Chromatic Symmetric Functions & LLT polynomials

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Polynomials

- $\mathbb{N} = \{0, 1, 2, \ldots\}$
- x will denote a *family* of variables: $x = (x_1, x_2, \ldots, x_n)$
- For $a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$,

$$x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \qquad\qquad (x_i^0 = 1)$$

- A polynomial $f(x)$ in the variables $x = (x_1, \ldots, x_n)$ is written as

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha = \sum_{(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n} \underbrace{\underbrace{c_{(a_1, a_2, \ldots, a_n)}}_{\text{coefficient}} \underbrace{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}_{\text{monomial}}}_{\text{term}}$$

Symmetric polynomials

- Let S_n denote the group of permutations of $[n] = \{1, 2, \ldots, n\}$.
- A polynomial $f(x_1, \ldots, x_n)$ is *symmetric* if

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots x_{\sigma(n)}) = f(x_1, x_2, \ldots, x_n) \qquad \text{for all } \sigma \in S_n.$$

- Example. Which of the following symmetric polynomials?

$$x_1^2 + x_2 x_3 \qquad x_1 x_2 + x_2 x_3 + x_1 x_3 \qquad x_1^2 + x_1 x_2 + x_2^2$$

- Algebra of symmetric polynomials:

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} = \big\{ f \in \mathbb{C}[x_1, \ldots, x_n] : f \text{ is symmetric} \big\}.$$

- Observation:

  If $17\, x_{20}^2\, x_{22}^5$ *is a term in a symmetric polynomial,*
  then so is $17\, x_1^2 x_2^5$ *and* $17\, x_2^2\, x_1^5$*, and* $\ldots$

- So, to construct a symmetric polynomial, symmetrize a monomial:

$$x_1^2 x_2^5 \xmapsto{\text{symmetrize}} x_1^2 x_2^5 + x_2^2 x_1^5 + x_1^2 x_3^5 + x_2^2 x_3^5 + x_3^2 x_1^5 + x_3^2 x_2^5$$

$$x_1^2 x_2^5 x_3^2 \xmapsto{\text{symmetrize}} x_1^2 x_2^5 x_3^2 + x_2^2 x_1^5 x_3^2 + x_1^2 x_3^5 x_2^2$$

- *Monomial symmetric functions*:

$$m_{(\lambda_1,\dots,\lambda_l)}(x_1,\dots,x_n)=\sum x_{i_1}^{\lambda_1}x_{i_2}^{\lambda_2}\cdots x_{i_l}^{\lambda_l},$$

  the sum over *all distinct* monomials with exponents $\lambda_1 \geqslant \dots \geqslant \lambda_l$.

- *partition of length l:*

  sequence $(\lambda_1,\dots,\lambda_l)$ of positive integers satisfying $\lambda_1 \geqslant \dots \geqslant \lambda_l$

- Theorem. Every symmetric polynomial in $\mathbb{C}[x_1,\dots,x_n]^{S_n}$ can be written uniquely as

$$f(x_1,\dots,x_n)=\sum_{\substack{\text{all partitions } \lambda \\ \text{of length } \leqslant n}} c_\lambda m_\lambda(x_1,\dots,x_n)$$

Recurring theme: symmetric polynomials from S_n-actions

Given

- a set of combinatorial objects $\mathcal{O}$
- a map $\varepsilon : \mathcal{O} \to \mathbb{N}^n$, written $\varepsilon(T) = \big(\varepsilon_1(T), \ldots, \varepsilon_n(T)\big)$
- an action of S_n on $\mathcal{O}$ *compatible with* ε

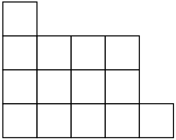
the following polynomial is symmetric:

$$f(x_1, \ldots, x_n) = \sum_{T \in \mathcal{O}} x_1^{\varepsilon_1(T)} \, x_2^{\varepsilon_2(T)} \, \cdots \, x_n^{\varepsilon_n(T)}$$

Tableaux

- Let λ be a partition of n; for example, $\lambda = (5, 4, 4, 1)$.

- The *(Young) diagram* of λ looks like this:



λ_1 elements in first row
λ_2 elements in second row
etc.

- A *semistandard (Young) tableau* of shape λ is a filling of the cells of the Young diagram of λ by positive integers with entries *weakly increasing in rows* and *strictly increasing in columns*:

6			
5	7		
4	4	5	7
2	2	4	5

Schur polynomial indexed by λ

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathsf{SSYT}(\lambda, [n])} x_1^{\varepsilon_1(T)} \cdots x_n^{\varepsilon_n(T)}$$

where

- $\mathsf{SSYT}(\lambda, [n]) =$ semistandard tableaux of shape λ and with entries in $[n] = \{1, 2, \ldots, n\}$
- $\varepsilon_i(T)$ is the number of copies of i in T

- Example. $\mathsf{SSYT}\left(\begin{array}{|c|c} \hline & \multicolumn{1}{c}{} \\ \hline & \multicolumn{1}{c|}{} \\ \hline \end{array}, \{1, 2, 3\}\right)$ consists of

$$\begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$$

$$s_{(2,1)}(x_1, x_2, x_3)$$
$$= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2 x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

Elementary symmetric functions

$$\begin{array}{ccccccccc} s_{(1,1,1)}(x) & = & x_1x_2x_3 & + & x_1x_2x_4 & + & x_1x_3x_4 & + & x_2x_3x_4 \\ & & \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} & & \begin{array}{|c|}\hline 4\\\hline 2\\\hline 1\\\hline\end{array} & & \begin{array}{|c|}\hline 4\\\hline 3\\\hline 1\\\hline\end{array} & & \begin{array}{|c|}\hline 4\\\hline 3\\\hline 2\\\hline\end{array} \end{array}$$

- *k-th elementary symmetric polynomial*:

$$e_k(x_1, \dots, x_n) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} \cdots x_{i_k} = s_{1^k}(x_1, \dots, x_n)$$

- *algebraically independent*: $e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}$ are linearly independent and form a basis of $\mathbb{C}[x_1, \dots, x_n]^{S_n}$

$$e_{(\lambda_1, \lambda_2, \dots, \lambda_l)} = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}$$

Complete symmetric functions

$$\begin{array}{cccccccc} s_{(3)}(x) & = & x_1^3 & + & x_1 x_2^2 & + \cdots + & x_3 x_3 x_4 & + \cdots \\ & & \boxed{1|1|1} & & \boxed{1|2|2} & & \boxed{3|3|4} & \end{array}$$

- *k-th complete symmetric polynomial*: sum of all degree k monomials

$$h_k(x_1, \ldots, x_n) = \sum_{i_1 \le i_2 \le \cdots \le i_k} x_{i_1} \cdots x_{i_k} = s_{(k)}(x_1, \ldots, x_n)$$

- *algebraically independent*: another basis of $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$

$$h_{(\lambda_1, \lambda_2, \ldots, \lambda_l)} = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l}$$

Power sum symmetric functions

- *k-th power sum symmetric polynomial*:

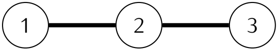
$$p_k(x_1, \ldots, x_n) = x_1^k + x_2^k + \cdots + x_n^k$$

- *algebraically independent*: another basis of $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$

$$p_{(\lambda_1, \lambda_2, \ldots, \lambda_l)} = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$$

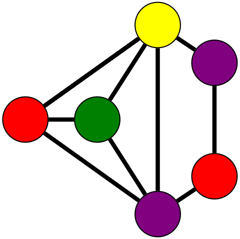
Chromatic Symmetric Functions

- A *graph* $\Gamma = (V, E)$ is a set of vertices V and a set of edges E.
- Example. $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}\}$, encodes the graph



- An *colouring* of Γ is a function $\kappa : V \to C$, with C a set of colours.
- A colouring κ is *proper* if adjacent vertices have different colours:

$$\{i, j\} \in E \implies \kappa(i) \neq \kappa(j).$$

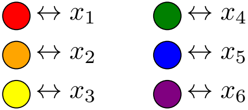


$$\longleftrightarrow \quad x_1^2\,x_3\,x_4\,x_6^2$$

combinatorial object

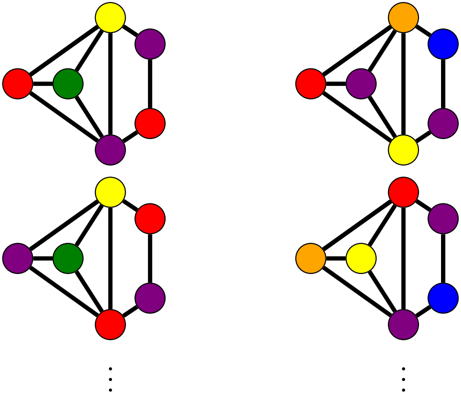
(proper colouring)

monomial



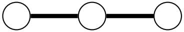
The *chromatic symmetric function* of Γ is a sum of monomials, one for each proper colouring of Γ:

$$28 \quad x_1^2 x_3 x_4 x_6^2 \quad + \quad 144\ x_1 x_2 x_3 x_5 x_6^2 \quad + \quad \cdots$$



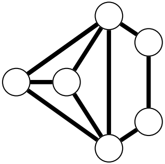
Another example

- Let's compute $X_\Gamma(x_1, x_2, x_3)$, where

$\Gamma \quad =$ 

- $3!$ ways to colour Γ with colours $\{1, 2, 3\}$, each giving $x_1x_2x_3$.
- 2 ways to colour Γ with colours $\{i, j\}$, giving $x_i^2x_j$ and $x_ix_j^2$.
- 0 ways to colour Γ with only one colour — no occurrences of x_i^3.

$$
\begin{aligned}
&X_\Gamma(x_1, x_2, x_3)\\
&\qquad = 6x_1x_2x_3 + x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2\\
&\qquad = 6m_{(1,1,1)} + m_{(2,1)}
\end{aligned}
$$



- Express $X_\Gamma(x)$ in a different basis:

$$\begin{aligned} X_\Gamma(x) &= 28\, x_1^2 x_3 x_4 x_6^2 + 144\, x_1 x_2 x_3 x_5 x_6^2 + \cdots \\ &= 720\, m_{111111} + 144\, m_{21111} + 28\, m_{2211} \\ &= 168\, s_{111111} + 60\, s_{21111} + 28\, s_{2211} \\ &= 28\, e_{42} + 32\, e_{51} + 108\, e_6 \end{aligned}$$

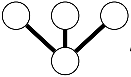
- Some numerology:

$$\begin{aligned} 28 + 32 + 108 &= \#\text{ acyclic orientations of } \Gamma \\ 28 + 32 &= \#\text{ acyclic orientations of } \Gamma \text{ with 2 sinks} \\ 108 &= \#\text{ acyclic orientations of } \Gamma \text{ with 1 sink} \end{aligned}$$

e-expansions and acyclic orientations

- Theorem (Stanley). If $X_\Gamma = \sum_\lambda c_\lambda e_\lambda$, then

$$\sum_{\ell(\lambda)=j} c_\lambda = \#\text{ acyclic orientations of } \Gamma \text{ with exactly } j \text{ sinks.}$$

- If $\Gamma =$ , then $X_\Gamma = e_{(2,1,1)} - 2e_{(2,2)} + 5e_{(3,1)} + 4e_{(4)}$.

- Open problem: Characterize the graphs for which X_Γ is e-positive.

- Conjecture (Stanley–Stembridge). If Γ is the incomparability graph of a $(3+1)$-free poset, then X_Γ is e-positive.

Chromatic *quasi*symmetric functions

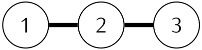
- Let $\Gamma = (V, E)$ be a finite graph on the vertex set $V = [n]$. We will consider colourings κ of the vertices by positive integers.

- Define the *ascent statistic* of κ as

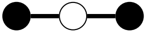
$$\mathrm{asc}_\Gamma(\kappa) = |\{(i,j) \in E : i < j \ \&\ \kappa(i) < \kappa(j)\}|$$

- The *chromatic quasisymmetric function* of Γ is

$$X_\Gamma(x;t) = \sum_{\substack{\text{proper} \\ \text{colourings} \\ \kappa:[n]\to\mathbb{N}^\times}} t^{\mathrm{asc}_\Gamma(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}.$$

- $\Gamma =$ 

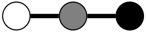
- There are two ways to colour Γ with colours $\{ i < j \}$:

 there is one ascent, giving $tx_i x_j^2$

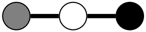
 there is one ascent, giving $tx_i^2 x_j$

This gives $t\sum_{i<j} x_i^2 x_j + t\sum_{i<j} x_i x_j^2$.

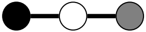
- There are $3!$ ways to colour Γ with colours $\{ i < j < k \}$:

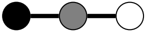
 $\rightsquigarrow$ 2 ascents $\rightsquigarrow$ $t^2 x_i x_j x_k$

 $\rightsquigarrow$ 1 ascents $\rightsquigarrow$ $t^1 x_i x_j x_k$

 $\rightsquigarrow$ 1 ascents $\rightsquigarrow$ $t^1 x_i x_j x_k$

 $\rightsquigarrow$ 1 ascents $\rightsquigarrow$ $t^1 x_i x_j x_k$

 $\rightsquigarrow$ 1 ascents $\rightsquigarrow$ $t^1 x_i x_j x_k$

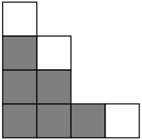
 $\rightsquigarrow$ 0 ascents $\rightsquigarrow$ $t^0 x_i x_j x_k$

This gives $(t^2 + 4t + 1)\sum_{i<j<k} x_i x_j x_k$.

- In the example, $X_\Gamma(x;t)$ is symmetric in the x variables
- However, this is not always the case!
- Shareshian & Wachs identified a class of graphs for which $X_\Gamma(x;t)$ is symmetric
- For this class of graphs, they conjecture $X_\Gamma(x;t)$ is e-positive

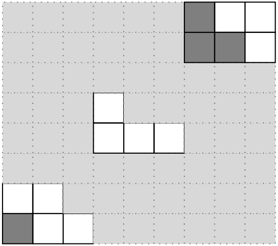
Tuples of skew-partitions

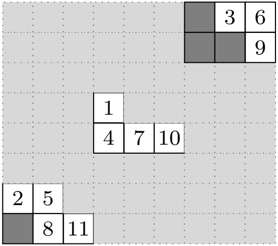
- If the diagram of λ contains the diagram of μ, then the *skew-partition* λ/μ consists of the cells of λ that are not in μ.



$\lambda = (4,2,2,1)$
$\mu = (3,2,1)$
λ/μ contains 3 cells

- *Tuples of skew-tableaux*, aligned according to diagonals:





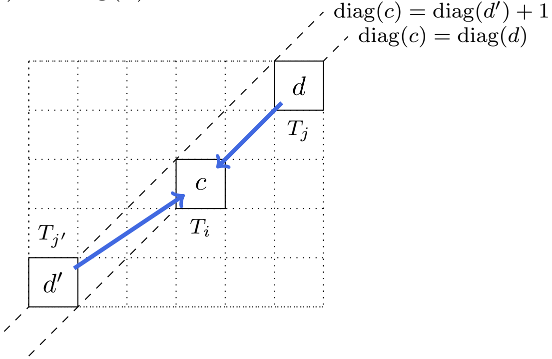
Inversions in tuples of skew-tableaux

Given a tuple of skew-tableaux $(T_1, \ldots, T_k)$,
a pair of cells $c \in T_i$ and $d \in T_j$ form an *inversion* if

- $T_i(c) > T_j(d)$, where $T_i(c)$ denotes the entry in cell c

and either:

- $i < j$ and $\mathrm{diag}(c) = \mathrm{diag}(d)$, or
- $i > j$ and $\mathrm{diag}(c) = \mathrm{diag}(d) + 1$.



LLT Polynomials

- For a *tuple of skew-partitions* $\vec{\nu} = (\nu^1, \nu^2, \ldots, \nu^k)$,

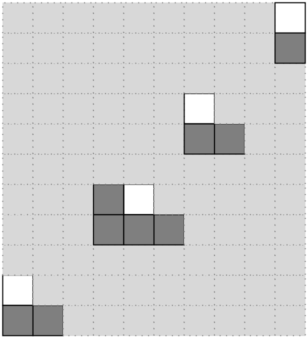
$$\mathrm{LLT}_{\vec{\nu}}(x;t) = \sum_{\substack{\vec{T}=(T_1,\ldots,T_k) \\ T^i \in \mathsf{SSYT}(\nu^i)}} t^{\mathrm{inv}(\vec{T})} x^{T_1} \cdots x^{T_k}$$

- $\mathrm{LLT}_{\vec{\nu}}(x;t)$ are symmetric in the x variables.

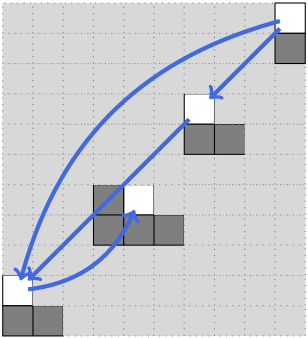
- Example. $s_{(3)} + 2t\, s_{(2,1)} + t^2\, s_{(1,1,1)}$

Unicellular LLT polynomials

If every ν^i in $\vec{\nu}$ is a single cell, then $\mathrm{LLT}_{\vec{\nu}}(x;t)$ is *unicellular*.



- Define a graph $\Gamma_{\vec{\nu}}$ on the cells of $\vec{\nu}$ with an edge connecting $c \in \nu^i$ and $d \in \nu^j$ whenever
  - $i < j$ and $\mathrm{diag}(c) = \mathrm{diag}(d)$; or
  - $i > j$ and $\mathrm{diag}(c) = \mathrm{diag}(d) + 1$.

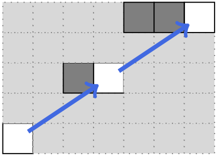
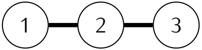


- $\mathrm{inv}(\vec{T})$ statistic equals the ascent statistic of the colouring

Proposition. If $\vec{\nu} = (\nu^1, \ldots, \nu^k)$ is unicellular, then

$$\mathrm{LLT}_{\vec{\nu}}(x;t) = \sum_{\substack{\textit{all}\text{ colourings} \\ \kappa:[k]\to\mathbb{N}^\times}} t^{\mathrm{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(k)}.$$

Example.

$\vec{\nu} =$  $\Gamma =$ 

$$\mathrm{LLT}_{\vec{\nu}}(x_1, x_2, x_3; t) = s_{(3)} + 2ts_{(2,1)} + t^2 s_{(1,1,1)}$$

From LLT to chromatic quasisymmetric polynomials

Theorem. If $\vec{\nu} = (\nu^1, \ldots, \nu^k)$ is unicellular, then

$$X_{\Gamma_{\vec{\nu}}}(x;t) = \frac{1}{(t-1)^k} \mathrm{LLT}_{\vec{\nu}}[(t-1)x;t].$$

- If f is a symmetric function, then $f[(t-1)x]$ denotes the *plethystic substitution* defined by

$$p_k[(t-1)x] = (t^k - 1)p_k(x).$$

- Attention: You have to switch bases first!

Representation theory

- A *representation (over $\mathbb{C}$)* of a group G is a morphism of groups

$$\rho : G \to \mathsf{GL}(V), \quad \text{where } V \text{ is a } \mathbb{C}\text{-vector space.}$$

- By fixing a basis of V, we get a *matrix representation* of G:

$$\rho : G \to \mathsf{GL}_n(\mathbb{C}),$$

and we define the *character* of the representation as

$$\chi_\rho(g) = \mathrm{trace}(\rho(g))$$

Symmetric functions from representations of S_n

- Let V be a representation of S_n with character χ.
- The *Frobenius characteristic* of V is the symmetric function

$$\mathrm{Frob}(V) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma)\, p_{\mathrm{cycletype}(\sigma)}$$

- A *graded representation* is a graded vector space $V = \bigoplus_{d \in \mathbb{N}} V_d$ equipped with an action of the group that maps each component V_d to itself. The *graded Frobenius characteristic* of $V = \bigoplus_d V_d$ is

$$\mathrm{Frob}(V)(t) = \sum_d \mathrm{Frob}(V_d) t^d \in \mathrm{Sym}[\![t]\!].$$

Theorem. Let $\mathrm{CF}(S_n)$ be the algebra of characters of S_n.

- $\mathrm{Frob} : \bigoplus_n \mathrm{CF}(S_n) \to \mathrm{Sym}$ is an algebra isomorphism.

- The Frobenius characteristic of an irreducible character is a Schur function s_λ; and conversely.

- If χ and ψ are characters of S_n and S_m, respectively, then

$$\mathrm{Frob}(\chi)\mathrm{Frob}(\psi) = \mathrm{Frob}\left(\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(\chi\psi)\right).$$

- If χ and ψ are characters of S_n, then

$$\langle \chi, \psi \rangle_{S_n} = \langle \mathrm{Frob}(\chi), \mathrm{Frob}(\psi) \rangle_{\mathrm{Sym}}.$$