Bounded generation and commutator width of Chevalley groups and Kac–Moody groups: function case

Boris Kunyavskiĭ, Bar-Ilan University, Israel
Eugene Plotkin, Bar-Ilan University, Israel

BIRS workshop "Arithmetic aspects of algebraic groups"
Banff, Canada

June 11, 2022

"Bounded generation and commutator width of Chevalley groups: function case"
arXiv:2204.10951 [math.GR], (2022) 54 pp.

joint work of B.Kunyavskii, E.Plotkin, N.Vavilov

Special thanks go to Inna Capdeboscq

Motivation
1. Kac-Moody groups.

• **Øre problem**: Commutator width of a finite simple group is 1?

Positive answer

For fields K, $|K| \geq 8$, — Ellers and Gordeev

Over small fields $\mathbb{F}_q$, $q = 2, 3, 4, 5, 7$, — Liebeck, O'Brien, Shalev, and Pham Huu Tiep.

However:

There exist finitely generated simple groups of infinite commutator width (A. Muranov, small-cancellation methods).

There exist finitely presented simple groups of infinite commutator width (Caprace-Fujiwara). In particular, these are all irreducible (non-spherical and non-affine) Kac–Moody groups over finite fields.

Question

Is the commutator width of affine Kac-Moody groups over finite fields finite?

Implies

Question

Is the commutator width of Chevalley groups over Laureant polynomials $G(\Phi, \mathbb{F}_q[t, t^{-1}])$ finite?

2. Model theory

Theorem (D.Segal-K.Tent, 2020)

Let $G(\Phi, R)$ be a Chevalley group, rkG ≥ 2. Let R be integral domain. Then R and $G(\Phi, R)$ are bi-interpretable provided either

- *$G(\Phi, R)$ is adjoint*
- *$G(\Phi, R)$ has finite elementary width*

Similar approach is developed in two recent papers by A.Myasnikov-M.Sohrabi.

Question

Is the commutator width of Chevalley groups over polynomials $G(\Phi, \mathbb{F}_q[t])$ and Laurent polynomials $G(\Phi, \mathbb{F}_q[t, t^{-1}])$ finite?

Notation

- Φ — reduced irreducible root system *usually* of rank ≥ 2;
- R — commutative ring with 1;
- $G(\Phi, R)$ — **Chevalley group** of type Φ over R;
- $T(\Phi, R)$ — split maximal torus in $G = G(\Phi, R)$;
- $x_\alpha(\xi)$, where $\alpha \in \Phi$, $\xi \in R$, — **elementary unipotents** with respect to $T(\Phi, R)$;
- $E(\Phi, R)$ — **elementary subgroup** in $G(\Phi, R)$:

$$E(\Phi, R) = \left\langle x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in R \right\rangle.$$

Let G be a group and X be a set of its generators. Usually one considers symmetric sets, for which $X^{-1} = X$.

- The **length** $l_X(g)$ of an element $g \in G$ with respect to X is the minimal k such that g can be expressed as the product

$$g = x_1 \dots x_k, \qquad x_i \in X.$$

- The **width** $w_X(G)$ of G with respect to X is the supremum of $l_X(g)$ over all $g \in G$.

We say that a group G has **bounded generation** with respect to X if the width $w_X(G)$ is finite.

Generating set X for $E(\Phi, R)$:

- The set of *elementary* root unipotents

$$\Omega = \left\{ x_\alpha(\xi) \mid \alpha \in \Phi, \xi \in R \right\}$$

relative to the choice of a split maximal torus T;

- The set of commutators

$$C = \left\{ [x, y] = xyx^{-1}y^{-1} \mid x \in G(\Phi, R),\ y \in E(\Phi, R) \right\}.$$

It is a classical theorem due to Suslin, Kopeiko and Taddei that for $\mathrm{rk}(R) \geq 2$ one indeed has $C \subseteq E(\Phi, R)$.

The width $w_\Omega(E(\Phi, R))$ is usually denoted $w_E(G(\Phi, R))$ and is called the **elementary width** of $G(\Phi, R)$.

Clearly, $w_E(G(\Phi, R))$ is the smallest such L that

$$E(\Phi, R) = E^L(\Phi, R).$$

Here, $E^L(\Phi, R)$ is the *subset* of $E(\Phi, R)$, consisting of products of $\leq L$ root unipotents.

The width $w_C(E(\Phi, R))$ is often called the **commutator width** of $G(\Phi, R)$ itself.

State of art

1. Fields, 0-dimensional rings

- **Bruhat decomposition**

$$w_E(G(\Phi, K)) \leq 2N + 4l.$$

where $N = |\Phi^+|$ is the number of positive roots, and $l = \mathrm{rk}(\Phi)$ is the rank of Φ.

- **Øre problem**:

$$w_C(G_{\mathrm{ad}}(\Phi, R)) = 1, \qquad \text{while} \qquad w_C(G_{\mathrm{sc}}(\Phi, R)) \leq 2.$$

- R — local ring. The same bound

$$w_E(G(\Phi, R)) \leq 2N + 4l.$$

as over a field.

- R — semilocal ring. Similar bound, about 1.5 times worse:

$$w_E(G(\Phi, R)) \leq 3N + 4l.$$

Follows from **Gauß decomposition** $G = DU^+U^-U^+$.

- It can be derived that

∗ $w_C(E(\Phi, R)) \leq 3$ for $\Phi = \mathrm{A}_l$ and F_4;

∗ $w_C(E(\Phi, R)) \leq 4$ for all other types, apart, maybe from E_6;

∗ $w_C(E(\Phi, R)) \leq 5$ for $G(\mathrm{E}_6, R)$.

Smolensky, 2019.

- The elementary width of rank ≥ 2 groups over a *Euclidean ring* can be infinite.

van der Kallen has proven that $\mathrm{SL}(3, \mathbb{C}[t])$ — and thus all $\mathrm{SL}(n, \mathbb{C}[t]$ — have INFINITE ELEMENTARY WIDTH.

Dennis and Vaserstein noticed that $\mathrm{SL}(3, \mathbb{C}[t])$ has INFINITE COMMUTATOR WIDTH.

Now from the work of Stepanov and others we know these results are equivalent.

Bounded generation is a rare phenomenon,and, in fact, an it arithmetic question.

Dedekind rings of arithmetic type

Borderline case are 1-dimensional rings.

K is a a global field, i.e. a finite extension of $\mathbb{Q}$ in charactersistic 0, or a finite extension of $\mathbb{F}_q(t)$, $q = p^m$, in positive characteristic p.

S is finite set of valuations of K, containing all Archimedean ones in the number case.

$R = \mathcal{O}_S$ — S-integers in K = **Dedekind rings of arithmetic type**.

When K is of characteristic 0 — **number case**.

When K is of characteristic $p > 0$ — **function case**.

Number case, rank ≥ 2 groups.

• Carter and Keller, 1983–1984, proved that $\mathrm{SL}(n, R)$, $n \geq 3$, are boundedly generated.

• Tavgen, 1990, the same for all Chevalley groups of rank ≥ 2.

With good bounds depending on the type of Φ and the class number of R alone.

• Carter, Keller, Paige, 1985 — model theoretic proof, redeveloped and illuminated by Morris.

SL$(2, R)$, for a Dedekind ring $R = \mathcal{O}_S$, with infinite multiplicative group.

• Cooke and Weinberger, 1975 —excellent bounds, **conditional**, modulo the Generalised Riemann Hypothesis.

• Liehl, 1981 — explicit unconditional bounds in some cases, grossly exaggerated.

• Vsemirnov [and Sury], 2012 — SL $\left(2, \mathbb{Z}\left[\frac{1}{p}\right]\right)$, the bound $w_E(\mathrm{SL}(2, R)) = 5$ *unconditionally*.

• Morgan, Rapinchuk and Sury, 2018:

$$w_E(SL(2, R)) \leq 9.$$

• Jordan and Zaytman, S contains at least one real or non-Archimedean valuation 2020:

$$w_E(SL(2, R)) \leq 8,$$

Dedekind rings of arithmetic type: function case

Note:

• The group $\mathrm{SL}(2, \mathbb{F}_q[t])$ is not even finitely generated.

• The groups $\mathrm{SL}(2, \mathbb{F}_q[t, t^{-1}])$ and $\mathrm{SL}(3, \mathbb{F}_q[t])$ are finitely generated but not finitely presented.

• Queen, 1975 — under some additional assumptions on R — which hold, for instance, for Laurent polynomial rings $\mathbb{F}_q[t, t^{-1}]$ with coefficients in a finite field — one has

$$w_E(\mathrm{SL}(2, R)) = 5,$$

• Nica, 2018, — bounded elementary generation of $\mathrm{SL}(n, \mathbb{F}_q[t])$, $n \geq 3$.

• Trost, 2021, — the ring of integers R of an arbitrary global function field K, with a bound of the form $L(d, q) \cdot |\Phi|$, where the factor L depends on q and of the degree d of K.

Results

Theorem (A)

Let $G(\Phi, R)$ be a simply connected Chevalley group of type Φ, $\mathrm{rk}(\Phi) \geq 2$. *Then*

$$w_E(G(\Phi, \mathbb{F}_q[t])) < \infty.$$

This result relies, in particular, on reduction to $A_1 \to C_2$, $C_2 \to C_3$, $B_2 \to B_3$. $A_1 \to G_2$ cases and the following theorems:

Theorem

The elementary width of $\mathrm{Sp}(4, \mathbb{F}_q[t])$ *is finite and, moreover,*

$$w_E\big(\mathrm{Sp}(4, \mathbb{F}_q[t]\big) \leq 79.$$

Theorem

The elementary width of $\mathrm{Sp}(6, \mathbb{F}_q[x])$ *is finite and, moreover,*

$$w_E\big(\mathrm{Sp}(6, \mathbb{F}_q[t]\big) \leq 72.$$

Theorem

The elementary width of $\mathrm{SO}(7, \mathbb{F}_q[x])$ *is finite and, moreover,*

$$w_E\big(\mathrm{SO}(7, \mathbb{F}_q[t]\big) \leq 65.$$

Few words about the ideas of the proof

- – Surjective stability of K_1-functor

Embedding $\Delta \subset \Phi$ implies $\varphi : K_1(\Delta, R) \to K_1(\Phi, R)$. Surjectivity of φ means $G(\Phi, R) = G(\Delta, R)E(\Phi, R)$, and this reduction is bounded.

- – Tavgen's rank reduction theorem
- – Arithmetic considerations adopted to function case like

Theorem

(*Kornblum–Artin*), *Let a, b be relatively prime polynomials in $\mathcal{O} = \mathbb{F}_q[t]$, $\deg a > 0$. Then there are infinitely many monic irreducible polynomials b' congruent to b modulo $a\mathcal{O}$. Moreover, such b' can be of arbitrary degree N, provided N is sufficiently large.*

and

Lemma. Any matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{F}_q[x])$$

can be moved to a matrix of the form

$$A = \begin{pmatrix} * & b_1^m \\ * & * \end{pmatrix}$$

where $m = q - 1$ or 2 by ≤ 3 elementary transformations.

The proof uses explicit power reciprocity laws, and other arithmetic tools.

• – The rest is playing with Mennicke symbols, in the style of Bass—Milnor—Serre, 1967, similar to what Carter—Keller, Tavgen, or Nica were doing. The difference is the use of root calculations whic control the explicit form of transformations.

Let $a, b, c, d \in R$, $ad - bc = 1$. The classes of matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & -c & d \end{pmatrix} \in \mathrm{Sp}(4, R)$$

modulo $\mathrm{Ep}(4, R)$ only depend on their first row (a, b).

These classes are denoted by $[a, b]$ and $\{a, b\}$, respectively, and are called **Mennicke symbols**.

Applications

Kac-Moody groups

Theorem (B)

Let $G(\Phi, R)$ be a simply connected Chevalley group of type Φ, $\mathrm{rk}(\Phi) \geq 2$ over $R = \mathbb{F}_q[t]$, Then $G(\Phi, R)$ is of finite commutator width.

The case $R = \mathbb{F}_q[t, t^{-1}]$ is much better than $R = \mathbb{F}_q[t]$. This yields the positive result for affine Kac-Moody groups:

Theorem (C)

The commutator width of an affine elementary untwisted Kac–Moody group $\widetilde{E}_{sc}(A, \mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ is $\leq L'$, where

- $L' = 5$ *for* $\Phi = \mathrm{F}_4$ *and* $\Phi = \mathrm{A}_l$, $l = 2k+1$, $k = 0, 1, \ldots$;
- $L' = 6$ *for* $\Phi = \mathrm{A}_l$, $l = 2k$, $k = 1, 2, \ldots$, $\Phi = \mathrm{B}_l, \mathrm{C}_l, \mathrm{D}_l$, *for* $l \geq 3$ *or* $\Phi = \mathrm{E}_7, \mathrm{E}_8$, *or, finally,* $\Phi = \mathrm{C}_2, \mathrm{G}_2$ *under the additional assumption that* 1 *is the sum of two units in* R (*which is automatically the case provided* $q \neq 2$);
- $L' = 7$ *for* $\Phi = \mathrm{E}_6$.

Model Theory

• – A model M of the theory T is called a **prime model** of T if it elementarily embeds in any model of T.

• – A model M of T is **atomic** if every type realized in M is principal.

• – A model M is **homogeneous** if for every two tuples $\bar{a} = (a_1, \ldots, a_n)$, $\bar{b} = (b_1, \ldots, b_n)$ in M^n that realize the same types in M there is an automorphism of M that takes $\bar{a}$ to $\bar{b}$.

• – A finitely generated group G is called **quasi-finite axiomatizable, or QFA** if the elementary theory $Th(G)$ is determined by a single formula φ, that is every finitely-generated group $H \in \mathcal{C}$ which satisfies φ is isomorphic to G.

Theorem

The groups $G = G(\Phi, \mathbb{F}_q[t])$, $\mathrm{rk}(\Phi) > 2$, *and* $G = G(\Phi, \mathbb{F}_q[t, t^{-1}])$, $\mathrm{rk}(\Phi) > 1$, *are QFA, first order rigid, prime, atomic, homogeneous. All their finitely generated subgroups are definable, and even uniformly definable. Their elementary theories are undecidable.*

The proof follows from Theorem A, Segal–Tent result, and the philosophy of rich groups introduced by Kharlampovich–Myasnikov–Sohrabi.

CONJECTURES

Conjecture

Let $E = E(\Phi, R)$ be a Chevalley group over an arbitrary commutative ring R, and $\mathrm{rank}(E) \geq 2$. Then $E = E(\Phi, R)$ is bi-interpretable with R.

This conjecture would be regarded as a far generalization of Segal-Tent's result:

Theorem (D.Segal-K.Tent, 2020)

Let $G(\Phi, R)$ be a Chevalley group, $rkG \geq 2$. Let R be integral domain. Then R and $G(\Phi, R)$ are bi-interpretable provided either

- *$G(\Phi, R)$ is adjoint*
- *$G(\Phi, R)$ has finite elementary width*

It is also genetically tied with the result of Bunina:

Theorem (Bunina, 2019)

Let $G_1 = G_\pi(\Phi, R)$ and $G_2 = G_\mu(\Psi, S)$ be two elementarily equivalent Chevalley groups. Here Φ, Ψ denote the root systems of rank $\geqslant 1$, R and S are commutative rings, and π, μ are weight lattices. Suppose that $2 \in R^*, S^*$ for A_2, B_l, C_l, and F_4, and $2, 3 \in R^*, S^*$ for G_2. Then root systems of Φ and Ψ coincide, while the rings are elementarily equivalent.

and with conjecture

Conjecture (Avni-Meiri)

Let Δ be an irreducible lattice in a semisimple group $\prod_\nu G(K_\nu)$. Then Δ is bi-interpretable with the ring of integers if and only if $\mathrm{rank}_S G \geq 2$.

Given a group G fix a word $w = w(x_1, x_2, \ldots, x_n)$ and consider the word map

$$w \colon G^n \to G. \qquad (1)$$

The map w is the evaluation map: one substitutes n-tuples of elements of the group G instead of the variables and computes the value. Assume the span $\langle w(G) \rangle = G$, where span means the verbal subgroup generated by $w(H)$ is G
Let $w(G)$ be the value set of the map w in the algebra H. Denote by $w(H)^k$ the set of elements $g \in H$ of the form $g = h_1 h_2 \cdots h_k$ where $h_i \in w(H)$, $i = 1, \ldots, k$.

Definition

The smallest k such that $w(H)^k = H$ is called the w-width of the group H. Denote it by $wd_w(H)$.

Conjecture

Let R be an arbitrary commutative ring with 1, $G = G(\Phi, R)$ be a Chevalley group, $rk(\Phi) > 1$, $w(x_1, \ldots, x_n) : G^n \to G$ be such that $\langle w(G) \rangle = G$, Then, $G(\Phi, R)$ is boundedly elementary generated if and only if it is boundedly verbally generated, that is $wd_w(H) < \infty$

Proposition

Bounded elementary width implies bounded verbal width.

Proof relies on decomposition of unipotents formulas by Vavilov-Stepanov.

In the same spirit

Theorem

A Chevalley group $G(\Phi, R)$, $rk(\Phi) > 1$ is strongly boundedly generated if and only if $G(\Phi, R)$ is boundedly elementary generated.

Modulo E_8 case.