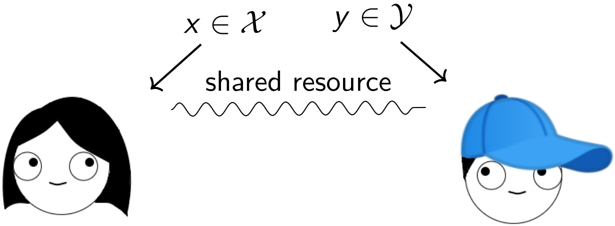

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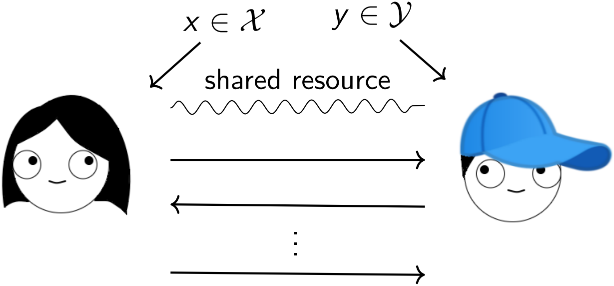

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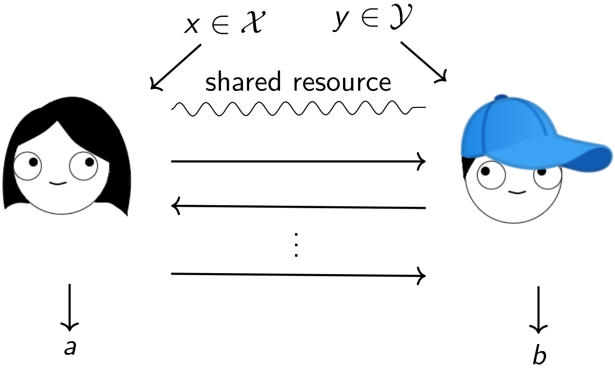

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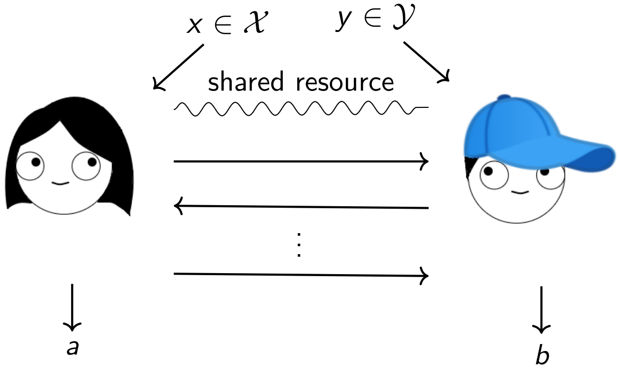

$\mathrm{QCC}(\mathcal{P}) =$ number of qubits communicated (resp. $\mathrm{CC}(\mathcal{P})$)

$$\Pr[\text{Success on } \mathcal{P}] = \min_{x,y} \Pr_{\mathcal{P}}[\mathrm{V}(x, y, a, b) = 1]$$

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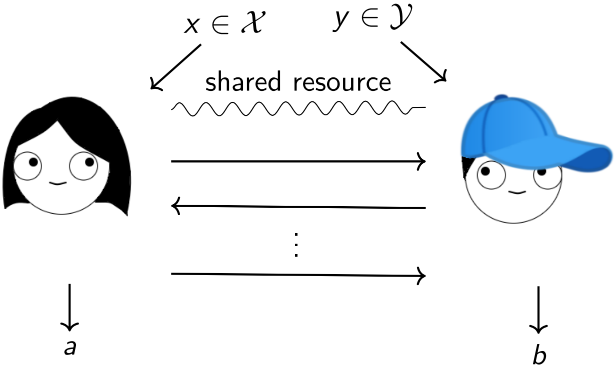

$$\Pr_\mu[\text{Success}] = \Pr_\mu \Pr_{\mathcal{S}}[\mathsf{V}(x, y, a, b) = 1]$$

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Yao's Lemma: $\mathsf{C}^{\mathsf{pub}}_\varepsilon(\mathsf{V}) = \max_\mu \mathsf{C}^{\mathsf{pub}}_\varepsilon(\mathsf{V}, \mu)$

Direct product for communication

- Direct product question: If c communication has success probability p on V, does $o(cn)$ communication have success probability $p^{\Omega(n)}$ on V^n?
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    - Lower bound for 2-party communication complexity of boolean functions

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$$\mathrm{suc}(\mathsf{V}^n, \mu^n, \mathcal{P}) \leq \left(1 - \frac{\nu}{2} + \sqrt{2\ell c}\right)^{\Omega(\nu^2 n/\ell^2)}$$
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2. If $1 \leq c = O(\varepsilon^2 \cdot \log \mathsf{eff}^*_{2\varepsilon}(\mathsf{V}, \mu)/\ell^3)$, then
$$\mathsf{suc}(\mathsf{V}^n, \mu^n, \mathcal{P}) \leq (1 - \varepsilon)^{\Omega(n)}$$
where $\mathsf{eff}^*(\mathsf{V}, \mu) =$ (relaxed) quantum partition bound or efficiency.

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With devices compatible with n copies of MS, it is possible to extract $\Omega(n)$ bits of key even in the presence of cn communication.

- Case 2. $\Rightarrow$ Direct product theorem in terms of $\max_{\text{product } \mu} \log \text{eff}^*(V, \mu)$
  - Not directly comparable to Sherstov's result for 2-party boolean functions
  - Works for more than 2 parties, non-boolean functions and predicates
  - Direct product theorem for generalized the inner-product function:

$$\mathrm{IP}_q^n = \sum_{i=1}^{n} x_i y_i \mod q.$$

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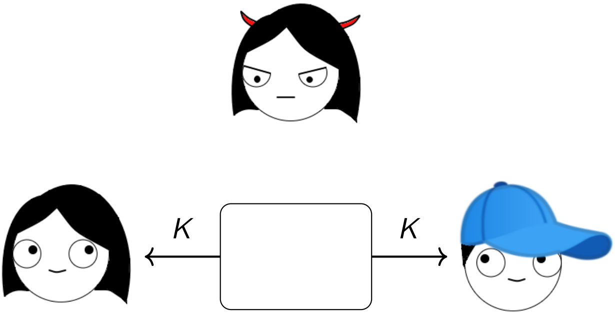

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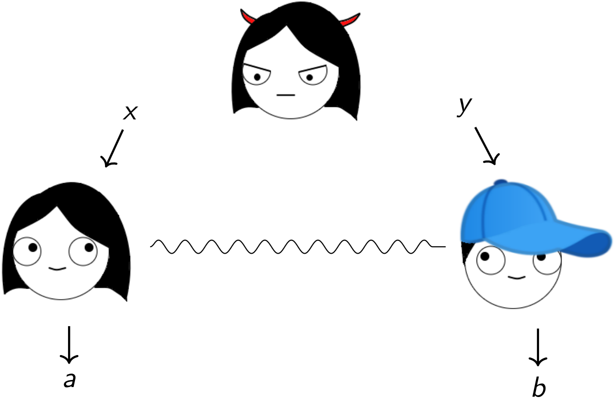

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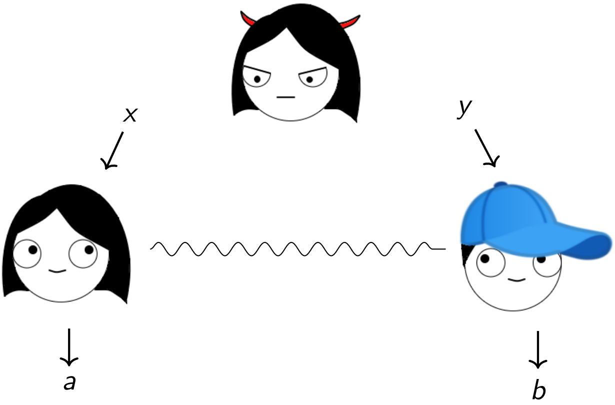

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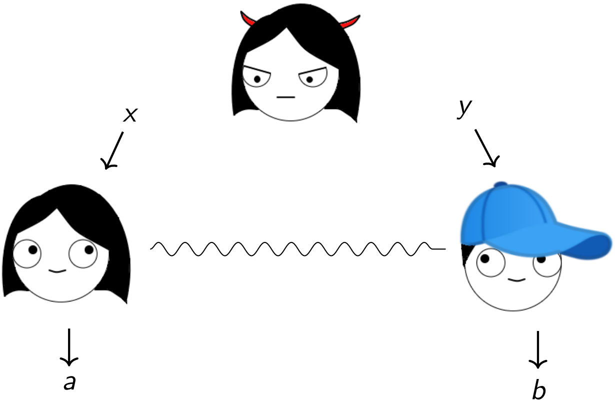

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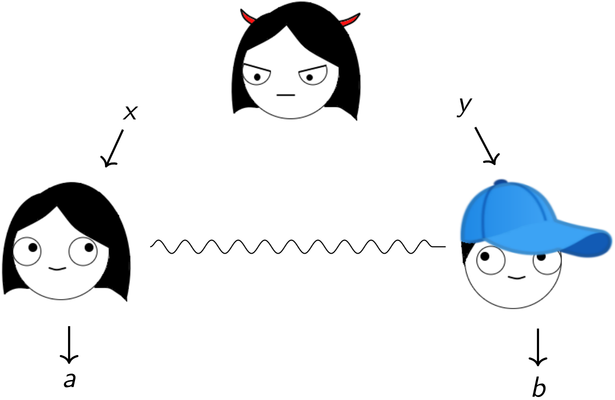

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- Using non-local games for security analysis requires no communication

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- Fully interactive leakage of cn qubits
  - Scenario modelled by communication complexity rather than non-local game
  - Case 1. of main theorem applies
  - Key rate with leakage = key rate without leakage $-O(\sqrt{c}n)$

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  2. Pr[Success in i|Success in S] $> 1 - \varepsilon \Rightarrow$ zero-communication protocol for V with efficiency $< \mathrm{eff}^*_\varepsilon(\mathsf{V}, \mu)$ and error probability ε Contradiction!

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Define "useful" state $|\varphi\rangle$:

- Contains registers $X_{\overline{S}} Y_{\overline{S}} A_{\overline{S}} B_{\overline{S}}$ in superposition
- $|\varphi\rangle_{x_i}, |\varphi\rangle_{y_i}, |\varphi\rangle_{x_i y_i}$: states obtained on measuring $X_i, Y_i, X_i Y_i$ registers
- Distribution $\mathrm{P}_{\hat{A}_i \hat{B}_i | X_i Y_i}$ on measuring $A_i B_i$ on $|\varphi\rangle_{x_i y_i}$:

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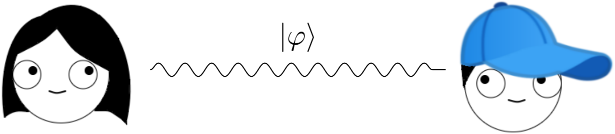

Proof idea: parallel repetition for product games

Define "useful" state $|\varphi\rangle$:

- Contains registers $X_{\overline{S}} Y_{\overline{S}} A_{\overline{S}} B_{\overline{S}}$ in superposition
- $|\varphi\rangle_{x_i}, |\varphi\rangle_{y_i}, |\varphi\rangle_{x_i y_i}$: states obtained on measuring $X_i, Y_i, X_i Y_i$ registers
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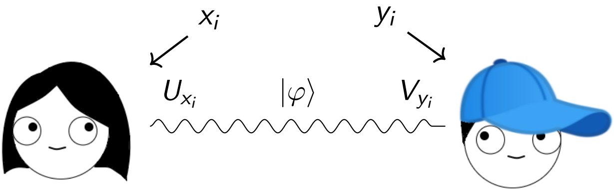

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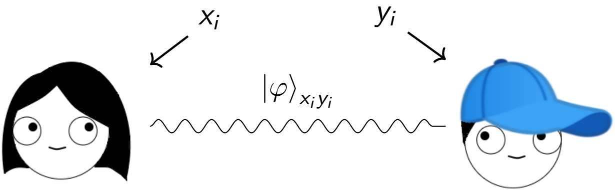

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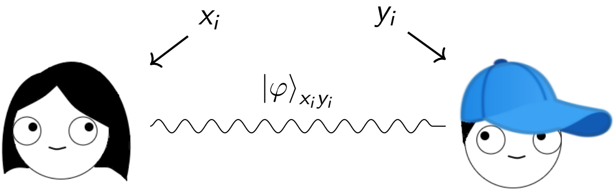

Proof also works with small communication! $(c < 1)$

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[J., Radhakrishnan, Sen 03]; [J. Nayak 12]: $\mathrm{I}(X : B)_\varphi \leq c \Rightarrow$

- $\exists \varphi'_{XB} \approx_\delta \varphi_{XB}$ s.t. $\varphi'_{XB} \leq 2^{O(c)}(\varphi_X \otimes \varphi_B)$
- $\forall X = x$, $\exists \Pi_x$ acting on A s.t. $\| \Pi_x |\varphi\rangle_{AB} \|_2^2 = 2^{-O(c)}$ and $2^{O(c)} \Pi_x |\varphi\rangle_{AB} = |\varphi'\rangle_{AB|x}$

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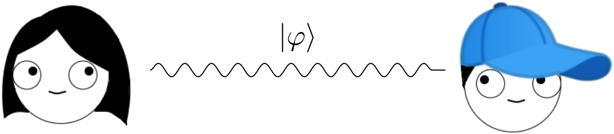

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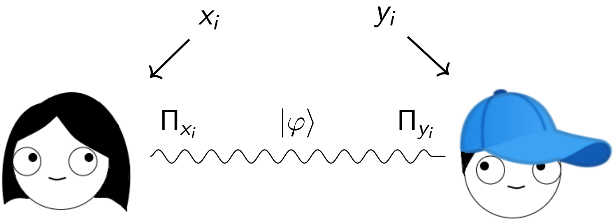

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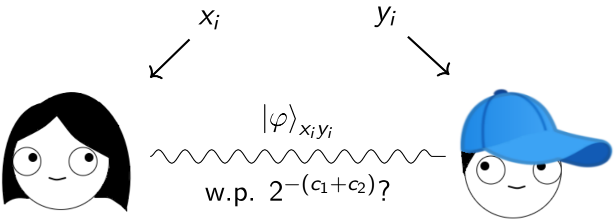

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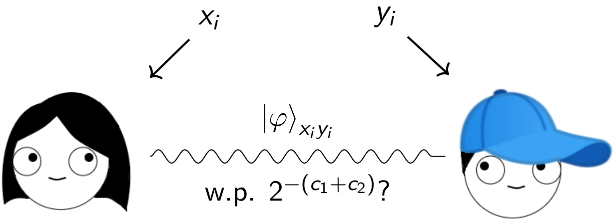

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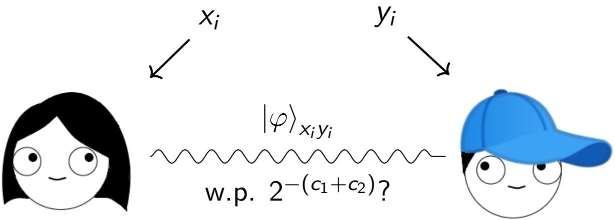

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Substate Perturbation Lemma:

$$\varphi'_{YA} \leq 2^{c}(\varphi_Y \otimes \varphi_A) \text{ and } \rho_A \approx_\delta \varphi_A$$
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Thanks for listening!