Actions of maximal tori on homogeneous spaces and applications to number theory

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Outline

1. Introduction

Particular setting: $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, H is a subgroup of G acting on G/Γ by

$$h\pi(g) = \pi(hg), \forall h \in H,$$

where $\pi : G \to G/\Gamma$ is the quotient map. G/Γ is endowed with the quotient topology and with a finite G-invariant (Haar) measure.

General $\mathcal{S}$-arithmetic setting: $\mathbf{G}$ is a simple algebraic group defined over a number field K, $\mathcal{S}$ a finite set of places of K containing the archimedean ones and $\mathcal{O}$ the ring of $\mathcal{S}$-integers in K (i.e. $\mathcal{O} = \{x \in K : |x|_v \leq 1 \text{ for all } v \notin \mathcal{S}\}$). Let K_v, $v \in \mathcal{S}$, be the completion of K with respect to v and by $K_{\mathcal{S}}$ the direct product of the topological fields K_v. $K_{\mathcal{S}}$ is a ring and K (and, therefore, also $\mathcal{O}$) is identified with its diagonal imbedding in $K_{\mathcal{S}}$. Moreover, K is dense in $K_{\mathcal{S}}$ and $\mathcal{O}$ is a co-compact lattice in $K_{\mathcal{S}}$. Put $G = \mathbf{G}(K_{\mathcal{S}})$, $G_v = \mathbf{G}(K_v)$, where $v \in \mathcal{S}$, and $\Gamma = \mathbf{G}(\mathcal{O})$. So, $G = \prod_{v \in \mathcal{S}} G_v$, Γ is a lattice in G, G/Γ is a homogeneous space endowed with the quotient topology and a subgroup H of G acts on G/Γ by left translations.

Theorem 1 (Moore)

If every G_v, $v \in \mathcal{S}$, is not compact and H is a non-compact subgroup of G then $\overline{H\pi(g)} = G/\Gamma$ for almost all with respect to the Haar measure $\pi(g) \in G/\Gamma$.

2. Some results and conjectures

The study of the behavior of the individual orbits is important for the applications. The closure $\overline{H\pi(g)}$ of $H\pi(g)$ in G/Γ is called *homogeneous* if $\overline{H\pi(g)} = L\pi(g)$ for a closed subgroup L of G.

The following two theorems are equivalent.

Theorem 2 (Margulis, 1987)
Let q be a real, non-degenerate, indefinite quadratic form in $n \geq 3$ variables which does not represent 0 over $\mathbb{Q}$ and 0 is an isolated point in $\overline{f(\mathbb{Z}^n)}$. Then q is proportional to a form with rational coefficeints.

Theorem 3 (Margulis, 1987)
If $SO(2,1)\pi(g)$ is bounded in $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ then $SO(2,1)\pi(g)$ is compact.

Theorem 2 confirms a conjecture of A.Oppenheim formulated around 1952. The observation that in order to prove the conjecture one needs to prove Theorem 3 is due to M.S.Raghunathan.

Let $q(\vec{x})$ be a quadratic form with coefficients in $K_{\mathcal{S}}$, equivalently, $q(\vec{x}) = (q_v(\vec{x}))_{v\in\mathcal{S}}$, where $q_v(\vec{x})$ is a quadratic form in $K_v[\ \vec{x}\ ]$.

The form $q(\vec{x})$ is *non-degenerated* (resp. *isotropic*) if each $q_v(\vec{x})$ is non-degenerated (resp. isotropic) and q is is irrational if $q \neq a \cdot q_0$, where $q_0 \in K[\vec{x}]$. The next result generalizes a result of Dani and Margulis for the real quadratic forms.

Theorem 4 (Borel-Prasad, 1992)
Let $n \geq 3$ and q be a non-degenerated, isotropic, and irrational quadratic form on $K_{\mathcal{S}}^n$. Then $q(\mathcal{O}^n)$ is dense in $K_{\mathcal{S}}$.

In 1955 Cassels and Swinnerton-Dyer formulated:

Conjecture 1 (Cassels and Swinnerton-Dyer, 1955) Let $f(\vec{x}) = l_1(\vec{x}) \cdots l_n(\vec{x})$ where $l_i(\vec{x})$ are linearly independent over $\mathbb{R}$ linear forms on $\mathbb{R}^n$. Suppose that $n \geq 3$, $f(\vec{x})$ does not represent 0 over $\mathbb{Q}$ and 0 is an isolated point in $\overline{f(\mathbb{Z}^n)}$. Then $f(\vec{x})$ is rational, i.e. $f = a \cdot f_0$, where $f_0 \in K[\vec{x}]$.

Conjecture 1 for $n = 3$ implies the famous Littlewood conjecture:

Conjecture 2 (Littlewood, 1930) Let $\| x \|$ denote the distance from $x \in \mathbb{R}$ to the closest integer. Then

$$\liminf_{n \to \infty} n \cdot \| n\alpha \| \cdot \| n\beta \| = 0$$

for any real numbers α and β.

Conjecture 1 is equivalent to the following conjecture about orbits of the diagonal group on $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$.

Conjecture 3 Let $n \geq 3$ and D be the group of all diagonal matrices in $\mathrm{SL}(n, \mathbb{R})$. If the orbit $D\pi(g)$ is bounded in $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ then it is compact.

3. A characterisation of the norm forms and a generalisation of the conjecture of Cassels and Swinnerton-Dyer

Let $K/\mathbb{Q}$ be a number field of degree $n = r + 2s$ where r is the number of real imbeddings and s is the number of complex imbeddings of the number field K. Let $K = \mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_m$, $m \geq n$, and $l_i(\vec{x}) = \sigma_i(x_1\alpha_1 + \cdots + x_m\alpha_m)$ where $\sigma_i : K \to \mathbb{C}$ are the imbeddings of K. Then $f(\vec{x}) = al_1(\vec{x}) \cdots l_m(\vec{x}), a \in \mathbb{R}$, is called a *norm form*. So, $f(\vec{x})$ is a decomposable over $\mathbb{C}$ real homogeneous form which does not represent 0 over $\mathbb{Q}$ and $f(\mathbb{Z}^m)$ is descrete in $\mathbb{R}$, that is, $f(\mathbb{Z}^m) \cap [a, b]$ is finite for all reals $a < b$.

Theorem 5 (T, 2022) Let $f(\vec{x}) \in \mathbb{R}[\vec{x}]$ be a decomposable over $\mathbb{C}$ homogeneous form and $\mathcal{Z}$ be a non-zero orbit for the action of $\mathrm{SL}(m, \mathbb{Z})$ on $\mathbb{Q}^m$. Suppose that $f(\vec{x})$ does not represent 0 over $\mathbb{Q}$, $f(\mathcal{Z})$ is discrete in $\mathbb{R}$ and that if $n = 2$ the form $f(\vec{x})$ splits over $\mathbb{R}$, i.e. $f(\vec{x}) = l_1(\vec{x}) \cdot l_2(\vec{x})$ where $l_1(\vec{x})$ and $l_2(\vec{x}) \in \mathbb{R}[\vec{x}]$. Then $f(\vec{x})$ is a norm form.

Remark: *The restriction in the above formulation when $n = 2$ is essential because of the example $f(x_1, x_2) = x_1^2 + ax_2^2$ where a is a positive irrational number.*

The proof uses the following general proposition.

Proposition (T, 2022) Let $p(\vec{x}) \in \mathbb{R}[\vec{x}]$ be any polynomial in m variables and H be the stabilizer of $p(\vec{x})$ for the natural action of $\mathrm{SL}(m, \mathbb{R})$ on $\mathbb{R}[\vec{x}]$. With $\mathcal{Z}$ as above, suppose that $p(\mathcal{Z})$ is discrete. Then $H\pi(e)$ is closed in $\mathrm{SL}(m, \mathbb{R})/\mathrm{SL}(m, \mathbb{Z})$.

Conjecture 5 Let $f(\vec{x}) = l_1(\vec{x}) \cdots l_n(\vec{x}) \in \mathbb{R}[\vec{x}]$ where $l_i(\vec{x})$ are linearly independent over $\mathbb{C}$ linear forms in m variables. Let $n = r + 2s$ where r is the number of $l_i(\vec{x})$ with real coefficients. Suppose that $r + s - 1 \geq 2$, $f(\vec{x})$ does not represent 0 over $\mathbb{Q}$ and 0 is an isolated point in $\overline{f(\mathbb{Z}^n)}$. Then $f(\vec{x})$ is a norm form.

Remark: *The restriction $r + s - 1 \geq 2$ in the formulation of the conjecture is essential and can not be omitted.*

Conjecture 5 is equivalent to the following

Conjecture 6 Let T be a maximal torus in $\mathrm{SL}(n, \mathbb{R})$ such that his $\mathbb{R}$-split part has dimension 2. If the orbit $T\pi(g) \subset \mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ is bounded then it is compact.

4. Main cases and some general results

If we compare Margulis and Borel-Prasad theorems with the conjecture of Cassels and Swinnerton-Dyer, we distinguish two main cases concerning the study of the orbit closures on homogeneous spaces: (a) H is spent by unipotent elements of G, and (b) H is a maximal split torus of G.

Remark: *Geometrically, the case (a) corresponds to the horocyclic flow on the unit tangent bundle on a Riemann manifold and the case (b) corresponds to the geodesic flow on this bundle.*

Definition. *An orbit $T\pi(g)$ in G/Γ is called divergent if the orbit map $T \to G/\Gamma, h \mapsto h\pi(g)$, is proper.*

The divergency properties represent the phenomena which separate the cases (a) and (b). *In case (b) there are plenty of divergent orbits and in case (a) the orbit is never divergent because of the following theorem.*

Theorem 6 (Margulis' recurrence theorem, 1971) *Let $\{u_t\}_{t\in\mathbb{R}}$ be a one-parameter group of unipotent matrices in $\mathrm{SL}(n,\mathbb{R})$. Then for every $g \in \mathrm{SL}(n,\mathbb{R})$ there exists a compact subset Ω of $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ such that $\{t \geq 0 : u_t\pi(g) \in \Omega\}$ is unbounded.*

In case (a) we have general results obtained using the following strategy:

if $U = \{u(t)\}$ is a 1-parameter unipotent group, we associate to every orbit $U\pi(g)$ an ergodic probability measure which turns to be algebraic, that is, a measure on a closed orbit $H\pi(g)$ of a subgroup H of G containing U and induced by the Haar mesure on H

The following result confirms conjectures of S.G.Dani and M.S.Raghunathan.

Theorem 7 (Marina Ratner, 1990) *Let G be a connected real Lie group, Γ its lattice and H a connected subgroup generated by unipotent elements. Then*

1. *Every H-invariant, ergodic, Borel probability measure is algebraic;*
2. *The closure of every H-orbit $H\pi(g)$ is homogeneous, that is, $\overline{H\pi(g)} = L\pi(g)$ where L is a subgroup containing H.*

A.Borel raised the question about the $\mathcal{S}$-adic generalization of Theorem 5.

Theorem 8 (Margulis-T, 1994; Ratner, 1995). *The analog of Theorem 5 is valid in $\mathcal{S}$-adic setting.*

What is the structure of the group L in the formulations of the above theorems? (This question arises in many applications, in particular, in the proof of our Theorem 5.) The answer is given by the next definition and theorem.

Definition Let $\mathbf{G}$ be a K-algebraic group, $\mathcal{S}$ a finite set of places of K containing the archimedean ones. A K-subgroup $\mathbf{L}$ of $\mathbf{G}$ is of class $\mathcal{F}$ with respect to $\mathcal{S}$ if for every normal K-algebraic subgroup $\mathbf{L}$ there exists $v \in \mathcal{S}$ such that $(\mathbf{L}/\mathbf{N})(K_v)$ contains a unipotent element different from the identity.

Theorem 9 (T, 2000). *Let H be a subgroup of $G = \mathbf{G}(K_\mathcal{S})$ generated by unipotent elements. Then $\overline{H\pi(g)} = gL'\pi(e)$ where L' is a subgroup of finite index of $L = \mathbf{L}(K_\mathcal{S})$ where $\mathbf{L}$ is of class $\mathcal{F}$.*

5. Description of the closed, divergent and locally divergent orbits of maximal split tori and applications

Definition. *An orbit Ta for the action of a group T on a space X is called divergent if the orbit map $t \mapsto ta$ is proper, that is, the sequence $\{t_i a\}$ leaves compacts in X whenever the sequence $\{t_i\}$ leaves compacts in T.*

Theorem 10 (Margulis' divergence theorem, 2001). *Let T be the sub-group of all diagonal matrices in $\mathrm{SL}(n,\mathbb{R})$. Then $T\pi(g)$ is divergent in $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ if and only if $g \in T \cdot \mathrm{SL}(n,\mathbb{Q})$.*

Theorem 11 (T.-Weiss 2003). *Let **G** be a semisimple $\mathbb{Q}$-algebraic group, let **T** be an $\mathbb{R}$-torus containing a maximal $\mathbb{R}$-split torus, let $T = \mathbf{T}(\mathbb{R})$, and let $g \in G$. Then:*

1. *$T\pi(g)$ is a closed orbit if and only if $g^{-1}\mathbf{T}g$ is a product of a $\mathbb{Q}$-subtorus and an $\mathbb{R}$-anisotropic $\mathbb{R}$-subtorus;*
2. *$T\pi(g)$ is a divergent orbit if and only if the maximal $\mathbb{R}$-split subtorus of $g^{-1}\mathbf{T}g$ is defined over $\mathbb{Q}$ and $\mathbb{Q}$-split.*

Question: What is the structure of $\overline{T\pi(g)}$, in general?

Example Let $G = \mathrm{SL}(2,\mathbb{R})$. Then $\mathrm{SO}(2)\backslash\mathrm{SL}(2,\mathbb{R})/\Gamma$ is a Riemann surface and the geodesic flow on it provides plenty of examples of orbits $T\pi(g)$ in $\mathrm{SL}(2,\mathbb{R})/\Gamma$ with non-homogeneous $\overline{T\pi(g)}$.

In his survey paper "*Problems and Conjectures in Rigidity Theory*" (2000) Margulis raised the following

Conjecture 6 If G is a semisimple real algebraic group, T an $\mathbb{R}$-split torus in G and Γ a lattice in G then $\overline{T\pi(g)}$ is homogeneous provided $\overline{T\pi(g)}$ does not admit *a quotient on which T acts as 1-dimensional torus*.

Example One of the simplest examples illustrating Margulis' conjecture is given by $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, $\Gamma = \mathrm{SL}(2,\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of a real quadratic extension $K/\mathbb{Q}$ (for example, $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$), and $T = D \times D$ where D is the group of diagonal matrices in $\mathrm{SL}(2,\mathbb{R})$.

In his paper "*Weyl chamber flow on irreducible quotients of* $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$", Transf. Groups, vol.11, 2006, pp. 17-28, Damien Ferte proved that in a number of cases the closure of $\overline{(D \times D)\pi(g)}$ is homogeneous in G/Γ. Other affirmative results are obtained by S.Mozes in the paper "*On closures of orbits and arithmetic of quaternions*", Israel J.Math. 86, 1994, pp.161-174.
But it turns out that $\overline{(D \times D)\pi(g)}$ is not homogeneous whenever $(D \times D)\pi(g)$ is so-called *locally divergent*, non-closed orbits. This is phenomenon which concerns arbitrary semisimple algebraic groups and is related with an $\mathcal{S}$-adique generalization of Theorem 10.

Recall that $\mathbf{G}$ is a semisimple algebraic group defined over K and K-isotromic, $\mathcal{S}$ is a finite set of places of K containing the archimedean ones, and $\mathcal{O}$ is the ring of $\mathcal{S}$-integers in K. Also K_v is the completion of K with respect to $v \in \mathcal{S}$, $G_v = \mathbf{G}(K_v)$, $K_{\mathcal{S}} = \prod_{v\in\mathcal{S}} K_v$, and $G = \mathbf{G}(K_{\mathcal{S}})(\equiv \prod_{v\in\mathcal{S}} G_v)$. The group $\Gamma = \mathbf{G}(\mathcal{O})$ is an $\mathcal{S}$-arithmetic lattice of G.
For simplicity, we suppose that $\mathrm{rank}_{K_v}\mathbf{G} = \mathrm{rank}_K\mathbf{G}$ for all $v \in \mathcal{S}$. We let $\mathbf{T}$ be a maximal K-split torus in $\mathbf{G}$. Denote $T_v = \mathbf{T}(K_v)$ where $v \in \mathcal{S}$. If $\mathcal{R}$ is a non-empty subset of $\mathcal{S}$ we denote $T_{\mathcal{R}} = \prod_{v\in\mathcal{R}} T_v$ and when $\mathcal{R} = \mathcal{S}$ write T instead of $T_{\mathcal{S}}$.

Definition. *An orbit $T_{\mathcal{R}}\pi(g)$ is called locally divergent if $T_v\pi(g)$ is divergent for every $v \in \mathcal{R}$.*

Theorem 12 (T. 2007). *With the above notation, we have the following statements.*

1. *If $\mathcal{R} \subsetneqq \mathcal{S}$ then the following conditions are equivalent:*
   (a) *$T_{\mathcal{R}}\pi(g)$ is closed,*
   (b) *$T_{\mathcal{R}}\pi(g)$ is divergent,*
   (c) *$\mathcal{R} = \{v\}$ and $g \in \mathcal{N}_G(T_v)\mathbf{G}(K)$ where $\mathcal{N}_G(T_v)$ is the normaliser of T_v in G;*
2. *$T\pi(g)$ is closed and divergent if and only if $g \in \mathcal{N}_G(T)\mathbf{G}(K)$ where $\mathcal{N}_G(T)$ is the normaliser of T in G;*
3. *$T\pi(g)$ is locally divergent if and only if $g \in \bigcap_{v\in\mathcal{S}} \mathcal{N}_G(T_v)\mathbf{G}(K)$;*
4. *If $|\mathcal{S}| \geq 2$ then $\mathcal{N}_G(T)\mathbf{G}(K) \subsetneqq \bigcap_{v\in\mathcal{S}} \mathcal{N}_G(T_v)\mathbf{G}(K)$ and $T\pi(g)$ is locally divergent and non-closed if and only if $g \in (\bigcap_{v\in\mathcal{S}} \mathcal{N}_G(T_v)\mathbf{G}(K)) \setminus \mathcal{N}_G(T)\mathbf{G}(K)$.*

Part 2 of the theorem is needed to obtain a result about values of homogeneous forms at integer points and its Part 4 is needed to get counter-examples to Margulis' Conjecture 6.

Rationality criterium:

Let $K_{\mathcal{S}}[\vec{x}]$ be the ring of polynomials with coefficients from $K_{\mathcal{S}}$ in n variables $\vec{x} = (x_1, \ldots, x_n)$. Note that $K_{\mathcal{S}}[\vec{x}] = \prod_{v \in \mathcal{S}} K_v[\vec{x}]$ and $K[\vec{x}]$ is diagonally imbedded and dense in $K_{\mathcal{S}}[\vec{x}]$. A polynomial $p(\vec{x}) \in K_{\mathcal{S}}[\vec{x}]$ is *rational* if $c \cdot p(\vec{x}) \in K_{\mathcal{S}}[\vec{x}]$ for some $c \in K_{\mathcal{S}}^*$.

Theorem 13 (T, 2007) *Let $f(\vec{x}) = l_1(\vec{x}) \ldots l_m(\vec{x}) \in K_{\mathcal{S}}[\ \vec{x}\ ]$ where $l_1(\vec{x}), \ldots, l_m(\vec{x})$ are linear forms which are linearly independent over $K_{\mathcal{S}}$. Then f is rational if and only if $f(\mathcal{O}^n)$ is discrete in K_S.*

In the particular case when $K = \mathbb{Q}$ and $\mathcal{S}$ consists of the unique archimedean place of $\mathbb{Q}$, we get:

Corollary. *Let $f(\vec{x}) = l_1(\vec{x}) \ldots l_m(\vec{x})$, where $l_1(\vec{x}), \ldots, l_m(\vec{x})$, are real linear forms in n variables $\vec{x} = (x_1, \ldots, x_n)$. Suppose that $l_1(\vec{x}), \ldots, l_m(\vec{x})$ are linearly independent over $\mathbb{R}$ and $f(\mathbb{Z}^n)$ is discrete in $\mathbb{R}$. Then $f(\vec{x})$ is rational, i.e. there exists $\alpha \in \mathbb{R}^*$ such that $\alpha \cdot f(\vec{x}) \in \mathbb{Z}[\vec{x}]$.*

6. Classification of the closures of the locally divergent orbits. Application.

We start our consideration with the simplest case when G is a product of $n \geq 2$ copies of $\mathrm{SL}(2,\mathbb{R})$

Theorem 14 (T, 2013) *Let* $G = \underbrace{\mathrm{SL}(2,\mathbb{R}) \times \cdots \times \mathrm{SL}(2,\mathbb{R})}_{n}$, $n \geq 2$, Γ *an irreducible non-uniform lattice in* G *and* $T = \underbrace{D \times \cdots \times D}_{n}$. *Let* $T\pi(g)$ *be a locally divergent* non-closed *orbit. Then the following holds:*

1. *if* $n = 2$, $\overline{T\pi(g)}$ *is a union of* $2 \leq s \leq 4$ *locally divergent* T*-orbits. In particular,* $\overline{T\pi(g)}$ *is never homogeneous.*
2. *if* $n > 2$, $\overline{T\pi(g)} = G/\Gamma$.

The case $n = 2$ *contradicts Conjecture 6. Completely different counter-examples to Conjecture 6 for tori action on* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ *were obtained by Francois Maucourant (2010) when* $n \geq 6$ *and* $\dim T = n-2$ *(so,* T *is not maximal) and by Uri Shapira (2011) when* $n = 3$ *and* T *is maximal.*

The closures of the locally divergent orbits for any semi-simple K-group $\mathbf{G}$ are described by the next two theorems.

Theorem 15 (T, ETDS 2021)
Let $\#\mathcal{S} = 2$. Then

1. $\overline{T\pi(g)}$ *is a union of finitely many T-orbits which are all locally divergent and stratified in terms of parabolic subgroups of $\mathbf{G} \times \mathbf{G}$;*
2. $T\pi(g)$ *is open in* $\overline{T\pi(g)}$;
3. *The following conditions are equivalent:*
   - (a) $\overline{T\pi(g)}$ *is closed,*
   - (b) $\overline{T\pi(g)}$ *is homogenous,*
   - (c) $g \in \mathcal{N}_G(T)\mathbf{G}(K)$.

Theorem 15 is a particular case of stronger but more technically formulated results. More precisely, the T-orbits contained in $\overline{T\pi(g)}$ are stratified in the following way:

1. *Given a locally divergent orbit $T\pi(g)$, we define a finite set $\mathcal{P}(g)$ of parabolic subgroups of $\mathbf{G} \times \mathbf{G}$ and associate to each $\mathbf{P} \in \mathcal{P}(g)$ a T-orbit $\mathrm{Orb}_g(\mathbf{P})$ contained in $\overline{T\pi(g)}$. We have $\mathbf{G} \times \mathbf{G} \in \mathcal{P}(g)$ and $T\pi(g) = \mathrm{Orb}_g(\mathbf{G} \times \mathbf{G})$.*
2. *If $\mathbf{P} \in \mathcal{P}(g)$ then $\overline{\mathrm{Orb}_g(\mathbf{P})} = \bigcup_{\mathbf{P}' \in \mathcal{P}(g),\ \mathbf{P}' \subset \mathbf{P}} \mathrm{Orb}_g(\mathbf{P}')$, in particular, $\overline{T\pi(g)} = \bigcup_{\mathbf{P} \in \mathcal{P}(g)} \mathrm{Orb}_g(\mathbf{P})$ and the closed T-orbits in $\overline{T\pi(g)}$ correspond to the minimal parabolic subgroups contained in $\mathcal{P}(g)$.*

Recall that *the semi-simple K-rank* of a reductive K-group $\mathbf{H}$, denoted by $\mathrm{s.s.rank}_K(\mathbf{H})$, is equal to $\mathrm{rank}_K \mathcal{D}(\mathbf{H})$ where $\mathcal{D}(\mathbf{H})$ is the derived subgroup of $\mathbf{H}$. Also, K is called a CM-*field* if it is a quadratic extension K/F where F is a totally real number field but K is totally imaginary. Note that a totally real number field is not a CM-field.

The main result for $\#\mathcal{S} > 2$ is the following.

Theorem 16 (T, ETDS 2021)
Let $\#\mathcal{S} > 2$ and K be not a CM-field. Then there exist h_1 and $h_2 \in \mathcal{N}_G(T)\mathbf{G}(K)$ and reductive K-subgroups $\mathbf{H}_1$ and $\mathbf{H}_2$ of $\mathbf{G}$ such that $\mathbf{H}_1 \subset \mathbf{H}_2$, $\mathrm{rank}_K(\mathbf{H}_1) = \mathrm{rank}_K(\mathbf{H}_2) = \mathrm{rank}_K(\mathbf{G})$,

$$\mathrm{s.s.rank}_K(\mathbf{H}_1) = \mathrm{s.s.rank}_K(\mathbf{H}_2), \tag{1}$$

and

$$h_2 H_2 \pi(e) \subseteq \overline{T\pi(g)} \subseteq h_1 H_1 \pi(e), \tag{2}$$

where $H_1 = \mathbf{H}_1(K_\mathcal{S})$, H_2 is a subgroup of finite index in $\mathbf{H}_2(K_\mathcal{S})$, and the orbits $h_1 H_1 \pi(e)$ and $h_2 H_2 \pi(e)$ are closed and T-invariant.

In the important case $\mathbf{G} = \mathbf{SL}_n$, Theorem 15 implies:

Theorem 17 (T, 2022) *Let* $\mathbf{G} = \mathbf{SL}_n$, $\#\mathcal{S} > 2$ *and* K *be not a* CM*-field. Then* $\overline{T\pi(g)} = H\pi(g)$, *where* H *is a closed subgroup of* G.

Remark: When $\mathbf{G} \neq \mathbf{SL}_n$ the closure $\overline{T\pi(g)}$ as in the formulation of Theorem 15 is, in general, not homogeneous. Also, if K is a CM-field then $\overline{T\pi(g)} \setminus T\pi(g)$ might be not contained in a countable union of closed orbits of proper subgroups of G in contrast to the orbits $T\pi(g)$ with non-homogeneous closures given in Maucourant, Shapira and the author where $\overline{T\pi(g)} \setminus T\pi(g)$ is always contained in a finite union of closed orbits of proper subgroups of G.

Closures of the set of values of irrational forms at integer points in the $\mathcal{S}$-adic case

As in Theorem 13, let $f(\vec{x}) = l_1(\vec{x}) \dots l_m(\vec{x}) \in K_{\mathcal{S}}[\,\vec{x}\,]$ where $l_1(\vec{x}), \dots, l_m(\vec{x})$ are linearly independent over $K_{\mathcal{S}}$ linear forms with coefficients from $K_{\mathcal{S}}$ in n variables $\vec{x} = (x_1, \dots, x_n)$. Recall that every $l_i(\vec{x}) = (l_i^{(v)}(\vec{x}))_{v\in S}$ and every $f_v(\vec{x}) = l_1^{(v)}(\vec{x}) \dots l_m^{(v)}(\vec{x})$. Suppose that $f(\vec{x})$ is irrational. Then, in view of Theorem 12, $f(\mathcal{O}^n)$ is not discrete. The following conjecture is plausible:

Conjecture 7. Suppose that $\#\mathcal{S} > 2$, $n \geq 2$, K is not a CM-field and f is irrational. Then $\overline{f(\mathcal{O}^n)} = K_{\mathcal{S}}$.

The form f is called locally K-decomposable if each of the linear forms $l_i^{(v)}$ is a multiple of a linear form with coefficients from K. Theorem 16 implies:

Theorem 18

Conjecture 5 holds for the locally K-decomposable homogeneous forms.

Remark: The analog of Theorem 18 (and, therefore, of Conjecture 7) is not true if $\#\mathcal{S} = 2$ or $\#\mathcal{S} \geq 2$ and K is a CM-field.

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Thank you !