Lectures on Symmetric Functions
with a view towards Hessenberg varieties — Draft
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Introduction

In 1995, Richard Stanley associated a symmetric function with every finite graph \( \Gamma \), called the *chromatic symmetric function* of \( \Gamma \), which is a generating function of the proper colourings of \( \Gamma \).

In 2011, John Shareshian and Michelle Wachs introduced a quasisymmetric refinement of the chromatic symmetric functions and they identified a significant class of graphs whose chromatic quasisymmetric functions are actually symmetric functions. Moreover, they conjectured a link between these functions and the characters of representations of symmetric groups on the cohomology space of Hessenberg varieties.

The purpose of these lectures is to survey the (quasi)symmetric function background required to understand the constructions and how these ideas tie together. The emphasis is on “how” things work rather than “why”; we will endeavour to include pointers the literature for proofs and further reading. It is hoped that these notes will provide a road map that the interested student or researcher can follow to gain the background required to appreciate some of the beautiful ideas driving the research in this area.
2 What are (chromatic) symmetric functions?

2.1 Symmetric polynomials

Monomials We will denote a family of variables as

\[ x = (x_1, x_2, \ldots, x_n). \]

For a sequence \( \alpha = (a_1, a_2, \ldots, a_n) \) of nonnegative integers, we write

\[ x^\alpha = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}. \]

Every monomial \( x^\alpha \) admits a unique expression of the form

\[ x_{i_1}^{a_{i_1}} x_{i_2}^{a_{i_2}} \cdots x_{i_l}^{a_{i_l}} \]

with \( i_1 < \cdots < i_l \) and \( a_{i_1}, \ldots, a_{i_l} \) nonzero;

the sequence \( (a_{i_1}, \ldots, a_{i_l}) \) is called its sequence of (nonzero) exponents.

Polynomials A polynomial in the variables \( (x_1, \ldots, x_n) \) is written as

\[ f(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha = \sum_{(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n} c(a_1, a_2, \ldots, a_n) x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \]

where the \( c_\alpha \) are coefficients that all belong to the same ring.

Symmetric polynomials Let \( S_n \) denote the group of permutations of \( [n] \).

A polynomial \( f(x_1, \ldots, x_n) \) is symmetric if

\[ f(\sigma x_1, \sigma x_2, \ldots, \sigma x_n) = f(x_1, x_2, \ldots, x_n) \]

for all \( \sigma \in S_n \).

Remarks 2.

1. Since \( S_n \) is generated by adjacent transpositions \((i, i + 1)\), to determine whether a polynomial is symmetric, it suffices to check

\[ f(\ldots, x_{i+1}, x_i, \ldots) = f(\ldots, x_i, x_{i+1}, \ldots) \]

for all \( i \in [n - 1] \).

2. Permuting variables has no effect on the multiset of exponents of a monomial. It follows that

\[ \text{if } f(x_1, \ldots, x_n) \text{ is a symmetric polynomial, then the coefficient of } x^\alpha \]

and the coefficient of \( x^\beta \) are equal whenever \( \{\alpha\} = \{\beta\} \).

Algebra of symmetric polynomials The sum and product of two symmetric polynomials in \( \mathbb{C}[x_1, \ldots, x_n] \) is again a symmetric polynomial in \( \mathbb{C}[x_1, \ldots, x_n] \); so we have an algebra of symmetric polynomials:

\[ \mathbb{C}[x_1, \ldots, x_n]^S = \{ f \in \mathbb{C}[x_1, \ldots, x_n] : f \text{ is symmetric} \}. \]

Some notational conventions:

- \( \mathbb{N} = \{0, 1, 2, \ldots\} \)
- \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, \ldots\} \)
- \( x_0^0 = 1 \)
- \( x^{a_1 \cdots a_n} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \)
- \([n] = \{1, 2, \ldots, n\} \)
- \( \{a_1, \ldots, a_n\} \) denotes a multiset.

Example 1. Which of the following polynomials are symmetric?

\[ x_1^2 + x_2 x_3 \]
\[ x_1 x_2 + x_2 x_3 + x_1 x_3 \]
\[ x_1^2 + x_1 x_2 + x_2^2 \]

N.B. The answer depends on the set of variables!
2.2 Monomial symmetric polynomials

Symmetrizing monomials  One way to construct a symmetric polynomial is to start with a monomial and symmetrize:

\[
\begin{align*}
    x_1x_2^4 &\mapsto x_1x_2^4 + x_1x_2 + x_1x_3 + x_2x_3^4 + x_3x_3^4, \\
    x_1x_2^3x_3 &\mapsto x_1x_2^4x_3 + x_2x_3^4x_3 + x_1x_3^3x_2.
\end{align*}
\]

These polynomials are determined by the multiset of exponents of the starting monomial, which we write as weakly-decreasing sequences:

\[
\begin{align*}
    m_{(4,1)}(x_1, x_2, x_3) &= x_1x_2^4 + x_1x_2 + x_1x_3 + x_2x_3^4 + x_3x_3^4, \\
    m_{(4,1,1)}(x_1, x_2, x_3) &= x_1x_2^4x_3 + x_2x_3^4x_3 + x_1x_3^3x_2.
\end{align*}
\]

These are the monomial symmetric polynomials:

\[
m_{(\lambda_1, \ldots, \lambda_l)}(x_1, \ldots, x_n) = \sum_{i} x_1^{\lambda_1}x_2^{\lambda_2} \cdots x_l^{\lambda_l},
\]

the sum over all distinct monomials with exponents \(\lambda_1 \geq \ldots \geq \lambda_l\).

Monomial basis  It turns out that every symmetric polynomial is a linear combination of the monomial symmetric polynomials.

**Theorem 3.** Every symmetric polynomial \(f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]^{S_n}\) can be written uniquely as

\[
f(x_1, \ldots, x_n) = \sum_{\lambda} c_{\lambda} m_{\lambda}(x_1, \ldots, x_n)
\]

with \(c_{\lambda} \in \mathbb{Z}\), where the sum is over all weakly-decreasing sequences \((\lambda_1, \ldots, \lambda_l)\) of positive integers of length \(l \leq n\).

**Example 4.** Here is a symmetric polynomial in the variables \(x_1, x_2, x_3\).

\[
f(x_1, x_2, x_3) = 2x_1^2 - 3x_1x_2 + 2x_2^2 - 3x_1x_3 - 3x_2x_3 + 2x_3^2.
\]

Pick a monomial and subtract the corresponding symmetric polynomial:

\[
f(x_1, x_2, x_3) - (-3)m_{(1,1)}(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 2x_3^2.
\]

The result is a symmetric polynomial with fewer monomials. Repeat:

\[
f(x_1, x_2, x_3) - (-3)m_{(1,1)}(x_1, x_2, x_3) - 2m_{(2)}(x_1, x_2, x_3) = 0,
\]

and so

\[
f(x_1, x_2, x_3) = 2m_{(2)}(x_1, x_2, x_3) - 3m_{(1,1)}(x_1, x_2, x_3).
\]

Some other examples:

\[
m_3 = x_1^3 + x_2^3 + x_3^3, \quad m_{21} = x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2, \quad m_{111} = x_1x_2x_3
\]
2.3 Partitions and Compositions

A sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$$

is called

- a composition of \(d\) if \(\lambda_1 + \lambda_2 + \cdots + \lambda_l = d\), in which case we write \(\lambda \in \text{Comp}_d\) or \(\lambda \vdash d\);
- a partition of \(d\) if \(\lambda_1 + \lambda_2 + \cdots + \lambda_l = d\) and \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l\), in which case we write \(\lambda \in \text{Part}_d\) or \(\lambda \vdash d\);

We will use the following notation:

- the size or weight of \(\lambda\) is \(|\lambda| = \lambda_1 + \cdots + \lambda_l = d\);
- the integers \(\lambda_1, \ldots, \lambda_l\) are called the parts of \(\lambda\);
- the length of \(\lambda\) is its number of parts, which we denote by \(\ell(\lambda) = l\);
- the (Young) diagram associated with \(\lambda\) is a collection of boxes or cells arranged in left-justified rows, with the first row containing \(\lambda_1\) cells, the second row containing \(\lambda_2\) cells, and so on.

**Corollary 5.** The symmetric polynomials

$$\{m_\lambda(x_1, \ldots, x_n) : \lambda \in \text{Part}_d \& \ell(\lambda) \leq n\}$$

form a basis of the space of symmetric polynomials in \(x_1, \ldots, x_n\) that are homogeneous\(^1\) of degree \(d\).

**Example 6.** Here is the monomial basis of the space of symmetric polynomials in \(x_1, x_2, x_3\) that are homogeneous of degree 3:

- \(m_3 = x_1^3 + x_2^3 + x_3^3\)
- \(m_{21} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2\)
- \(m_{111} = x_1x_2x_3\)

N.B. This basis is not multiplicative:

$$m_2m_1 = m_{21} + m_3 \neq m_{21}$$

**Figure 1:** The Young diagram of the partition \(\lambda = (5, 4, 4, 1) \vdash 14\).

\(^1\) Recall that a polynomial \(f(x_1, \ldots, x_n)\) is homogeneous of degree \(d\) if the sum of the multiset of exponents of every monomial appearing in \(f\) is \(d\). That is, \(f = \sum c_\alpha x^\alpha\) and \(c_\alpha \neq 0\) implies \(|\alpha| = d\).
2.4 Elementary symmetric polynomials

The elementary symmetric polynomials appear in the expansion of a product of linear monic polynomials:

\[
(t - x_1)(t - x_2)(t - x_3) = t^3 - (x_1 + x_2 + x_3)t^2 + (x_1x_2 + x_1x_3 + x_2x_3)t - x_1x_2x_3
\]

Define the \( k \)-th elementary symmetric polynomial as

\[
e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} \cdots x_{i_k} = m_k(x_1, \ldots, x_n).
\]

These polynomials are algebraically independent, which implies that they are linearly independent and give rise to another basis of \( \mathbb{C}[x_1, \ldots, x_n]^{S_n} \).

Fundamental theorem of symmetric polynomials

**Theorem 7.** Every symmetric polynomial \( f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]^{S_n} \) can be written uniquely as

\[
f(x_1, \ldots, x_n) = \sum_{\lambda} c_{\lambda} e_\lambda(x_1, \ldots, x_n) \quad \text{with} \quad c_\lambda \in \mathbb{Z}.
\]

In other words, every symmetric polynomial is a polynomial in the elementary symmetric polynomials \( e_1, e_2, \ldots, e_n \):

\[
\mathbb{Z}[e_1, \ldots, e_n] = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}. \tag{1}
\]

**Example 8.** We have the following expansion of \( m_{(2,1)}(x_1, x_2, x_3) \):

\[
m_{(2,1)}(x_1, x_2, x_3) = e_2(x_1, x_2, x_3)e_1(x_1, x_2, x_3) - 3e_3(x_1, x_2, x_3).
\]

Thus, \( m_{(2,1)} = f(e_1, e_2, e_3) \), where \( f(y_1, y_2, y_3) = y_1y_2 - 3y_3 \).

Monomial expansion of elementary symmetric polynomials

In the expansion of \( e_2e_2 \), one finds the product

\[
\begin{pmatrix}
  x_1^0 & x_1^1 & x_1^2 & x_1^3 \\
  x_2^0 & x_2^1 & x_2^2 & x_2^3 \\
  x_3^0 & x_3^1 & x_3^2 & x_3^3
\end{pmatrix}
\begin{pmatrix}
  x_1^0 & x_1^1 & x_1^2 & x_1^3 \\
  x_2^0 & x_2^1 & x_2^2 & x_2^3 \\
  x_3^0 & x_3^1 & x_3^2 & x_3^3
\end{pmatrix}
\]

leading to \( \begin{pmatrix} 1 & 1 & 0 & 1 \ 0 & 1 & 0 & 1 \end{pmatrix} \).

These are in bijection with 0/1-matrices whose row sums are \((3, 2, 2)\).

**Proposition 9.** Let \( \lambda \in \text{Part}_n \),

\[
e_\lambda = \sum_{\mu \in \text{Part}_n} M_{\lambda, \mu} m_\mu,
\]

where \( M_{\lambda, \mu} \) is the number of matrices with entries in \( \{0, 1\} \) whose rows sum to \( \lambda \) and columns sum to \( \mu \). Consequently, the transition matrix between the bases \( \{m_\lambda : \lambda \vdash n\} \) and \( \{e_\lambda : \lambda \vdash n\} \) is symmetric.

---

Generating function. After substituting \( t \mapsto 1/t \) and multiplying by \( t^3 \), we get

\[
(1 + tx_1)(1 + tx_2)(1 + tx_3) = 1 + e_1t + e_2t^2 + e_3t^3.
\]

In general,

\[
\sum_{k \geq 0} e_k(x_1, \ldots, x_n)t^k = \prod_{i=1}^n (1 + tx_i).
\]

Some other examples:

\[
e_3 = x_1x_2x_3 \quad e_{21} = (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3) \quad e_{111} = (x_1 + x_2 + x_3)^3
\]

---

1 There are two ways to interpret the notation on the left of Equation (1), and luckily they agree here.

- \( \mathbb{Z}[e_1, \ldots, e_n] \) is the subring of \( \mathbb{Z}[x_1, \ldots, x_n] \) generated by the polynomials \( e_k(x_1, \ldots, x_n) \); the theorem states that it is \( \mathbb{Z}[x_1, \ldots, x_n]^{S_n} \).
- \( \mathbb{Z}[e_1, \ldots, e_n] \) can also be interpreted as a polynomial ring because \( \mathbb{Z}[y_1, \ldots, y_n] \rightarrow \mathbb{Z}[e_1, \ldots, e_n] \)
  
  \[
y_i \mapsto e_i(x_1, \ldots, x_n)
\]
  
  is an isomorphism of rings.

---

2 Here are the details of the expansion:

\[
m_{(2,1)}(x_1, x_2, x_3) = x_1^3 + x_1x_2^2 + x_1x_3^2 + x_2^3 + x_2x_3 + x_3^2
\]

\[
= x_1x_2(x_1 + x_3) + x_1x_3(x_1 + x_2) + x_2x_3(x_1 + x_2)
\]

\[
= x_1x_2(x_1 + x_2 + x_3) + x_1x_3(x_1 + x_2 + x_3)
\]

\[
= x_1x_2x_3(x_1 + x_2 + x_3) - 3x_1x_2x_3
\]

\[
= e_2(x_1, x_2, x_3)e_1(x_1, x_2, x_3) - 3e_3(x_1, x_2, x_3)
\]
2.5 Complete symmetric polynomials

The $k$-th complete homogeneous symmetric polynomial is the sum of all monomials of degree $k$:

$$h_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k} = \sum_{\lambda \vdash k} m_{\lambda}(x_1, \ldots, x_n).$$

These polynomials are also algebraically independent, and we define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l}.$$

**Theorem 10.** Every symmetric polynomial $f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]_{S_n}$ can be written uniquely as

$$f(x_1, \ldots, x_n) = \sum_{\lambda} c_\lambda h_\lambda(x_1, \ldots, x_n) \quad \text{with} \quad c_\lambda \in \mathbb{Z}.$$

In other words,

$$\mathbb{Z}[h_1, \ldots, h_n] = \mathbb{Z}[x_1, \ldots, x_n]_{S_n}.$$

**Monomial expansion of homogeneous symmetric polynomials**  Let $\lambda \in \text{Part}_n$.

$$h_\lambda = \sum_{\mu \in \text{Part}_n} N_{\lambda,\mu} m_\mu,$$

where $N_{\lambda,\mu}$ is the number of matrices with entries in $\mathbb{N}$ whose rows sum to $\lambda$ and columns sum to $\mu$. Consequently, the transition matrix between the bases $\{m_\lambda : \lambda \vdash n\}$ and $\{h_\lambda : \lambda \vdash n\}$ is symmetric.

**The involution $\omega$**  Since $h_k$ is a symmetric polynomial in $x_1, \ldots, x_n$, it can be expressed as a polynomial in $e_1, \ldots, e_n$. It turns out that

$$h_k = f(e_1, \ldots, e_n) \quad \text{iff} \quad e_k = f(h_1, \ldots, h_n).$$

For example,

$$h_3 = e_{111} - 2e_{21} + e_3 \quad \text{and} \quad e_3 = h_{111} - 2h_{21} + h_3.$$

**Proposition 11.** The algebra morphism defined by

$$\mathbb{Z}[e_1, \ldots, e_n] \xrightarrow{\omega} \mathbb{Z}[h_1, \ldots, h_n]$$

$$e_k \mapsto h_k$$

is an involution (that is, $\omega^2 = \text{Id}$). Consequently,

$$\mathbb{Z}[e_1, \ldots, e_n] = \mathbb{Z}[h_1, \ldots, h_n] = \mathbb{Z}[x_1, \ldots, x_n]_{S_n}.$$

**Generating function.** Like the $e_k$, the $h_k$ can also be obtained by expanding a product:

$$\sum_{k \geq 0} h_k(x_1, \ldots, x_n) t^k = \prod_{i=1}^n \frac{1}{1 - tx_i}.$$

Some other examples:

$$h_3 = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$$

$$x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_3^3$$

$$h_{21} = (x_1 + x_2 + x_3) \times (x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_3^2)$$

$$h_{111} = (x_1 + x_2 + x_3)^3$$

$^4$This can be proved by considering

$$e(t) = \sum_{k \in \mathbb{N}} e_k t^k \quad \text{and} \quad h(t) = \sum_{k \in \mathbb{N}} h_k t^k;$$

noting that $e(t) h(-t) = 1$, which gives

$$\sum_{i+j=k} (-1)^i e_i h_j = 0;$$

and deducing the result by induction.
2.6 Power sum symmetric polynomials

The $k$-th power sum symmetric polynomial is

$$p_k(x_1, \ldots, x_n) = x_1^k + x_2^k + \cdots + x_n^k = m_k(x_1, \ldots, x_n).$$

These polynomials are also algebraically independent, and we define

$$P_\lambda = p_{\lambda_1}p_{\lambda_2} \cdots p_{\lambda_l}.$$

**Theorem 12.** Every symmetric polynomial $f(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]^{S_n}$ can be written uniquely as

$$f(x_1, \ldots, x_n) = \sum \lambda c_\lambda p_\lambda(x_1, \ldots, x_n) \quad \text{with} \quad c_\lambda \in \mathbb{Q}.$$

**Remark 13.** The result does not hold over $\mathbb{Z}$ because $m_{11} = \frac{1}{2}p_{11} - \frac{1}{2}p_2$.

**Monomial expansion of power sum symmetric polynomials**

Let $\lambda \in \text{Part}_n$,

$$p_\lambda = \sum_{\mu \in \text{Part}_n} R_{\lambda, \mu} m_\mu,$$

where $R_{\lambda, \mu}$ is the number of ordered set partitions $(B_1, \ldots, B_k)$ of $[\ell(\lambda)]$ such that $\mu = \sum_{i \in B_j} \lambda_i$ for all $j \in [k]$.

**Example 14.** For instance, if $\lambda = (2, 1)$, then we consider the ordered set partitions of $[2]$:

$$\{(1, 2)\}, \{(1), (2)\}, \{(2), (1)\}$$

which result in the following sequences of partial sums:

$$\lambda_1 + \lambda_2 = (3) \quad \lambda_1, \lambda_2 = (2, 1) \quad \lambda_2, \lambda_1 = (1, 2)$$

Thus,

$$p_{21} = m_3 + m_{21}.$$

**Power sum expansion of elementary and homogeneous symmetric polynomials**

$$h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda$$

$$e_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \frac{1}{z_\lambda} p_\lambda$$

where $z_\mu = 1^{d_1} d_1! 2^{d_2} d_2! \cdots n^{d_n} d_n!$ and $d_i = \text{mult}_i(\mu)$.

**Example 15.**

$$h_1 = p_1$$

$$h_2 = \frac{1}{2}p_{11} + \frac{1}{2}p_2$$

$$h_3 = \frac{1}{6}p_{111} + \frac{1}{2}p_{21} + \frac{1}{3}p_3$$

Some other examples:

$$p_3 = x_1^3 + x_2^3 + x_3^3$$

$$p_{21} = (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3)$$

$$p_{111} = (x_1 + x_2 + x_3)^3$$

Other examples:

$$p_3 = m_3$$

$$p_{21} = m_{21} + m_3$$

$$p_{111} = 6m_{111} + 3m_{21} + m_3$$

For $\lambda = (1, 1, 1)$, there are 13 ordered set partitions of $[3]$:

$$\{(1), (2), (3)\}, \{(1), (3), (2)\}$$

$$\{(2), (1), (3)\}, \{(3), (1), (2)\}$$

$$\{(2), (3), (1)\}, \{(3), (2), (1)\}$$

$$\{(1), (2), (3)\}, \{(2), (1), (3)\}$$

$$\{(3), (1), (2)\}, \{(1), (2), (3)\}$$

$$\{(1, 3), (2)\}, \{(2, 3), (1)\}$$

$$\{(1, 2, 3)\}$$

which result in the following sequences:

$$\{(1, 1, 1)\}, \{(1, 1, 1)\}, \{(1, 1, 1)\}$$

$$\{(1, 2), (1)\}, \{(2, 1), (1)\}$$

$$\{(2, 1), (2)\}, \{(2, 1), (2)\}$$

$$\{(2, 1), (3)\}, \{(2, 1), (3)\}$$

Thus,

$$p_{111} = 6m_{111} + 3m_{21} + m_3.$$
2.7 Schur polynomials

Recurring theme: symmetric polynomials from actions of $S_n$. An eloquent way to construct symmetric polynomials is from (certain) group actions of the symmetric group. More precisely, if we have

- a set of combinatorial objects $\mathcal{O}$, a map $\alpha : \mathcal{O} \to \mathbb{N}^n$, and
- an action of $S_n$ on $\mathcal{O}$ satisfying $\alpha_i(\sigma(T)) = \alpha_{\sigma(i)}(T),$

then the following polynomial is symmetric:

$$f(x_1, \ldots, x_n) = \sum_{T \in \mathcal{O}} x^{\alpha(T)} = \sum_{T \in \mathcal{O}} x_1^{a_1(T)} x_2^{a_2(T)} \cdots x_n^{a_n(T)}.$$

We will apply this in a few different settings.

Tableaux. Let $\lambda \vdash n$.

- A (Young) tableau of shape $\lambda$ is a filling of the cells of the Young diagram of $\lambda$ by positive integers, called the entries of the tableau.

- A tableau is semistandard if its entries are weakly increasing in rows when read from left-to-right; and strictly increasing in columns when read from longest-to-shortest row.

- A tableau of shape $\lambda$ is standard if it is a semistandard whose cells are filled with 1, 2, $\ldots$, $n$ (each occurring exactly once).

- The weight or content of a tableau $T$ is the vector $(a_1, a_2, \ldots)$ with $a_i = \text{mult}(T)$, where $\text{mult}(T)$ is the number of times $i$ occurs in $T$.

Schur polynomials. The Schur polynomial indexed by $\lambda$ is

$$s_{\lambda}(x_1, \ldots, x_n) = \sum_{T \in \text{SYT}(\lambda, [n])} x^{\text{weight}(T)},$$

where $\text{SYT}(\lambda, [n])$ is the set of semistandard tableau of shape $\lambda$ and with entries in $[n]$.

For example, if $\lambda = (2, 1)$ and $n = 2$, then $\text{SYT}(\lambda, [n])$ consists of

$$\begin{array}{cc}
1 & 1 \\
2 & 2
\end{array} \quad \text{and} \quad \begin{array}{cc}
1 & 2 \\
2 & 2
\end{array}$$

so that

$$s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2 = m_{(2,1)}(x_1, x_2);$$

and if $n = 3$, then $\text{SYT}(\lambda, [n])$ consists of

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 \\
2 & 3 & 2 & 2 & 3 & 3 & 2 & 3 & 3
\end{array}$$

so that

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

$$= m_{(2,1)}(x_1, x_2, x_3) + 2m_{(1,1,1)}(x_1, x_2, x_3).$$

Figure 2: A semistandard tableau of shape $(4, 4, 2, 1)$.
Schur polynomials are symmetric

First note that it suffices to show that $s_{\lambda}(x_1, \ldots, x_n)$ is invariant under interchanging $x_i$ and $x_{i+1}$. Hence, we need a bijection on SSYT($\lambda$, $[n]$) that maps

- a tableau with $a$ occurrences of $i$ and $b$ occurrences of $i+1$ to
- a tableau with $b$ occurrences of $i$ and $a$ occurrences of $i+1;

and fixes all other elements of the tableau. This is the Bender–Knuth involution:

1. Let $T \in$ SSYT$(\lambda, [n])$. We say an occurrence of $i$ or $i+1$ in $T$ is fixed if it belongs to a column that contains both $i$ and $i+1$.

   \[
   \begin{array}{cccccccc}
   1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
   2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
   3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
   \end{array}
   \]

   2 and 3 are fixed
   2 and 3 are non-fixed

2. In each row, interchange the number of non-fixed $i$s and $(i+1)$s:

   \[
   \begin{array}{cccccccc}
   1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
   2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
   3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
   \end{array}
   \]

   row 1: one 2 and three 3s
   \[\mapsto\] three 2s and one 3
   row 2: two 2s and one 3s
   \[\mapsto\] one 2 and two 3s

Monomial expansion of Schur polynomials

Let $\lambda \in \text{Part}_n$.

\[
\begin{align*}
s_{\lambda} &= \sum_{\mu \in \text{Part}_n} K_{\lambda, \mu} m_{\mu},
\end{align*}
\]

where $K_{\lambda, \mu}$ is the number of SSYT of shape $\lambda$ and content $\mu$.

For example, SSYT((2, 1), [3]) consists of

\[
\begin{array}{cccc}
1 & 1 & 1 & 2 \\
2 & 2 & 1 & 2 \\
3 & 3 & 3 & 3 \\
\end{array}
\]

One 3 of these have content that is a partition. Hence,

\[
s_{(2, 1)} = m_{(2, 1)} + 2m_{(1, 1, 1)}.
\]

Note that this formula is independent of the number of variables.

\[
\begin{align*}
s_{(2, 1)}(x_1, x_2, x_3) &= m_{(2, 1)}(x_1, x_2, x_3) + 2m_{(1, 1, 1)}(x_1, x_2, x_3) \\
s_{(2, 1)}(x_1, x_2) &= m_{(2, 1)}(x_1, x_2) + 2m_{(1, 1, 1)}(x_1, x_2, x_3)
\end{align*}
\]
2.8 Chromatic symmetric polynomials

- A **graph** \( \Gamma = (V, E) \) consists of a finite set of vertices \( V \) and a set of edges \( E \), which are unordered pairs of vertices.

- A **colouring** of \( \Gamma \) is a function \( \kappa : V \to C \), where \( C \) is a set of colours.

- A colouring \( \kappa \) is **proper** if adjacent vertices have different colours:
  \[ \{i, j\} \in E \implies \kappa(i) \neq \kappa(j). \]

- The **chromatic symmetric function** of \( \Gamma \) is a sum of monomials, one for each proper colouring of \( \Gamma \):
  \[ X_\Gamma(x_1, \ldots, x_n) = \sum_{\text{proper colourings } \kappa : V \to [n]} \prod_{v \in V} x_{\kappa(v)}. \]

**Example 16.** For \( \Gamma \) equal to the path graph \( \circ \rightarrow \circ \rightarrow \circ \) there are
- 3! ways to colour \( \Gamma \) with 3 colours, each giving the monomial \( x_1 x_2 x_3 \);
- 2 ways to colour \( \Gamma \) with 2 colours, giving the monomials \( x_1^2 x_3 \) and \( x_1 x_2^2 \).

Thus,
\[
X_\Gamma(x_1, x_2, x_3) = 6x_1 x_2 x_3 + x_1^2 x_2 + x_1 x_2^2 + x_1^3 x_3 + x_1 x_3^2 + x_2^3 x_3 + x_2 x_3^2
= 6m(1,1,1) + m(2,1)
\]

**Proposition 17.**
1. \( X_\Gamma(x_1, \ldots, x_n) \) is a symmetric polynomial.
2. \( X_\Gamma(1, \ldots, 1) \) is equal to the number of proper \( n \)-colourings of \( \Gamma \).

**Proof.**
- The second statement follows from the observation that there is exactly one monomial for each proper \( n \)-colouring of \( \Gamma \).

- The first statement follows from the fact that if \( \sigma \in S_n \) and \( \kappa : V \to [n] \) is a proper colouring, then \( \sigma \circ \kappa \) is again a proper colouring. \( \square \)

**Example 18.** For \( \Gamma \) in Figure 3,
\[
X_\Gamma = 720 m_{11111} + 144 m_{21111} + 28 m_{2211}
= 168 s_{11111} + 60 s_{21111} + 28 s_{2211}
= 28 e_4 + 32 e_5 + 108 e_6
\]

Some numerology:
\[
28 + 32 + 108 = \# \text{ acyclic orientations of } \Gamma
\]
\[
28 + 32 = \# \text{ acyclic orientations of } \Gamma \text{ with 2 sinks}
\]
\[
108 = \# \text{ acyclic orientations of } \Gamma \text{ with 1 sink}
\]
2.9 The Stanley–Stembridge conjecture

*e-expansion and acyclic orientations* The numerology from the previous example holds in general.

**Theorem 19** (Stanley). Let $\Gamma$ be a graph and suppose the expansion of $X_\Gamma$ in the basis of elementary symmetric polynomials is

$$X_\Gamma = \sum \lambda c_\lambda e_\lambda.$$  

Then

$$\sum_{\ell(\lambda)=j} c_\lambda = \text{number of acyclic orientations of } \Gamma \text{ with exactly } j \text{ sinks}.$$  

**Example 20.** This holds even if $X_\Gamma$ is not $e$-positive. For example, the chromatic symmetric polynomial of the claw graph $K_{1,3}$ is

$$X_{K_{1,3}} = e_{(2,1,1)} - 2e_{(2,2)} + 5e_{(3,1)} + 4e_{(4)}.$$  

And $K_{1,3}$ admits: 1 acyclic orientation with exactly 3 sinks; 3 = 5 - 2 acyclic orientations with exactly 2 sinks; 6 acyclic orientations with exactly 1 sinks.

**Trees and chromatic symmetric polynomials** Given that the coefficients of the $e$-expansion of $X_\Gamma$ is related to interesting properties of the graph, one can inquire which properties of $\Gamma$ are encoded by $X_\Gamma$. One question posed by Stanley (1995) is whether a tree can be reconstructed from $X_\Gamma$. This has been verified for trees up to 29 vertices.

This is not true of every graph, since there are different (non-tree) graphs that share the same chromatic symmetric polynomial.

**The Stanley–Stembridge conjecture** It is an open problem to characterize graphs $\Gamma$ for which $X_\Gamma$ is $e$-positive. A conjecture of Stanley and Stembridge, which predates the introduction of chromatic symmetric polynomials, posits $e$-positivity for a certain graphs defined as follows.

The *incomparability graph* of a poset $P$ is the graph on $P$ with edges

$$E(\text{Inc}(P)) = \{\{i,j\} : i \text{ and } j \text{ are incomparable in } P\}.$$  

A poset is *(3 + 1)-free* if it does not contain an induced subposet isomorphic to the direct sum of a 3-chain and a 1-chain; see Figure 4.

**Conjecture 21** (Stanley–Stembridge). If $\Gamma$ is the incomparability graph of a *(3 + 1)-free* poset, then $X_\Gamma$ is $e$-positive.

A significant step towards proving this conjecture was made by Mathieu Guay-Paquet, who proved it is sufficient to prove the conjecture for posets that are both *(3 + 1)-free* and *(2 + 2)-free*. This smaller class of posets will help establish the link with Hessenberg varieties.
2.10 Symmetric polynomials versus symmetric functions

A word of caution  A word of caution is warranted at this point.

- I have deliberately restrained myself from using the terminology “symmetric function”. However, many people refer to symmetric polynomials as symmetric functions. (*But they are not the same thing!*)

- One reason for this is that symmetric functions can be viewed as formal power series in countably many variables \( \{x_1, x_2, x_3, \ldots\} \) that are invariant under permutations of the variables.

- I find this leads to confusion when one is first learning about symmetric functions, even more so once one realizes that symmetric functions aren’t really functions at all.\(^5\)

Motivation  After working a bit with symmetric polynomials, one starts to recognize that many relations between symmetric polynomials do not depend on the number of variables. For example, we saw that

\[
m(2,1)(x_1, x_2, x_3) = e_2(x_1, x_2, x_3)e_1(x_1, x_2, x_3) - 3e_3(x_1, x_2, x_3),
\]

which remains valid if we work over any number of variables:

\[
m(2,1)(x_1, \ldots, x_n) = e_2(x_1, \ldots, x_n)e_1(x_1, \ldots, x_n) - 3e_3(x_1, \ldots, x_n).
\]

We can summarize this observation by dropping any reference to the variables and writing

\[
m(2,1) = e_{(2,1)} - 3e_{(3)}.
\]

This raises the question of whether there exists a ring that formalizes this idea. We are looking for a ring \( \text{Sym} \) with the following properties:

1. \( \text{Sym} \) has various bases \( \{m_\lambda\}, \{e_\lambda\}, \text{etc.} \)—all indexed by partitions;

2. symmetric polynomials are “shadows” of elements of \( \text{Sym} \):

\[
m_\lambda \mapsto m_\lambda(x_1, \ldots, x_n);
\]

3. identities in \( \text{Sym} \) reflect identities among symmetric polynomials that are independent of the number of variables.

\(^5\) At least, not in the obvious sense.
Formal power series  Let \( \mathbb{C}[x] \) denote the \( \mathbb{C} \)-algebra of all formal power series in the variables \( x = (x_1, x_2, x_3, \ldots) \). The elements of \( \mathbb{C}[x] \) are (possibly infinite) linear combinations of the form

\[
\sum_{\alpha=(a_1,a_2,\ldots)}^\text{finite support} c_\alpha x_1^{a_1} x_2^{a_2} \cdots
\]

A power series \( \sum_\alpha c_\alpha x^\alpha \) is said to be

1. **homogeneous of degree \( d \)** if \( c_\alpha \neq 0 \) implies \( \deg(x^\alpha) = d \) for all \( \alpha \);
2. **of bounded degree** if there is a \( d \) such that \( c_\alpha \neq 0 \) implies \( \deg(x^\alpha) \leq d \).

Each symmetric group \( S_n \) acts on \( \mathbb{C}[x] \) by permuting \( \{x_1, \ldots, x_n\} \).

**Definition 22.** The *algebra of symmetric functions in \( x = (x_1, x_2, \ldots) \)* is

\[
\operatorname{Sym} = \operatorname{Sym}(x) = \left\{ f \in \mathbb{C}[X] : \begin{array}{l}
 f \text{ is of bounded degree,} \\
 \text{and for all } n \in \mathbb{N} \text{ we have } \\
 \sigma \cdot f = f \text{ for all } \sigma \in S_n
\end{array} \right\}.
\]

The elements of \( \operatorname{Sym} \) are called *symmetric functions*. It is a graded ring:

\[
\operatorname{Sym} = \bigoplus_{d \in \mathbb{N}} \operatorname{Sym}_d,
\]

where

\[
\operatorname{Sym}_d(x) = \{ f \in \operatorname{Sym}(x) : f \text{ is homogeneous of degree } d \}.
\]

**Monomial symmetric functions**  In analogy with monomial symmetric polynomials, define the *monomial symmetric function* \( m_\lambda \) as the sum over all distinct monomials of exponent \( \lambda \):

\[
\begin{align*}
 m_{(3)} &= x_1^3 + x_2^3 + x_3^3 + \cdots + x_{19}^3 + \cdots \ + x_3^3 + \cdots \\
 m_{(2,1)} &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \cdots + x_{19} x_{23} + x_{19} x_{23}^2 + \cdots \\
 m_{(1,1,1)} &= x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_5 + \cdots + x_{19} x_{23} x_{74} + \cdots
\end{align*}
\]

**Theorem 23.** Every symmetric function \( f \in \operatorname{Sym} \) can be written uniquely as a finite* linear combination of the form

\[
f = \sum_{\lambda \in \operatorname{Part}} c_\lambda m_\lambda,
\]

where the sum ranges over all partitions \( \lambda \). If \( f \in \operatorname{Sym}_d \), then

\[
f = \sum_{\lambda \in \operatorname{Part}_d} c_\lambda m_\lambda.
\]

*The finiteness follows from the fact that \( f \) is of bounded degree.
Other bases of symmetric functions  The other bases of Sym are defined in analogy with the other bases of symmetric polynomials.

\[ e_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} \cdots x_{i_k} = m_1^k \]
\[ h_k = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} = \sum_{\lambda \vdash k} m_\lambda \]
\[ p_k = \sum_{t} x_{i_1}^k = m_k \]
\[ s_\lambda = \sum_{T \in SSYT(\lambda)} x_{\text{weight}(T)} = \sum_{\mu \vdash |\lambda|} K_{\lambda, \mu} m_\mu \]

Relationship with symmetric polynomials  For each \( n \in \mathbb{N} \), there is an algebra morphism obtained by setting the variables \( x_{n+1}, x_{n+2}, \ldots \) to 0:

\[ \text{Sym} \xrightarrow{\text{eval}_n} \mathbb{C}[x_1, \ldots, x_n]^{S_n}. \]

For \( d \leq n \), this restricts to an isomorphism of vector spaces

\[ \text{Sym}_d \xrightarrow{\text{eval}_n} \mathbb{C}[x_1, \ldots, x_n]^{S_n}_d \]

because

- \( \{ m_\lambda : \lambda \in \text{Part}_d \} \) is a basis of Sym\(_d\), and
- \( \{ m_\lambda(x_1, \ldots, x_n) : \lambda \in \text{Part}_d \} \) is a basis\(^7\) of \( \mathbb{C}[x_1, \ldots, x_n]^{S_n}_d \).

Thus,\(^8\)

identities that are valid in \( \mathbb{C}[x_1, \ldots, x_n]^{S_n} \) for all \( n \)

are also valid in Sym.

\(^7\) Recall from Corollary 5 that the polynomials \( m_\lambda(x_1, \ldots, x_n) \) as \( \lambda \) ranges over all partitions of \( d \) of length at most \( n \) form a basis of \( \mathbb{C}[x_1, \ldots, x_n]^{S_n}_d \). Since \( d \leq n \), we have \( \ell(\lambda) \leq d \leq n \) and so the second condition is superfluous.

\(^8\) In other words, the algebra of symmetric functions is an inverse limit:

\[ \text{Sym} = \lim_{\rightarrow} \mathbb{C}[x_1, \ldots, x_n]^{S_n} \]
2.11 Symmetric functions via algebraic independence

Another way to construct the algebra of symmetric functions is based on the algebraic independence of the elementary symmetric polynomials:

\[ C[x_1, \ldots, x_n]^{S_n} = C[e_1, \ldots, e_n]. \]

One defines Sym to be a commutative ring generated by countably many indeterminates denoted \( e_1, e_2, \ldots, \)

\[ \text{Sym} = C[e_1, e_2, e_3, \ldots]. \]

For each \( n \in \mathbb{N}, \) there is an algebra morphism

\[ \text{Sym} \xrightarrow{\text{eval}_n} C[x_1, \ldots, x_n]^{S_n}, \]

\[ e_k \mapsto e_k(x_1, \ldots, x_n). \]

The other bases of Sym are defined in relation to the \( e_\lambda; \) for example,

\begin{align*}
  e_k &= m_{1^k} \\
  p_k &= m_k \\
  h_k &= \sum_{\lambda \vdash k} m_\lambda \\
  s_\lambda &= \sum_{\mu \vdash |\lambda|} K_{\lambda, \mu} m_\mu
\end{align*}

Remark 24. This is the approach used in SageMath’s implementation of Sym: one defines an algebra with a basis indexed by partitions and then other bases are defined with respect to this basis.

2.12 Symmetric functions, TL;DR

There is an abstract algebra of symmetric functions called Sym that admits a basis indexed by partitions,

\[ \{ e_\lambda : \lambda \in \text{Part} \}, \]

and projections onto the algebras of symmetric polynomials

\[ \text{Sym} \xrightarrow{\text{eval}_n} C[x_1, \ldots, x_n]^{S_n}, \]

\[ e_\lambda \mapsto e_\lambda(x_1, \ldots, x_n). \]

If an identity holds in each of the projections, then it holds in Sym:

for \( f, g \in \text{Sym}, \)

\[ f = g \iff \text{eval}_n(f) = g(x_1, \ldots, x_n) \text{ for all } n \in \mathbb{N}. \]

To verify an identity in Sym, it suffices to verify it in \( C[x_1, \ldots, x_n]^{S_n} \) for a sufficiently large \( n: \)

\[ f = g \iff f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \]

for some \( n \geq \max\{\deg(f), \deg(g)\}. \)
2.13 Some expansions

Monomial expansion of elementary symmetric functions Let \( \lambda \in \Part_n \).

\[ e_\lambda = \sum_{\mu \in \Part_n} M_{\lambda, \mu} m_\mu, \]

where \( M_{\lambda, \mu} \) is the number of matrices with entries in \( \{0, 1\} \) whose rows sum to \( \lambda \) and columns sum to \( \mu \). Consequently, the transition matrix between the bases \( \{ m_\lambda : \lambda \vdash n \} \) and \( \{ e_\lambda : \lambda \vdash n \} \) is symmetric.

Monomial expansion of homogeneous symmetric functions Let \( \lambda \in \Part_n \).

\[ h_\lambda = \sum_{\mu \in \Part_n} N_{\lambda, \mu} m_\mu, \]

where \( N_{\lambda, \mu} \) is the number of matrices with entries in \( \mathbb{N} \) whose rows sum to \( \lambda \) and columns sum to \( \mu \). Consequently, the transition matrix between the bases \( \{ m_\lambda : \lambda \vdash n \} \) and \( \{ h_\lambda : \lambda \vdash n \} \) is symmetric.

Monomial expansion of power sum symmetric functions Let \( \lambda \in \Part_n \).

\[ p_\lambda = \sum_{\mu \in \Part_n} R_{\lambda, \mu} m_\mu, \]

where \( R_{\lambda, \mu} \) is the number of ordered set partitions \( (B_1, \ldots, B_k) \) of \( [\ell(\lambda)] \) such that \( \mu_j = \sum_{i \in B_j} \lambda_i \) for all \( j \in [k] \).

Power sum expansion of elementary and homogeneous symmetric functions

\[ h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda \]

\[ e_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \frac{1}{z_\lambda} p_\lambda \]

where

\[ z_\mu = 1^{d_1} d_1! 2^{d_2} d_2! \cdots n^{d_n} d_n! \quad \text{where} \quad d_i = \text{mult}_i(\mu). \]

Monomial expansion of Schur functions Let \( \lambda \in \Part_n \).

\[ s_\lambda = \sum_{\mu \in \Part_n} K_{\lambda, \mu} m_\mu, \]

where \( K_{\lambda, \mu} \) is the number of SSYT of shape \( \lambda \) and content \( \mu \).

More generally, if \( \lambda/\nu \) is a skew partition of size \( n \), then

\[ s_{\lambda/\nu} = \sum_{\mu \in \Part_n} K_{\lambda/\nu, \mu} m_\mu, \]

where \( K_{\lambda/\nu, \mu} \) is the number of SSYT of shape \( \lambda/\nu \) and content \( \mu \).
2.14 What to read next

A very delightful introduction to generating functions that does an excellent job of (visually) presenting the underlying ideas appears in

- **Concrete Mathematics: A Foundation for Computer Science**, by Ronald Graham, Donald Knuth, and Oren Patashnik; see Chapter 7.

Afterwards, I strongly recommend

  - Chapter 4, sections 4.1–4.6, on symmetric functions.
  - Chapter 5, section 5, on chromatic symmetric functions.

Other nice introductions to symmetric functions can by found in:

- **Combinatorics: The Art of Counting**, by Bruce E. Sagan.
- **Enumerative Combinatorics, volume 2**, by Richard P. Stanley.

After having read these notes, you should have enough background to start reading Stanley’s paper in which $X_Γ$ is introduced and to begin appreciating some of the results. It contains formulas for expansions of the $X_Γ$ in other bases, the connection with acyclic orientations, and other nice properties.


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9 One does not read a paper in one sitting, but in phases. See
1. *How to Read Mathematics*, by Shai Simonson and Fernando Gouvea
3 Symmetric functions from representations of symmetric groups

3.1 Initiation to representation theory

A representation (over \( \mathbb{C} \)) of a group \( G \) is a morphism of groups
\[ \rho : G \to \text{GL}(V), \]
where \( V \) is a \( \mathbb{C} \)-vector space.

By fixing a basis of \( V \), we get a matrix representation of \( G \):
\[ \rho : G \to \text{GL}_n(\mathbb{C}). \]

**G-modules** It is important to view \( \rho \) as a linear action of \( G \) on \( V \) via
\[ g \bullet \vec{v} = \rho(g)(\vec{v}) \quad (g \in G, \vec{v} \in V) \]

With this definition, on has
\[ g \bullet (a\vec{v} + b\vec{u}) = a(g \bullet \vec{v}) + b(g \bullet \vec{u}) \quad (\vec{v}, \vec{u} \in V, g \in G, a, b \in \mathbb{C}) \]
(2)
\[ (gh) \bullet \vec{v} = g \bullet (h \bullet \vec{v}) \quad (\vec{v} \in V, g, h \in G) \]
(3)
\[ e \bullet \vec{v} = \vec{v} \quad (\vec{v} \in V) \]
(4)

A CG-module, or more simply a G-module, is a \( \mathbb{C} \)-vector space \( V \) equipped with an action of \( G \) verifying the conditions (2), (3), (4).

**From modules to representations** Let \( V \) be a G-module with basis \( B \) and action denoted by \( g \bullet \vec{v} \). Then the function \( \rho : G \to \text{GL}_n(\mathbb{K}) \)
\[ \rho(g) = [g]_B, \]
where \([g]_B\) is the matrix of \( \vec{v} \mapsto g \bullet \vec{v} \) in \( B \), is a representation of \( G \).

**Illustrating example**

- Let \( G \) be the group of isometries of the triangle in \( \mathbb{R}^3 \) whose vertices are \( \vec{e}_1, \vec{e}_2 \) and \( \vec{e}_3 \); cf. Figure 6.
- \( \mathbb{R}^3 \) is a G-module: \( g \bullet \vec{v} \) is the image of \( \vec{v} \) under the isometry \( g \).
- Each isometry is a linear transformation, so it can be represented by a matrix (with respect to the basis \( (\vec{e}_1, \vec{e}_2, \vec{e}_3) \)):
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

- Expressing these in the basis \( B = (\vec{e}_1 - \vec{e}_2, \vec{e}_2 - \vec{e}_3, \vec{e}_1 + \vec{e}_2 + \vec{e}_3) \), we get block diagonal matrices:
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

- Thus, we have a direct sum decomposition of our vector space
\[ \mathbb{R}^3 = \text{vect}\{\vec{e}_1 - \vec{e}_2, \vec{e}_2 - \vec{e}_3\} \oplus \text{vect}\{\vec{e}_1 + \vec{e}_2 + \vec{e}_3\} \]
into subspaces that are stable\(^{10}\) for the action of \( G \).
- **General objective**: find the finest such decomposition.

\(^{10}\) \( W \subseteq V \) is stable for the action of \( G \) if \( g \bullet \vec{w} \in W \) for all \( g \in G, \vec{w} \in W \).
**Maschke’s Theorem**  A module/representation \( V \) is **irreducible** if

- it is non-trivial; and
- it does not admit a proper subspace that is stable for the action of \( G \).

**Theorem 25.** Let \( G \) be a finite group.

For every complex representation \( \rho : G \to \text{GL}(V) \), there exists

- a basis \( B \) of \( V \) and
- irreducible matrix representations \( \rho_1, \ldots, \rho_k \) of \( G \)

such that, for every \( g \in G \),

\[
[rho(g)]_B = \begin{bmatrix}
\rho_1(g) & 0 & \cdots & 0 \\
0 & \rho_2(g) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \rho_k(g)
\end{bmatrix}.
\]  

(5)

**Main problems of representation theory**

- Given a group \( G \), determine all its irreducible representations.

- Given a representation, determine its irreducible subrepresentations up to isomorphism (this is its **character**).

- Given a representation \( V \), decompose it into a direct sum of irreducible subrepresentations; in other words, find a basis \( B \) with respect to which the matrices \( [rho(g)]_B \) are block diagonal as in (5).

**Characters**  The **character** of a representation \( \rho : G \to \text{GL}_n(\mathbb{C}) \) is the function \( \chi_\rho : G \to \mathbb{C} \) defined by

\[
\chi_\rho(g) = \text{trace}(\rho(g)) \quad (g \in G).
\]

Here are a few important properties of characters.

1. Representations are isomorphic iff they have the same character.

2. Characters are constant on conjugacy classes:

   
   if \( g \) and \( h \) are conjugate in \( G \), then \( \chi_\rho(g) = \chi_\rho(h) \).

3. Since conjugacy classes of \( S_n \) correspond to partitions of \( \lambda \) of \( n \), it is customary to write:

   \[
   \chi(\lambda) = \chi(w) \quad \text{for any permutation } w \text{ of cycle type } \lambda.
   \]

**The cycle type** of a permutation is the size of the cycles appearing in its (disjoint) cycle decomposition.

\[
\begin{align*}
[31524] &= (13542) \\
\text{cycletype([31524])} &= (5) \\
[52431] &= (15)(2)(34) \\
\text{cycletype([53241])} &= (2,2,1)
\end{align*}
\]
3.2 Frobenius characteristic

The Frobenius characteristic establishes a correspondence between characters of symmetric groups and symmetric functions.

Frobenius characteristic Let $V$ be a representation of $S_n$ with character $\chi_V$. The Frobenius characteristic of $V$ is

$$\text{Frob}(V) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_V(\sigma) p_{\text{cycle type}(\sigma)}.$$

By grouping together terms with the same value of cycle type, we get

$$\text{Frob}(V) = \sum_{\mu \vdash n} \chi_V(\mu) \frac{p_\mu}{z_\mu},$$

where

$$z_\mu = 1^{d_1} d_1! 2^{d_2} d_2! \cdots n^{d_n} n!$$

where $d_i$ is the size of the conjugacy class of permutations of cycle type $\mu$; equivalently, $z_\mu$ is the size of the centralizer of a permutation of cycle type $\mu$.

Example 26. Returning to our illustrating example of Figure 6, we get

$$\frac{1}{2} \left( 3p_{111} + 0p_3 + 0p_3 + p_{21} + p_{21} + p_{21} \right)$$

$$= \frac{1}{2} p_{111} + \frac{1}{2} p_{21}$$

$$= s_{21} + s_3$$

Theorem 27. Let $\text{CF}(S_n)$ be the algebra of characters of $S_n$. The Frobenius characteristic $\text{Frob} : \bigoplus S_n \text{CF}(S_n) \to \text{Sym}$ is an algebra isomorphism.

1. The Frobenius characteristic of an irreducible representation of $S_n$ is a Schur function $s_\lambda$ with $\lambda \vdash n$; and conversely.

2. If $\chi$ and $\psi$ are characters of $S_n$ and $S_m$, respectively, then

$$\text{Frob}(\chi) \text{Frob}(\psi) = \text{Frob} \left( \text{Ind}_{S_n \times S_m}^{S_{n+m}}(\chi \psi) \right).$$

3. If $\chi$ and $\psi$ are characters of $S_n$, then

$$\langle \chi, \psi \rangle_{S_n} = \langle \text{Frob}(\chi), \text{Frob}(\psi) \rangle_{\text{Sym}}.$$

Remark 28. Notably, the coefficients in the expansion of $\text{Frob}(V)$ in the Schur basis are the multiplicities of the irreducible representations appearing in a direct sum decomposition of $V$ into irreducibles.
**Graded Frobenius characteristic**  We will often work with infinite dimensional representations that break down into finite-dimensional pieces.

A vector space $V$ is **graded** if there are subspaces $V_d$ of $V$, one for each $d \in \mathbb{N}$, such that

$$V = \bigoplus_{d \in \mathbb{N}} V_d.$$  

A **graded representation** is a graded vector space $V = \bigoplus_d V_d$ equipped with an action of the group that maps each component $V_d$ to itself.

In this case, the **graded Frobenius characteristic** of $V = \bigoplus_d V_d$ is

$$\text{Frob}(V)(t) = \sum_d \text{Frob}(V_d) t^d \in \text{Sym}[t].$$

### 3.3 Example: Coinvariant algebra

Consider $\mathbb{C}[x_1, x_2, x_3]$ with an action of $S_3$ given by

$$\sigma f(x_1, x_2, x_3) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

The ideal generated by the elementary symmetric polynomials

$$\langle e_1, e_2, e_3 \rangle = \langle x_1 + x_2 + x_3, x_1x_2 + x_1x_3 + x_2x_3, x_1x_2x_3 \rangle$$

is invariant under the action of $S_3$ (it consists of symmetric functions). Thus, the quotient space is also a representation of $S_3$:

$$\mathbb{C}[x_1, x_2, x_3]/\langle e_1, e_2, e_3 \rangle.$$

A basis of this vector space is given by the **Schubert polynomials**.

**Schubert polynomials**  Schubert polynomials are a family of polynomials indexed by permutations. They can be defined recursively as follows.

1. If $w_0$ is the permutation $[n, n-1, \ldots, 2, 1]$, define

   $$S_{w_0} = x_1^{n-1}x_2^{n-2} \cdots x_{n-1}.$$  

2. If $S_w$ is defined and $w(i) > w(i+1)$, define

   $$S_{w|i} = \partial_i(S_w),$$

   where the $i$-th **divided difference operator** is defined as

   $$\partial_i(f(x_1, \ldots, x_n)) = \frac{f(x_1, \ldots, x_n) - s_i(f(x_1, \ldots, x_n))}{x_i - x_{i+1}}.$$

**Theorem 29.** The Schubert polynomials determine a vector space basis of the quotient space $\mathbb{C}[x_1, \ldots, x_n]/I_n$, where $I_n = \langle e_1, \ldots, e_n \rangle$. More precisely,

$$\{ S_w + I_n : w \in S_n \}$$

is a basis of $\mathbb{C}[x_1, \ldots, x_n]/I_n$.  

![Figure 7: Schubert polynomials for permutations of size $n = 3$.](image)
Action of $S_3$ in the Schubert basis  It turns out that the subspaces

\[
\text{vect}\{x_1^2 x_2 + I_3\} \text{ vect}\{x_1 x_2 + I_3, x_1^2 + I_3\} \\
\text{vect}\{1 + I_3\} \text{ vect}\{x_1 + I_3, x_1 + x_2 + I_3\}
\]

are stable for the action of $S_3$; cf. Figure 8.

For example,

\[s_2(x_1^2 x_2) = x_1^2 x_3,\]

and since

\[
(x_1 x_2 + x_1 x_3 + x_2 x_3) x_1 = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3,
\]

we have

\[x_1^2 x_3 + I_3 = -x_1^2 x_2 + I_3.\]

As another example,

\[s_2(x_1 x_2) = x_1 x_3,\]

and since

\[
(x_1 + x_2 + x_3) x_1 = x_1^2 + x_1 x_2 + x_1 x_3
\]

we have

\[x_1 x_3 + I_3 = -(x_1 x_2 + I_3) - (x_1^2 + I_3).\]

The following table presents the representation matrices on these subspaces.

<table>
<thead>
<tr>
<th>$S_3$</th>
<th>$\text{vect}{x_1^2 x_2}$</th>
<th>$\text{vect}{x_1 x_2, x_1^2}$</th>
<th>$\text{vect}{x_1, x_1 + x_2}$</th>
<th>$\text{vect}{1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>$\begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \ 1 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>132</td>
<td>$\begin{bmatrix} -1 \ -1 \ 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; 0 \ -1 &amp; 1 \ 1 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>213</td>
<td>$\begin{bmatrix} -1 \ 0 \ -1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 &amp; -1 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; 0 \ 1 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>231</td>
<td>$\begin{bmatrix} 1 \ 0 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; 1 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; 1 \ -1 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>312</td>
<td>$\begin{bmatrix} 1 \ -1 \ -1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; 0 \ -1 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; 0 \ -1 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>321</td>
<td>$\begin{bmatrix} 1 \ -1 \ 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; 0 \ -1 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; 0 \ -1 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

Graded Frobenius characteristic

\[
\operatorname{Frob}(C[x_1, x_2, x_3]/\langle e_1, e_2, e_3 \rangle) = s_{111}f_3 + s_{21}f_2 + s_{21}f_2 + s_3.
\]
3.4 Example: Hessenberg ring

(Big) Hessenberg ring Consider the ring whose elements are tuples of polynomials from \( C[t_1, t_2, t_3] \), one for each permutation \( w \in S_3 \). We visualize elements as

\[
\begin{array}{c}
\scriptstyle f_1^2(t_2 - t_3) \\
\scriptstyle t_2 t_3 \\
\scriptstyle 5 t_1 \\
\scriptstyle 0
\end{array}
\]

Addition and multiplication are defined component-wise.

Action of \( S_n \) If \( f(w, t_1, \ldots, t_n) \) denotes the polynomial associated with \( w \in S_n \), then for all \( v \in S_n \), we have

\[
(v \cdot f)(w, t_1, \ldots, t_n) = f(v^{-1} w, t_{v(1)}, \ldots, t_{v(n)}).
\]

In other words,

the polynomial at the vertex \( w \) in \( v \cdot f \), is obtained from the polynomial at the vertex \( v^{-1} w \) by permuting its variables according to \( v \).

Example 30. To simplify notation, let us write:

\[
t_{ij} = t_i - t_j
\]

Then

Figure 9: Action of \( S_3 \) on elements of the Hessenberg ring.
A subquotient of the Hessenberg ring

Consider the quotient ring \( R/I \), where \( R \) is the subring generated by

\[
\begin{align*}
&b_1 = 1, \\ &b_2 = 0, \\ &b_3 = 0, \\ &b_4 = 0, \\ &b_5 = 0, \\ &b_6 = 0,
\end{align*}
\]

and \( I \) is the ideal generated by

\[
\begin{align*}
&t_1 = 0, \\ &t_2 = 0, \\ &t_3 = 0, \\ &t_4 = 0, \\ &t_5 = 0, \\ &t_6 = 0
\end{align*}
\]

It turns out that \( \{b_1 + I, \ldots, b_6 + I\} \) is a vector space basis of \( R/I \).

**Action on \( b_1 \).** We begin by observing \( w \cdot b_1 = b_1 \) for all \( w \in S_3 \); thus,

\[
S_3 \text{ acts trivially on the vector space } \text{vect}\{b_1 + I\}.
\]

**Action on \( b_2 \).** We compute

\[
\begin{align*}
&t_1 t_2 t_3 = b_2 + (t_1 - t_2) b_4 + (t_2 - t_3) b_6, \\ &t_3 t_1 t_2 = b_2 + (t_1 - t_2) b_4 + (t_2 - t_3) b_6.
\end{align*}
\]

It follows that \( w \cdot (b_2 + I) = b_2 + I \) for all \( w \in S_3 \), and so

\[
S_3 \text{ acts trivially on } \text{vect}\{b_2 + I\}.
\]
Figure 11: Action of $s_i$ on $b_3, b_4, b_5, b_6$. 
Action on $b_3, \ldots, b_6$. In Figure 11, we have the action of $s_1$ and $s_2$ on $b_3, \ldots, b_6$; from which the matrices representing the action of $s_1$, $s_2$ and $s_1s_2$ on $\text{vect}\{b_3 + I, b_4 + I, b_5 + I, b_6 + I\}$ are

$$[s_1] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad [s_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [s_1s_2] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}$$

Hence, the Frobenius characteristic of this representation is

$$\frac{1}{6} (4p_{111} + (3 \times 2)p_{21} + (2 \times 1)p_3)$$

$$= \frac{2}{3}p_{111} + p_{21} + \frac{1}{3}p_3$$

$$= 2s_3 + s_{21}$$

and the graded Frobenius characteristic of $R/I$ is

$$\text{Frob}(R/I) = s_3t^2 + (2s_3 + s_{21})t + s_3.$$  

A second subquotient of the Hessenberg ring Consider instead the quotient ring $R/J$, with $R$ as above and $J$ the ideal generated by

Combining the identities from Figure 11 with

it follows that the graded Frobenius characteristic of $R/J$ is

$$\text{Frob}(R/J) = s_{111}t^2 + 2s_{21}t + s_3.$$
3.5 What to read next

For the representation theory of the symmetric group, I recommend

  - Chapter 1, Group Representations
  - Chapter 2, Representations of the Symmetric Group

For a general introduction to representation theory of finite groups, I recommend:

- Representations and Characters of Groups,
  by Gordon James and Martin Liebeck.

There are many other great resources, but these two are excellent starting points into the theory, and include many examples and exercises.
What are (chromatic) quasisymmetric functions?

Quasisymmetric functions are refinements of symmetric functions.

4.1 Quasisymmetric functions

Recall that a symmetric function can be viewed as a formal power series of bounded degree such that monomials with the same multiset of nonzero exponents have the same coefficient:

**Symmetric functions**: the coefficient of \( x^{a_1} x^{a_2} \cdots x^{a_l} \) equals the coefficient of \( x^{\beta_1} x^{\beta_2} \cdots x^{\beta_l} \) whenever \( \{a_1, \ldots, a_l\} = \{\beta_1, \ldots, \beta_l\} \).

A quasisymmetric function requires that monomials with the same sequence of nonzero exponents have the same coefficient:

**Quasisymmetric functions**: the coefficient of \( x^{\alpha_1} x^{\alpha_2} \cdots x^{\alpha_l} \) equals the coefficient of \( x^{\beta_1} x^{\beta_2} \cdots x^{\beta_l} \) whenever \( (\alpha_1, \ldots, \alpha_l) = (\beta_1, \ldots, \beta_l) \).

Monomial quasisymmetric functions Let \( \alpha \) be a composition. The monomial quasisymmetric function \( M_\alpha \) is defined as

\[
M_\alpha = \sum_{i_1 < i_2 < \cdots < i_l} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_l}^{\alpha_l}.
\]

For example,

\[
M_{(2,1)} = \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots
\]

\[
M_{(1,2)} = \sum_{i<j} x_i x_j^2 = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \cdots
\]

\[
m_{(2,1)} = M_{(2,1)} + M_{(1,2)}
\]

More generally, every \( m_\lambda \) is refined by the \( M_\alpha \):

\[
m_\lambda = \sum_{\text{rearrangements } \alpha \text{ of } \lambda} M_\alpha.
\]

Algebra structure Let \( \text{QSym} \) denote the set of quasisymmetric functions. It is closed under sums and products, thus it is an algebra.

**Proposition 31.**

\[
M_\alpha M_\beta = \sum_\gamma c_{\alpha,\beta}^{\gamma} M_\gamma,
\]

where \( c_{\alpha,\beta}^{\gamma} \) is the number of ways of writing \( \gamma \) as a shuffle sum of \( \alpha \) and \( \beta \).

**Example 32.**

\[
M_{[1,2]} M_{[1]} = M_{[1,2,1]} + M_{[1,1+2]} + M_{[1,1,2]} + M_{[1,1+2]} + M_{[1,1,2]}
\]

\[
= M_{[1,2,1]} + M_{[1,3]} + M_{[1,1,2]} + M_{[2,2]} + M_{[1,1,2]}
\]
Fundamental quasisymmetric functions

The (Gessel) fundamental quasisymmetric function $F_\alpha$ indexed by the composition $\alpha$ is

$$F_\alpha = \sum_{\beta \text{ refines } \alpha} M_\beta.$$ 

For example,

$$F_{(2,2)} = M_{(2,2)} + M_{(1,1,2)} + M_{(2,1,1)} + M_{(1,1,1,1)}$$

$$F_{(1,3)} = M_{(1,3)} + M_{(1,1,2)} + M_{(1,2,1)} + M_{(1,1,1,1)}$$

Some special cases of the fundamental quasisymmetric functions are

$$F_{(1^n)} = M_{(1^n)} = m_{(1^n)} = e_n$$

$$F_{(n)} = \sum_{\alpha \in \Comp_n} M_\alpha = \sum_{\lambda \in \Part_n} = h_n$$

The fundamental quasisymmetric functions form a basis of $\mathbb{Q}\Sym$ because

$$M_\alpha = \sum_{\beta \text{ refines } \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} F_\beta.$$ 

For example,

$$M_{(2,2)} = F_{(2,2)} - F_{(1,1,2)} - F_{(2,1,1)} + F_{(1,1,1,1)}$$

$$M_{(1,3)} = F_{(1,3)} - F_{(1,1,2)} - F_{(1,2,1)} + F_{(1,1,1,1)}$$

Fundamental quasisymmetric expansion of Schur functions

There is a bijection between compositions of $n$ and subsets of $[n-1]$:

$$\Comp_n \leftrightarrow \mathcal{P}([n-1])$$

$$(\alpha_1, \alpha_2, \ldots, \alpha_l) \overset{\text{Des}}{\rightarrow} \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{l-1}\}$$

$$(i_1, i_2 - i_1, \ldots, n - i_l) \overset{\text{comp}}{\leftarrow} \{i_1 < i_2 < \cdots < i_l\}$$

With this bijection, we can express $F_\alpha$ directly as a formal power series:

$$F_\alpha = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n, \atop i_j < i_{j+1} \text{ if } j \in \Des(\alpha)} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

**Theorem 33** (Gessel). For any partition $\lambda$,

$$s_\lambda = \sum_{T \in \SYT(\lambda)} F_{\comp(\Des(T))}$$

where $\Des(T)$ is the set of $i$ in $T$ such that $i + 1$ appears below $i$ in $T$ (not necessarily in the same column).

For example, from Figure 14,

$$s_{(3,2)} = F_{(1,2,2)} + F_{(1,3,1)} + F_{(2,2,1)} + F_{(2,3)} + F_{(3,2)}.$$
4.2 Chromatic quasisymmetric functions

Let \( \Gamma = (V, E) \) be a finite graph on the vertex set \( V = [n] \). We will consider colourings of the vertices of \( \Gamma \) by positive integers.

N.B. We require here that both the vertex set and the set of colours is ordered. Recall that a function \( \kappa : [n] \to \mathbb{N}^* \) is a proper colouring of \( \Gamma \) if adjacent vertices are assigned different colours:

\[
\{i, j\} \in E \implies \kappa(i) \neq \kappa(j).
\]

Define the ascent set and ascent statistic of \( \kappa \) as

\[
\text{Asc}_\Gamma(\kappa) = \{(i, j) \in E : i < j \& \kappa(i) < \kappa(j)\}
\]

\[
\text{asc}_\Gamma(\kappa) = |\text{Asc}_\Gamma(\kappa)|.
\]

The chromatic quasisymmetric polynomial\(^{14}\) of \( \Gamma \) is

\[
X_\Gamma(x; t) = \sum_{\kappa : [n] \to \mathbb{N}^*} t^{\text{asc}_\Gamma(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}.
\]

**Example** Let \( \Gamma \) be the path graph \( \Gamma = \begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
2 & \rightarrow & 3
\end{array} \).

- There are two ways to colour \( \Gamma \) with colours \( \{\bigcirc < \bullet\} \):
  - \( \bullet \bigcirc \bigcirc \bigcirc \) There is one ascent, giving \( tM_{(1,2)} \).
  - \( \bigcirc \bigcirc \bigcirc \bigcirc \) There is one ascent, giving \( tM_{(2,1)} \).

- There are \( 3! \) ways to colour \( \Gamma \) with colours \( \{1, 2, 3\} \), one for each \( w \in S_3 \); the number of ascents of the associated colouring is the number of ascents of \( w \). This gives \( (t^2 + 4t + 1)M_{(1,1,1)} \).

Thus,

\[
X_\Gamma(x; t) = (t^2 + 4t + 1)M_{(1,1,1)} + tM_{(1,2)} + tM_{(2,1)} = (t^2 + 4t + 1)m_{(1,1,1)} + tm_{(2,1)}
\]

**Problem** In the above example, \( X_\Gamma(x; t) \) turned out to be a symmetric function. This is not always that case.\(^{15}\) An open problem is to characterize the (ordered) graphs \( \Gamma \) for which the quasisymmetric chromatic function \( X_\Gamma(x; t) \) is symmetric.

\(^{14}\) One can think of this as either:
- a quasisymmetric function in \( x_1, x_2, \ldots \) with coefficients in \( \mathbb{C}[t] \), i.e. as an element of \( \text{QSym}_C[t] \); or
- a polynomial in \( t \) with coefficients in \( \text{QSym} \), i.e. as an element of \( \text{QSym}[t] \).

\(^{15}\) Exercise. Compute the chromatic quasisymmetric function of the graph

\[ \begin{array}{ccc}
1 & \rightarrow & 3 \\
\downarrow & & \downarrow \\
3 & \rightarrow & 2
\end{array} \]
Some other examples

Example 34. Let $\Gamma = ([n], \emptyset)$. Every $\kappa : [n] \to \mathbb{N}^*$ is a proper colouring, and $\text{asc}(\kappa) = 0$ for all $\kappa$, so

$$X_\Gamma(x; t) = \sum_{k_1, k_2, \ldots, k_n} x_{k_1} x_{k_2} \cdots x_{k_n}.$$ 

Note that the indices $k_1, \ldots, k_n$ are independent and run through $\mathbb{N}^*$; hence

$$X_\Gamma(x; t) = \left( \sum_k x_k \right)^n = e_1^n = e_{\{1^n\}}.$$ 

Example 35. Let $\Gamma$ be the complete graph on $n$ vertices. Every $\kappa : [n] \to \mathbb{N}^*$ that is a proper colouring of $\Gamma$ is an injective map, hence

$$X_\Gamma(x; t) = \sum_{\text{injective } \kappa : [n] \to \mathbb{N}^*} t^{\text{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}$$

Since $\text{asc}(\kappa)$ depends only on the relative order of $(\kappa(1), \kappa(2), \ldots, \kappa(n))$, we have in this case that

$$\text{asc}(\kappa) = \text{asc}(\text{std}(\kappa))$$

where $\text{std}(\kappa)$ is the unique permutation in $S_n$ whose elements appear in the same relative order as $(\kappa(1), \kappa(2), \ldots, \kappa(n))$. So we group together the $\kappa$ with the same value for std:

$$X_\Gamma(x; t) = \sum_{\sigma \in S_n} t^{\text{asc}(\sigma)} \sum_{\text{injective } \kappa : [n] \to \mathbb{N}^*} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}$$

Note that the monomials in the inner sum are in bijection with the set of all square-free monomials: if $x_{i_1} x_{i_2} \cdots x_{i_n}$ is square-free with $i_1 < \cdots < i_n$, then $\kappa(j) = i_{\sigma(j)}$ is a proper colouring of $\Gamma$ and $\text{std}(\kappa) = \sigma$. Thus,

$$X_\Gamma(x; t) = e_n \sum_{\sigma \in S_n} t^{\text{asc}(\sigma)} = [n]! e_n,$$

where in the last equality we use a well-known identity.$^{16}$

$^{16}$ Notation:

$$[n]! = [n][n-1][n-2]\cdots[1]$$

$$[i]_t = 1 + t + t^2 + \cdots + t^{i-1}$$
4.3 Natural unit interval orders

**Hessenberg vectors**  \( h = (h_1, \ldots, h_n) \in \mathbb{N}^n \) is a **Hessenberg vector** if

\[
    h_1 \leq h_2 \leq \cdots \leq h_n \quad \text{and} \quad i \leq h_i \leq n \text{ for all } i \in [n].
\]

Given a Hessenberg vector \( h = (h_1, \ldots, h_n) \), define a poset \( P_h \) on \([n]\) by

\[
    i <_h j \quad \text{iff} \quad h_i < j.
\]

and let \( \Gamma_h = \text{Inc}(P_h) \) be the **incomparability graph** of \( P_h \):

- \( i \) and \( j \) are connected by an edge in \( \Gamma_h \) iff
- \( i \) and \( j \) are incomparable in \( P_h \).

To simplify notation, for Hessenberg vectors \( h \), we write

\[
    X_h(x; t) = X_{\Gamma_h}(x; t).
\]

**Example**  Let \( h = (2, 3, 3) \).

- \( P_{(2,3,3)} \) is the poset on \([1, 2, 3]\) with exactly one relation \( 1 <_h 3 \).
- Its incomparability graph \( \Gamma_{(2,3,3)} \) is

\[
    \begin{array}{ccc}
    1 & 2 & 3 \\
    \end{array}
\]

- Thus,

\[
    X_{(2,3,3)}(x; t) = (t^2 + 4t + 1) n_{(1,1,1)} + tm_{(2,1)}
\]

\( X_h(x; t) \) is **symmetric**

**Theorem 36** (Shareshian–Wachs). If \( h \) is a Hessenberg vector, then the chromatic quasisymmetric function \( X_h(x; t) \) is a symmetric function.

The idea of the proof is reminiscent of the Bender–Knuth involution on semistandard tableaux.

1. Note it suffices to prove invariance under exchanging \( x_c \) and \( x_{c+1} \).
2. Fix a proper colouring \( \kappa \) of \( \Gamma_h \). It turns out that the (induced) subgraph on the vertices coloured \( c \) and \( c+1 \) is a disjoint union of paths \( v_1 - v_2 - \cdots - v_j \) with vertices satisfying \( v_1 < v_2 < \cdots < v_j \).
3. For each of these connected components \( v_1 - v_2 - \cdots - v_j \):
   - if \( j \) is even, then do nothing;
   - if \( j \) is odd, then exchange the colours \( c \) and \( c+1 \).
4. The result is a proper colouring with the same ascent statistic.

![Figure 15: The Shareshian–Wachs involution interchanges these two colourings.](image-url)
Example 37 (Shareshian–Wachs involution). Let
\[ h = (3, 3, 5, 5, 7, 7, 9, 9, 11, 11, 11). \]

A proper colouring of \( \Gamma_h \) with \{\( \bullet \) \( < \) \( \bigcirc \) \( < \) \( \bullet \)\} is

The subgraph induced by the colours \{\( \bullet \), \( \bigcirc \)\}, which we assume adjacent, is

The Shareshian–Wachs involution maps this subgraph to

and thus the image of \( \Gamma_h \) is

Note the ascent statistic of \( \Gamma_h \) and that of its image are equal (even though the ascent set changes).

4.4 Strengthenings of the Stanley–Stembridge conjecture

Conjecture 38 (Stanley–Stembridge). If \( \Gamma \) is the incomparability graph of a \((3 + 1)\)-free poset, then the chromatic symmetric function \( X_\Gamma(x) \) is e-positive.

Theorem 39 (Guay-Paquet). If \( X_h(x) \) is e-positive for all Hessenberg vectors \( h \), then the Stanley–Stembridge conjecture holds.

Conjecture 40 (Shareshian–Wachs). For every Hessenberg vector \( h \), the chromatic quasisymmetric function \( X_h(x; t) \) is e-positive.
4.5 Some expansions of chromatic quasisymmetric functions

**Schur-expansion** Gasharov proved that the chromatic symmetric function $X_\Gamma(x)$ is Schur-positive whenever $\Gamma$ is a $(3+1)$-free poset. This result was extended to chromatic quasisymmetric functions of natural unit interval orders by Shareshian–Wachs. \footnote{N.B. Incomparability graphs of natural unit interval orders are both $(3+1)$-free and $(2+2)$-free.}

Let $P$ be a poset and $\lambda$ a partition. A $P$-tableau of shape $\lambda$ is a filling of the diagram of $\lambda$ with elements of $P$ such that

1. every element of $P$ appears exactly once;
2. if $y$ appears immediately to the right of $x$, then $x <_P y$;
3. if $y$ appears immediately below $x$, then $y \not<_P x$.

For a $P$-tableau $T$ and $\Gamma = \text{Inc}(P)$, define

$$\text{inv}_\Gamma(T) = \left| \left\{ \{i,j\} \in E(\Gamma) : i < j \text{ and } i \text{ is in a row below } j \text{ in } T \right\} \right|$$

**Theorem 41** (Shareshian–Wachs, Theorem 6.3). Let $h$ be a Hessenberg vector. Recall that $\Gamma_h$ denotes the incomparability graph of the natural unit interval order $P_h$. Then

$$X_h(x;t) = \sum_{P_h\text{-tableaux }T} t^{\text{inv}_\Gamma(T)} s_{\text{shape}(T)}$$

$$= \sum_{\lambda} \left( \sum_{P_h\text{-tableaux }T \atop \text{shape}(T)=\lambda} t^{\text{inv}_\Gamma(T)} \right) s_{\lambda}.$$

**Example 42.** Let $h = (2,3,3)$. Then $P_{(2,3,3)}$ and $\Gamma_{(2,3,3)}$ are as follows.

\[
\begin{array}{c}
\text{3} \\
\text{1} \\
\text{2}
\end{array}
\quad
\begin{array}{c}
\text{3} \\
\text{2} \\
\text{1}
\end{array}
\]

The $P$-tableaux, with their inversions highlighted, are

\[
\begin{array}{ccc}
1 & 3 & 2 \\
2 & 3 & 1 \\
3 & 2 & 1
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 3 & 2 \\
2 & 1 & 3
\end{array}
\]

Thus,

$$X_{(2,3,3)}(x;t) = ts_{(2,1)} + (1 + 2t + t^2)s_{(1,1,1)}.$$
Fundamental-expansion  Chow described an expansion of $X_\Gamma(x)$ in the $F$-basis of $QSym$, which Shareshian and Wachs\(^9\) generalized to an expansion of $X_\Gamma(x; t)$.

- For a graph $\Gamma$ with vertex set $[n]$ and a permutation $\sigma \in S_n$, define
  \[
  \text{inv}_\Gamma(\sigma) = \left\{ \{u, v\} \in E(\Gamma) : u < v \text{ and } v \text{ appears before } u \text{ in } \sigma \right\}
  \]

- For a poset $P$ on $[n]$ and a permutation $\sigma \in S_n$, define
  \[
  \text{Asc}_P(\sigma) = \left\{ i \in [n-1] : i \text{ and } i+1 \text{ are incomparable in } P \right\}
  \]
  \[
  \text{ascomp}_P(\sigma) = \text{comp}_n(\text{Asc}_P(\sigma))
  \]

**Theorem 43** (Shareshian–Wachs, Theorem 3.1). Let $\Gamma$ be the incomparability graph of a poset $P$ on $[n]$.

\[
X_\Gamma(x; t) = \sum_{\sigma \in S_n} t^{\text{inv}_\Gamma(\sigma)} F_{\text{rev}(\text{ascomp}_P(\sigma))}^P
\]

**Example 44.** Let $P$ be the poset on $[3]$ whose only relation is $1 <_P 3$. Then $\Gamma = \text{Inc}(P)$ is the graph $1 - 2 - 3$ with edges $\{1, 2\}$ and $\{2, 3\}$.

<table>
<thead>
<tr>
<th>$S_3$</th>
<th>(1,3)</th>
<th>(2,3)</th>
<th>$\text{inv}_\Gamma$</th>
<th>1</th>
<th>2</th>
<th>$\text{Asc}_P$</th>
<th>$\text{ascomp}_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[123]</td>
<td>no</td>
<td>no</td>
<td>o</td>
<td>yes</td>
<td>yes</td>
<td>${1, 2}$</td>
<td>(1,1,1)</td>
</tr>
<tr>
<td>[132]</td>
<td>no</td>
<td>yes</td>
<td>1</td>
<td>yes</td>
<td>yes</td>
<td>${1, 2}$</td>
<td>(1,1,1)</td>
</tr>
<tr>
<td>[213]</td>
<td>yes</td>
<td>no</td>
<td>1</td>
<td>yes</td>
<td>no</td>
<td>${1}$</td>
<td>(1,2)</td>
</tr>
<tr>
<td>[231]</td>
<td>yes</td>
<td>no</td>
<td>1</td>
<td>yes</td>
<td>no</td>
<td>${1}$</td>
<td>(1,2)</td>
</tr>
<tr>
<td>[312]</td>
<td>no</td>
<td>yes</td>
<td>1</td>
<td>no</td>
<td>yes</td>
<td>${2}$</td>
<td>(2,1)</td>
</tr>
<tr>
<td>[321]</td>
<td>yes</td>
<td>yes</td>
<td>2</td>
<td>yes</td>
<td>yes</td>
<td>${1, 2}$</td>
<td>(1,1,1)</td>
</tr>
</tbody>
</table>

\[
X_\Gamma(x; t) = F_{(1,1,1)} + (2F_{(1,1,1)} + F_{(2,1)} + F_{(1,2)})t + F_{(1,1,1)}t^2
\]

\[
= e_{(3)} + (e_{(3)} + e_{(2,1)})t + e_{(3)}t^2
\]

**Example 45.** Let $P$ be the poset on $[3]$ whose only relation is $1 <_P 2$. Then $\Gamma = \text{Inc}(P)$ is the graph $1 - 3 - 2$ with edges $\{1, 3\}$ and $\{2, 3\}$.

<table>
<thead>
<tr>
<th>$S_3$</th>
<th>(1,3)</th>
<th>(2,3)</th>
<th>$\text{inv}_\Gamma$</th>
<th>1</th>
<th>2</th>
<th>$\text{Asc}_P$</th>
<th>$\text{ascomp}_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[123]</td>
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<td>no</td>
<td>o</td>
<td>yes</td>
<td>yes</td>
<td>${1, 2}$</td>
<td>(1,1,1)</td>
</tr>
<tr>
<td>[132]</td>
<td>no</td>
<td>yes</td>
<td>1</td>
<td>yes</td>
<td>yes</td>
<td>${1, 2}$</td>
<td>(1,1,1)</td>
</tr>
<tr>
<td>[213]</td>
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<td>yes</td>
<td>1</td>
<td>yes</td>
<td>no</td>
<td>${2}$</td>
<td>(2,1)</td>
</tr>
<tr>
<td>[231]</td>
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<td>no</td>
<td>1</td>
<td>yes</td>
<td>no</td>
<td>${1}$</td>
<td>(1,2)</td>
</tr>
<tr>
<td>[312]</td>
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<td>yes</td>
<td>2</td>
<td>yes</td>
<td>yes</td>
<td>${1, 2}$</td>
<td>(1,1,1)</td>
</tr>
<tr>
<td>[321]</td>
<td>yes</td>
<td>yes</td>
<td>2</td>
<td>yes</td>
<td>no</td>
<td>${1}$</td>
<td>(1,2)</td>
</tr>
</tbody>
</table>

\[
X_\Gamma(x; t) = (F_{(1,1,1)} + F_{(2,1)})t + 2F_{(1,1,1)}t^2 + (F_{(1,1,1)} + F_{(1,2)})t^2
\]

\[
= (1 + 2t + t^2)F_{(1,1,1)} + F_{(1,2)} + t^2F_{(2,1)}
\]

\[^9\text{Note that the Shareshian–Wachs paper uses a nonstandard definition of } F_n; \text{ see their footnote 3.}\]

If we define

\[
\text{Des}_P(\sigma) = \left\{ i \in [n-1] : \sigma(i) >_P \sigma(i+1) \right\},
\]

then

\[
\text{Asc}_P(\sigma) = [n-1] \setminus \text{Des}_P(\sigma).
\]

\[
e_{(3)} = F_{(1,1,1)}
\]

\[
e_{(2,1)} = F_{(1,1,1)} + F_{(1,2)} + F_{(2,1)}
\]

\[
e_{(1,1,1)} = F_{(1,1,1)} + 2F_{(2,1)} + 2F_{(2,1)} + F_{(3)}
\]

**Takeaway messages:**

1. $X_\Gamma(x; t)$ depends on the labelling of the vertices of $\Gamma$.
2. If $\Gamma'$ is a relabelling of $\Gamma$, it is possible that $X_{\Gamma'}(x; t)$ is symmetric and $X_\Gamma(x; t)$ is not symmetric.
5 LLT polynomials / Plethysm

5.1 (Unicellular) LLT polynomials

LLT polynomials are a family of symmetric functions introduced by A. Lascoux, B. Leclerc, and J-Y. Thibon as a means to study plethysm coefficients. There are multiple ways to index the LLT polynomials, and we will follow the definition as presented in Haglund–Haiman–Loeher, in which the LLT polynomials are indexed by skew-partitions.

Skew-partitions If λ and µ are partitions such that the diagram of λ contains the diagram of µ, then the skew-partition λ/µ consists of the cells of λ that do not belong to µ.

If a cell c lies in row i and column j, then diagonal index$^{20}$ is
\[ \text{diag}(c) = j - i. \]

Tuple of skew-tableaux A semistandard skew-tableau of shape λ/µ is a filling of the cells of the diagram of λ/µ such that the entries are weakly increasing in rows and strictly increasing in columns.

Given a tuple of semistandard skew-tableaux
\[
(\bar{T}^1, \ldots, \bar{T}^k) \in \text{SSYT}(\bar{\nu}^1) \times \text{SSYT}(\bar{\nu}^2) \times \cdots \times \text{SSYT}(\bar{\nu}^k)
\]
we say that cells $c \in \nu^i$ and $d \in \nu^j$ form an inversion of $\bar{T}$ if

- $\bar{T}(c) > \bar{T}(d)$, where $\bar{T}(c)$ denotes the entry in cell $c$; and either
- $i < j$ and $\text{diag}(c) = \text{diag}(d)$, or $i > j$ and $\text{diag}(c) = \text{diag}(d) + 1$.

Let $\text{Inv}(\bar{T})$ denote the set of inversions of $\bar{T}$, and $\text{inv}(\bar{T}) = |\text{Inv}(\bar{T})|$.

(Unicellular) LLT polynomials The LLT polynomial indexed by $\bar{\nu}$ is
\[
\text{LLT}_{\bar{\nu}}(x; t) = \sum_{\bar{T} \in \text{SSYT}(\bar{\nu})} t^{\text{inv}(\bar{T})} x^{\bar{T}}.
\]

If every $\nu^i$ in $\bar{\nu}$ is a single cell, then $\text{LLT}_{\bar{\nu}}(x; t)$ is unicellular.

Expansion in the fundamental quasisymmetric basis
\[
\text{LLT}_{\bar{\nu}}(x; t) = \sum_{\bar{T} \in \text{SYT}(\nu)} t^{\text{inv}(\bar{T})} \text{idescomp}(\bar{T})(x),
\]
where the inverse descent composition is computed from the reading word of $\bar{T}$ (which is obtained by traversing the cells of $\bar{\nu}$ in the opposite order of the one we define below).
5.2 Unicellular LLT polynomials and chromatic quasisymmetric functions

It turns out there is an intriguing relationship between unicellular LLT polynomials and chromatic quasisymmetric functions. To understand this, we begin by reformulating $\text{Inv}(\vec{T})$ in terms of the ascent statistic of a certain graph associated with $\vec{v}$.

Order on the cells of $\vec{v}$ Given a tuple of skew-partitions

$$\vec{v} = (v^1, v^2, \ldots, v^k),$$

align the diagrams on a grid so that cells with the same diagonal index lie on the same diagonal. Order the cells by scanning along diagonals from right-to-left, starting with the bottom-most diagonal:

Graph associated to $\vec{v}$ Define a graph $\Gamma_{\vec{v}}$ (cf. Figure 19 and Figure 20) on the cells of $\vec{v}$ with an edge connecting $c \in v^i$ and $d \in v^j$ whenever

- $i < j$ and $\text{diag}(c) = \text{diag}(d)$; or
- $i > j$ and $\text{diag}(c) = \text{diag}(d) + 1$.

Since these are precisely the conditions from the definition of $\text{Inv}(\vec{T})$,

$$\text{Inv}(\vec{T}) = \{\{u, v\} \in E(\Gamma_{\vec{v}}) : u < v \& \vec{T}(u) < \vec{T}(v)\} = \text{Asc}_{\vec{v}}(\vec{T}),$$

where we view $\vec{T}$ as a function from the cells of $\vec{v}$ into $\mathbb{N}^*$.

**Proposition 46.** If $\vec{v}$ is unicellular, then

$$\text{LLT}_{\vec{v}}(x; t) = \sum_{\text{all colourings } \kappa} \text{asc}(\kappa) x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(k)},$$

where $k = \ell(\vec{v})$ and $\text{asc}(\kappa) = |\text{Asc}_{\vec{v}}(\kappa)|$.

**Proof.** If $\vec{v}$ is unicellular, then $\vec{T} \in \text{SSYT}(\vec{v})$ associates a positive integer $\kappa(i)$ to the unique cell in component $i$ of $\vec{v}$; in other words, it defines a map $\kappa : [k] \to \mathbb{N}^*$, and conversely.

There is an operation on the algebra of symmetric functions that has the effect of eliminating the improper colourings in the summation.
Theorem 47. If $\bar{v}$ is unicellular, then

$$X_{\Gamma_{\bar{v}}}(x; t) = \frac{1}{(t - 1)^{\ell(\bar{v})}} \text{LLT}_{\bar{v}}[(t - 1)x; t].$$

Example 48. Consider $\bar{v} = ((1)/\emptyset, (2)/(1), (3)/(2))$. Then

$$\text{LLT}_{\bar{v}} = \frac{1}{1} \sqrt{2} \frac{1}{3} = \Gamma_{(2,3,3)}.$$

We compute $\text{LLT}_{\bar{v}}$ explicitly by working with the variables $(x_1, x_2, x_3)$. There are $3^3 = 27$ possible colourings, which we represent as words.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\text{asc}(\kappa)$</th>
<th>$\kappa$</th>
<th>$\text{asc}(\kappa)$</th>
<th>$\kappa$</th>
<th>$\text{asc}(\kappa)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>111</td>
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<td>211</td>
<td>0</td>
<td>311</td>
<td>0</td>
</tr>
<tr>
<td>112</td>
<td>1</td>
<td>212</td>
<td>1</td>
<td>312</td>
<td>1</td>
</tr>
<tr>
<td>113</td>
<td>1</td>
<td>213</td>
<td>1</td>
<td>313</td>
<td>1</td>
</tr>
<tr>
<td>121</td>
<td>1</td>
<td>221</td>
<td>0</td>
<td>321</td>
<td>0</td>
</tr>
<tr>
<td>122</td>
<td>1</td>
<td>222</td>
<td>0</td>
<td>322</td>
<td>0</td>
</tr>
<tr>
<td>123</td>
<td>2</td>
<td>223</td>
<td>1</td>
<td>323</td>
<td>1</td>
</tr>
<tr>
<td>131</td>
<td>1</td>
<td>231</td>
<td>1</td>
<td>331</td>
<td>0</td>
</tr>
<tr>
<td>132</td>
<td>1</td>
<td>232</td>
<td>1</td>
<td>332</td>
<td>0</td>
</tr>
<tr>
<td>133</td>
<td>1</td>
<td>233</td>
<td>1</td>
<td>333</td>
<td>0</td>
</tr>
</tbody>
</table>

Thus,

$$\text{LLT}_{\bar{v}}(x_1, x_2, x_3; t) = (x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1^2 x_3 + x_2 x_3^2 + x_3^3)\]

$$2(x_1^2 x_2 + x_1 x_2^2 + x_2^3 x_3 + 2x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2) + x_1 x_2 x_3 t^2$$

$$= s(3) + 2s(2,1)t + s(1,1,1)t^2$$

$$= \frac{1}{6} (t^2 + 4t + 1) p_{(1,1,1)} - \frac{1}{2} (t - 1) (t + 1) p_{(2,1)} + \frac{1}{3} (t - 1)^2 p_{(3)}.$$

The plethystic substitution satisfies

$$p_k[(t - 1)x] = (t^k - 1)p_k(x)$$

so that

$$p_{(3)}[(t - 1)x] = (t^3 - 1)p_{(3)}(x)$$

$$p_{(2,1)}[(t - 1)x] = (t^2 - 1)(t - 1)p_{(2,1)}(x)$$

$$p_{(1,1,1)}[(t - 1)x] = (t - 1)^3 p_{(1,1,1)}(x)$$

and hence

$$\frac{1}{(t - 1)^2} \text{LLT}_{\bar{v}}[(t - 1)x; t]$$

$$= \frac{1}{6} (t^2 + 4t + 1) p_{(1,1,1)} - \frac{1}{2} (t + 1)^2 p_{(2,1)} + \frac{1}{3} (t^2 + t + 1) p_{(3)}$$

$$= (t^2 + 2t + 1)s_{(1,1,1)} + ts_{(2,1)} = X_{(2,3,3)}(x; t).$$
5.3 What is plethysm?

Characters of $\text{GL}_n(\mathbb{C})$ For each partition $\lambda$ such that $\ell(\lambda) \leq n$, there exists an irreducible $\text{GL}_n(\mathbb{C})$-representation $\rho_{\lambda}$ that admits a basis$^{23}$

$$E_{\lambda} = \{ e_T : T \in \text{SSYT}(\lambda, [n]) \}$$

satisfying

$$\text{diag}(x_1, \ldots, x_n) \cdot e_T = x^T e_T.$$

Its character is trace of matrix representing this action:

$$\chi_{\lambda}(x_1, \ldots, x_n) = \sum_{T \in \text{SSYT}(\lambda, [n])} x^T = s_\lambda(x_1, \ldots, x_n).$$

Illustrating example

1. For $n = 2$ and $\lambda = (2)$,

$$\rho_{(2)} \left( \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1^2 & 0 & 0 \\ 0 & x_1 x_2 & 0 \\ 0 & 0 & x_2^2 \end{bmatrix} = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_2 \end{bmatrix}$$

whose trace is

$$x_{\begin{array}{c} u \\ v \end{array}} + x_{\begin{array}{c} u \\ v \end{array}} + x_{\begin{array}{c} u \\ v \end{array}} = \sum_{T \in \text{SSYT}((2,2))} x^T = s_{(2)}(x_1, x_2).$$

2. For $n = 3$ and $\lambda = (2, 1)$,

$$\rho_{(2,1)} \left( \begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix} \right)$$

whose trace is

$$\sum_{T \in \text{SSYT}((2,1,1))} y^T = s_{(2,1)}(y_1, y_2, y_3).$$

3. Observe that we can compose these representations and that the character of the composition $\rho_{(2,1)} \circ \rho_{(2)}$ is

$$s_{(2,1)}(x_{\begin{array}{c} u \\ v \end{array}}, x_{\begin{array}{c} u \\ v \end{array}}, x_{\begin{array}{c} u \\ v \end{array}}) = s_{(2,1)}(x_1, x_1 x_2, x_2^2)$$

$$= (s_{(3,2,1)} + s_{(4,2)} + s_{(5,1)}) (x_1, x_2).$$
Composition of GL-characters

Let

\[
\text{GL}_n(K) \xrightarrow{\rho} \text{GL}_m(K) \quad \text{and} \quad \text{GL}_m(K) \xrightarrow{\tau} \text{GL}_l(K)
\]

be two representations such that the composition \(\tau \circ \rho\) is defined.

The character \(\chi_{\tau}(\chi_{\rho})\) of \(\tau \circ \rho : \text{GL}_n(K) \rightarrow \text{GL}_l(K)\) is obtained by replacing the \(i\)-th variable of \(\chi_{\tau}(y_1, \ldots, y_m)\) by the \(i\)-th monomial in of \(\chi_{\rho}(x_1, \ldots, x_n)\).

This operation on polynomials admits an extension to \(\text{Sym}\) (i.e., so that the result is independent of the number of variables).

Motivation of the axiomatic definition

What should \(f[g]\) look like?

1. If \(\rho = \rho_1 \oplus \rho_2\), then \((\rho_1 \oplus \rho_2) \circ \tau = (\rho_1 \circ \tau) \oplus (\rho_2 \circ \tau)\), so that

\[
\chi_{(\rho_1 \oplus \rho_2) \circ \tau} = \chi_{\rho_1 \oplus \rho_2} \circ \chi_{\tau} = (\chi_{\rho_1} + \chi_{\rho_2}) \circ \chi_{\tau}
\]

So one would like

\[
(f_1 + f_2)[g] = f_1[g] + f_2[g] \quad (f_1, f_2, g \in \text{Sym})
\]

Similarly, by considering tensor products of representations,

\[
(f_1f_2)[g] = f_1[g]f_2[g] \quad (f_1, f_2, g \in \text{Sym})
\]

2. Under the projections \(\text{Sym} \rightarrow C[x_1, \ldots, x_n]^{S_n}\), we would expect

\[
p_\tau[p_\rho](x_1, \ldots, x_n) = p_\tau[x_1^r + \cdots + x_n^r] = x_1^{sr} + \cdots + x_n^{sr}
\]

\[
= p_\rho(x_1, \ldots, x_n) = p_\rho(p_\tau)(x_1, \ldots, x_n)
\]

\[
p_\tau[p_\rho + p_\tau](x_1, \ldots, x_n) = p_\tau[x_1^r + \cdots + x_n^r + x_1^t + \cdots + x_n^t]
\]

\[
= (x_1^{sr} + \cdots + x_n^{sr}) + (x_1^{tr} + \cdots + x_n^{tr})
\]

\[
= (p_\rho[p_\tau] + p_\tau[p_\rho])(x_1, \ldots, x_n)
\]

\[
p_\tau[p_\rho p_\tau](x_1, \ldots, x_n) = p_\tau[(x_1^r + \cdots + x_n^r)(x_1^t + \cdots + x_n^t)]
\]

\[
= \sum_{i,j} (x_i^ry_j^t)^r = \sum_{i} x_i^{sr} \sum_{j} y_j^{tr}
\]

\[
= (p_\rho[p_\tau]p_\tau[p_\rho])(x_1, \ldots, x_n)
\]

These conditions uniquely define an operation on \(\text{Sym}\), called plethysm.
Plethysm. There exists a unique operation on \( \text{Sym} \), called plethysm and denoted by \( f[g] \), satisfying:

1. \( f \mapsto f[g] \) is an algebra morphism for all \( g \in \text{Sym} \).
2. \( g \mapsto p_r[g] \) is an algebra morphism for all \( r \geq 1 \).
3. \( p_r[p_s] = p_{rs} \) for all \( r, s \geq 1 \).

Furthermore, this operation satisfies the following properties.

1. \( p_r[f](x_1, \ldots, x_n) = f(x_1^r, \ldots, x_n^r) \) for all \( f \in \text{Sym} \) and \( r \geq 1 \).
2. \( p_r[f] = f[p_r] \) for all \( f \in \text{Sym} \) and \( r \geq 1 \).
3. \( p_1[f] = f = f[p_1] \) for all \( f \in \text{Sym} \).
4. \( f[g[h]] = (f[g])[h] \) for all \( f, g, h \in \text{Sym} \).

5.4 Plethystic notation

Another widespread way to define plethysm makes use of the notion of an “alphabet”. The guiding principal here is that:

an expression of the form \( f[A] \) represents the image of \( f \in \text{Sym} \) under a ring morphism \( \phi_A : \text{Sym} \rightarrow R \) that is determined by an element \( A \) of the ring \( R \).

The rings and ring morphisms here turn out to be quite special.

Lambda rings. A \( \lambda \)-ring is a commutative ring \( R \) equipped with a family of operators \( \lambda^k : R \rightarrow R \), one for each \( k \in \mathbb{N} \), that verify conditions that abstract the operations of direct sum \( \oplus \), tensor product \( \otimes \), and the exterior product \( \Lambda^k \) on vector spaces. For instance, one has

\[
\lambda^k(u + v) = \sum_{i=0}^{n} \lambda^i(u)\lambda^{n-i}(v)
\]

because for vector spaces \( U \) and \( V \) one has

\[
\Lambda^k(U \oplus V) = \bigoplus_{i=0}^{n} \Lambda^i(U) \otimes \Lambda^{n-i}(V).
\]

Sym is a \( \lambda \)-ring. Sym is a \( \lambda \)-ring with operations \( \lambda^k(e_1) = e_k \), which parallels the fact that the \( \text{GL}_n(\mathbb{C}) \)-character of \( \Lambda^k \mathbb{C}^n \) is \( e_k \). Moreover,

Sym is the free \( \lambda \)-ring on one generator; it is generated by \( e_1 \).

Consequently, each element \( A \) of a \( \lambda \)-ring \( R \) determines a unique morphism of \( \lambda \)-rings \[^5\] \( \phi_A : \text{Sym} \rightarrow R \) such that

\[
\phi_A(e_1) = A \quad \text{and} \quad f[g] := \phi_g(f).
\]

Attention: the function \( g \mapsto f[g] \) is not in general an algebra map.

One also has

\[
\lambda^2(uv) = \lambda^1(u)\lambda^2(v) + \lambda^2(u)\lambda^1(v) - 2\lambda^2(u)\lambda^2(v)
\]

which parallels the vector space isomorphism

\[
[\Lambda^2(U \otimes V)] \cong [\Lambda^1(U) \otimes \Lambda^1(U) \otimes \Lambda^1(V)]
\]

\[
\cong [\Lambda^1(U) \otimes \Lambda^1(U) \otimes \Lambda^1(V)] + [\Lambda^2(U) \otimes \Lambda^1(V) \otimes \Lambda^1(V)]
\]

[^5]: A morphism of \( \lambda \)-rings is a ring morphism that commutes with the \( \lambda^k \) operations.

\( f[g] \) is characterized by \( \phi_{f[g]} = \phi_g \circ \phi_f \), which says that plethysm corresponds to the composition of \( \lambda \)-ring endomorphisms of Sym.
Example 49. Here are some common examples that one finds in the literature.

- It is customary to write $X = x_1 + x_2 + x_3 + \cdots$, where \{x_1, x_2, \ldots\} is the set of variables. In this case, $X$ is precisely $e_1$ so that $\phi = \text{Id}_{\text{Sym}}$ and
  \[
  f[X] = f[x_1 + x_2 + x_3 + \cdots] = f[e_1] = f.
  \]

- For a finite set of variables, commonly written as $X_n = x_1 + x_2 + \cdots + x_n$, one has
  \[
  f[x_1 + x_2 + \cdots + x_n] = f(x_1, x_2, \ldots, x_n).
  \]
  Here, $\phi$ is the canonical projection of $\text{Sym}$ onto $\mathbb{C}[x_1, \ldots, x_n]^{\text{Sym}}$.

- The notation $f[X + Y]$ corresponds to the coproduct \(^26\) of $f$, defined by
  \[
  f[X + Y] = \Delta(f) = f[p_1 \otimes 1 + 1 \otimes p_1],
  \]
  where $\phi = \Delta : \text{Sym} \to \text{Sym} \otimes \text{Sym}$, $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$.

- The notation $f[-X]$ is the image of $f$ under the endomorphism of $\text{Sym}$ defined by $\phi(p_k) = -p_k$ (this is the antipode of $\text{Sym}$).

The $(t-1)$-transform of $\text{Sym}$ So how does one interpret $f[(t-1)X]$?

- Firstly, $(t-1)X = (t-1)e_1$, so we are working with the $\lambda$-ring morphism that maps $e_1$ to $(t-1)e_1$.

- However, one needs to be careful because the base ring is $\mathbb{C}[t]$, which is a $\lambda$-ring via the operations
  \[
  \lambda^k(z) = \binom{z}{k} \quad \text{and} \quad \lambda^k(t) = \begin{cases} 1, & \text{if } k = 0, \\ t, & \text{if } k = 1, \\ 0, & \text{if } k \geq 2. \end{cases}
  \]

With this in mind, we compute $p_2[(t-1)X]$ as follows.

\[
\begin{align*}
p_2[(t-1)X] & = \phi_{(t-1)e_1}(p_2) \\
& = \phi_{(t-1)e_1}(e_{11}) - 2\phi_{(t-1)e_1}(e_2) \\
& = \phi_{(t-1)e_1}(e_{11}) - 2\phi_{(t-1)e_1}(\lambda^2(e_1)) \\
& = \phi_{(t-1)e_1}(e_{11})^2 - 2\lambda^2(\phi_{(t-1)e_1}(e_1)) \\
& = (t-1)^2e_{11} - 2\lambda^2((t-1)e_1) \\
& = (t-1)^2e_{11} - 2((t-1)^2e_2 + \lambda^2(t-1)e_{11} - 2\lambda^2(t-1)e_2) \\
& = (t-1)^2e_{11} - 2((t-1)^2e_2 - (t-1)e_{11} + 2(t-1)e_2) \\
& = (t-1)^2e_{11} - 2((t-1)^2e_2 - (t-1)e_{11}) \\
& = (t^2 - 1)e_{11} - 2(t^2 - 1)e_2 \\
& = (t^2 - 1)(e_{11} - 2e_2) \\
& = (t^2 - 1)p_2.
\end{align*}
\]

\(^{26}\)Sym is a Hopf algebra, so admits a product, coproduct, antipode, etc.

\[
\begin{align*}
p_2 & = e_{11} - 2e_2 \\
\lambda^2(uv) & = \lambda^1(u)\lambda^2(v) + \lambda^2(u)\lambda^1(v) - 2\lambda^2(u)\lambda^2(v) \\
\lambda^2(t-1) & = (1)(2^2) + (1)(-1) + (0)(1) \\
& = 1 - t.
\end{align*}
\]
**λ-anneaux** Voici la définition complète de λ-anneaux.

**Définition 50.** Un **λ-anneau** est un anneau unitaire commutatif $R$ muni d’une famille d’opérateurs $\lambda^k : R \to R$, un pour chaque $k \in \mathbb{N}$, qui vérifient les axiomes suivants pour tous $x, y \in R$:

1. $\lambda^0(x) = 1$
2. $\lambda^1(x) = x$
3. $\lambda^0(1) = 0$ pour $n \geq 2$
4. $\lambda^u(x + y) = \sum_{i=0}^{n} \lambda^i(x) \lambda^{n-i}(y)$
5. $\lambda^u(xy) = P_n(\lambda^1(x), \lambda^2(x), \ldots, \lambda^u(x); \lambda^1(y), \lambda^2(y), \ldots, \lambda^u(y))$
6. $\lambda^u(\Lambda_m(x)) = P_{n,m}(\lambda^1(x), \lambda^2(x), \ldots, \lambda^u(x))$

où $P_n$ et $P_{n,m}$ sont les polynômes définis par

$$\sum_{n \geq 0} P_n(e_1(x), \ldots, e_n(x)) t^n = \prod_{i \geq 1} (1 + x_i t_i)$$

$$\sum_{n \geq 0} P_{n,m}(e_1(x), e_2(x), \ldots, e_m(x)) t^n = \prod_{i_1 < i_2 < \cdots < i_m} (1 + x_{i_1} x_{i_2} \cdots x_{i_m})$$

**Exemple 51.** Voici certains des polynômes $P_n$ et $P_{n,m}$:

$$P_1(x_1; y_1) = x_1 y_1$$

$$P_2(x_1, x_2; y_1, y_2) = x_1^2 y_1^2 + x_2 y_1^2 - 2x_1 y_2$$

**Exemple 52 (λ-anneaux).**  

- L’anneau des entiers $\mathbb{Z}$ est un λ-anneau, où

$$\lambda^k(n) = \binom{n}{k}.$$ 

- L’anneau de polynômes symétriques à $n$ variables et à coefficients dans $\mathbb{Z}$ est un λ-anneau, où

$$\lambda^k(f) = e_k[f] \quad \text{et} \quad \lambda^k(m) = \binom{m}{k}$$

pour tout polynôme symétrique $f$ et tout $m \in \mathbb{Z}$.

- L’anneau de fonctions symétriques $\Lambda_\mathbb{Z}$ est un λ-anneau, où

$$\lambda^k(f) = e_k[f] \quad \text{et} \quad \lambda^k(n) = \binom{n}{k}$$

pour tout $f \in \Lambda_\mathbb{Z}$ et tout $n \in \mathbb{Z}$.

**Proposition 53.** $\Lambda$ est engendré comme λ-anneau par $e_1$; en outre, il est le λ-anneau libre engendré par un seul générateur. Ainsi, pour définir un morphisme de λ-anneaux $\phi : \Lambda \to R$, il suffit de préciser l’image de $e_1$, car

$$\phi(e_k) = \phi(\lambda^k(e_1)) = \lambda^k(\phi(e_1)).$$