Algebraic Groups with Good Reduction and Applications

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Banff June 2022

- One major direction of my current work is studying the arithmetic theory of algebraic groups over various classes of higher-dimensional fields.

- A property that has recently come to the forefront is good reduction of reductive algebraic groups.

Let K be a field equipped with discrete valuation v.

Definition

A reductive K-group G has *good reduction* at v if there exists a reductive group scheme $\mathcal{G}$ over valuation ring $\mathcal{O}_v \subset K_v$ such that

$$\textit{generic fiber } \mathcal{G} \otimes_{\mathcal{O}_v} K_v \text{ is isomorphic to } G \otimes_K K_v.$$

Then *special fiber* (reduction)

$$\underline{G}^{(v)} = \mathcal{G} \otimes_{\mathcal{O}_v} K^{(v)}$$

is *a connected reductive* group over residue field $K^{(v)}$.

Examples.

0. If G is K-split then G has a good reduction at *any* v (follows from Chevalley's construction).

1. For a central simple K-algebra A, group $G = \mathrm{SL}_{1,A}$ has good reduction at v if there exists an Azumaya algebra $\mathcal{A}$ over $\mathcal{O}_v$ such that

$$A \otimes_K K_v \simeq \mathcal{A} \otimes_{\mathcal{O}_v} K_v$$

(in other words, A is *unramified* at v).

2. $G = \mathrm{Spin}_n(q)$ has good reduction at v if (over K_v)

$$q \sim \lambda(a_1x_1^2 + \cdots + a_nx_n^2) \quad \text{with} \quad \lambda \in K_v^\times, \ \ a_i \in \mathcal{O}_v^\times$$

(assuming that $\mathrm{char}\, K^{(v)} \neq 2$).

A K-group G' is a *K-form* (or $\overline{K}/K$-form) of G if

$$G' \otimes_K \overline{K} \simeq G \otimes_K \overline{K} \quad \text{(where } \overline{K} \text{ is a sep. closure of } K\text{)}.$$

Example.

If A is a central simple algebra of degree n over K, **then**

$$A \otimes_K \overline{K} \simeq M_n(\overline{K})$$

and $G' = \mathrm{SL}_{1,A}$ is a K-form of $G = \mathrm{SL}_n$.

General problem

Let

- K be a field equipped with a set V of discrete valuations;
- G a reductive K-group.

We are interested in analyzing

K-forms of G that have good reduction at all $v \in V$.

To make this question meaningful, one needs to specialize

K, V, and G.

- Previous work has dealt mainly with the case where K is fraction field of *Dedekind ring* R, and V consists of valuations associated with *maximal ideals* of R.

- This situation was first studied in detail by G. Harder (Invent. math. **4**(1967), 165-191) and J.L. Colliot-Thélène & J.J. Sansuc (Math. Ann. **244** (1979), no. 2, 105-134).

- Case $R = \mathbb{Z}$: B.H. Gross (Invent. math. **124**(1996), 263-279) and B. Conrad (Autours des schémas en groupes, Vol. II, 193-253, 2015)

- Case $R = k[x]$: Raghunathan - Ramanathan (Proc. Indian Acad. Sci. Math. Sci. **93** (1984), no. 2-3, 137-145)

- Case $R = k[x, x^{-1}]$: Chernousov - Gille - Pianzola (Amer. J. Math. **134** (2012), no. 6, 1541-1583)

We initiated the analysis of the following higher-dimensional situation.

- Let K be a finitely generated field.
- Pick a normal integral affine model $\mathfrak{X}$ for K.
- Let V = set of discrete valuations of K associated with prime divisors on $\mathfrak{X}$ (*divisorial* set).

Algebraically: We find $R \subset K$ such that $K = \mathrm{Frac}(R)$ and

- R is a finitely generated $\mathbb{Z}$-algebra (or $\mathbb{F}_p$-algebra);
- R is integrally closed in K.

Then: V corresponds to height one prime ideals of R.

Main Finiteness Conjecture

Let:

- K a finitely generated field;
- V a divisorial set of places of K;
- G a (connected) reductive K-group.

Main Conjecture for Groups with Good Reduction

If $\mathrm{char}\, K$ *is "good," then the set of* K*-isomorphism classes of (inner)* K*-forms* G' *of* G *having good reduction at all* $v \in V$ *is finite.*

(If G is absolutely almost simple, $\mathrm{char}\, K = p$ is "good" for G if $p = 0$ or p does not divide order of Weyl group of G. For non-semisimple reductive groups only char. 0 is "good.")

Connections and applications of the Main Conjecture

This conjecture has close connections to:

- Local-global principles for algebraic groups.
- Finiteness properties of unramified cohomology.
- Study of simple algebraic groups having same isomorphism classes of maximal tori (genus problem).
- Analysis of weakly commensurable Zariski-dense subgps and applications to classical problems on locally symmetric spaces (G. Prasad-A. Rapinchuk).

Thus, study of groups with good reduction occupies a central place in the emerging arithmetic theory of algebraic groups over higher-dimensional fields.

Set-up: Global-to-local map in Galois cohomology

Let

- K be a field
- V a set of (discrete) valuations of K
- G an algebraic group over K.

One says that the Hasse principle holds if global-to-local map

$$\theta_{G,V} \colon H^1(K, G) \to \prod_{v \in V} H^1(K_v, G)$$

is *injective*.

Kernel of $\theta_{G,V}$ is called *Tate-Shafarevich set*

$$Ш(G, V) := \ker \theta_{G,V}.$$

Hasse principle over number fields

Let k = *number field*, V = set of all places of k.

- If G is *simply-connected* or *adjoint* alg. k-group, then

$$\theta_{G,V} \colon H^1(k,G) \to \prod_{v \in V} H^1(k_v, G)$$

  is injective (i.e. Hasse principle holds).

- For arbitrary alg. k-group G, the map $\theta_{G,V}$ may not be injective, but it is always *proper*; in particular, $Ш(G,V)$ is finite.

Our recent results strongly suggest the following properness conjecture for reductive groups over finitely generated fields.

Properness conjecture

Suppose

- K a finitely generated field;
- V a divisorial set of places of K.

Properness Conjecture.

If G is a (connected) reductive algebraic K-group, then $\theta_{G,V}$ is proper. In particular, the Tate-Shafarevich set $\text{Ш}(G,V)$ is finite.

Connection to groups with good reduction:

Proposition 1.

Assume Main Conjecture holds for an absolutely almost simple simply connected K-group G and all divisorial sets of places of K. **Then** *$\theta_{\overline{G},V}$ is proper for corresponding adjoint group $\overline{G}$ and any divisorial set V.*

Definition of the genus

• Let G_1 and G_2 be semisimple groups over a field K.

We say: G_1 & G_2 have *same isomorphism classes of maximal K-tori* **if** every maximal K-torus T_1 of G_1 is K-isomorphic to a maximal K-torus T_2 of G_2, and vice versa.

• Let G be an absolutely almost simple K-group.

$\mathbf{gen}_K(G)$ = set of isomorphism classes of K-forms G' of G having same K-isomorphism classes of maximal K-tori as G.

Question A. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question B. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 2. (G. Prasad-A. Rapinchuk)

Let G be an absolutely almost simple simply connected algebraic group over a number field K.

(1) $\mathbf{gen}_K(G)$ *is finite;*

(2) *If G is not of type A_n, D_{2n+1}, or E_6, then* $|\mathbf{gen}_K(G)| = 1$.

Conjectures about the genus

Conjecture 3.

(1) *For* $K = k(x)$, *k a number field, and G an absolutely almost simple simply connected K-group with* $|Z(G)| \leqslant 2$, *we have* $|\mathbf{gen}_K(G)| = 1$;

(2) *If G is an absolutely almost simple group over a finitely generated field K of "good" characteristic, then* $\mathbf{gen}_K(G)$ *is finite.*

The genus and good reduction

Theorem 4.

Let G be an absolutely almost simple simply connected group over K, and v be a discrete valuation of K.

Assume that residue field $K^{(v)}$ is finitely generated, and G has good reduction at v.

Then *every* $G' \in \mathbf{gen}_K(G)$ *has good reduction at v, and reduction* $\underline{G'}^{(v)} \in \mathbf{gen}_{K^{(v)}}(\underline{G}^{(v)})$.

Proof is based on characterizing existence of good reduction in terms of existence of (generic) maximal tori with special properties.

The theorem remains valid whenever residue field is Hilbertian. (I.R. — work in progress)

Corollary.

Let K be a finitely generated field, V a divisorial set of places of K, and G an absolutely almost simple simply connected K-group. There exists a finite subset $S \subset V$ (depending on G) such that every $G' \in \mathbf{gen}_K(G)$ has good reduction at all $v \in V \setminus S$.

Consequently, if Main Conjecture holds for all divisorial sets, then $\mathbf{gen}_K(G)$ is finite.

Thus, Main Conjecture provides a uniform approach to both the Properness Conjecture and the finiteness of the genus.

We have resolved all conjectures for algebraic tori.

Theorem 5.

Suppose K is a finitely generated field of char. 0, and V is a divisorial set of places. **Then** *for any $d \geqslant 1$, the set of K-isomorphism classes of d-dimensional K-tori having good reduction at all $v \in V$ is finite.*

- Similar result when $\operatorname{char} K = p > 0$ for tori T for which degree of splitting field $[K_T : K]$ is prime to p.

Theorem 6.

Suppose K is a finitely generated field and V is a divisorial set of places. **Then** *for any linear algebraic K-group D whose connected component D° is a torus, the global-to-local map*

$$\theta_{D,V}\colon H^1(K,D) \to \prod_{v \in V} H^1(K_v, D)$$

is proper.

In particular, for a K-torus T, the Tate-Shafarevich group

$$\text{Ш}(T,V) = \ker\left(H^1(K,T) \to \prod_{v \in V} H^1(K_v, T)\right)$$

is finite.

Classical proof of this fact for tori over number fields relies on Tate-Nakayama duality, which is not available in general.

Our proof is based on adelic considerations.

In particular, the argument shows that finiteness of $Ш(T, V)$ over number fields follows from finiteness of class number and finite generation of group of S-units.

Here is one application of Theorem 6:

Theorem 7.

Suppose K is a finitely generated field, V is a divisorial set of places, and G a connected reductive K-group. Fix a maximal K-torus $T \subset G$, and let $\mathcal{C}(T)$ be set of all maximal K-tori $T' \subset G$ such that T and T' are $G(K_v)$-conjugate for all $v \in V$. **Then** *$\mathcal{C}(T)$ consists of finitely many $G(K)$-conjugacy classes.*

(Proved by P. Gille & L. Moret-Bailly over global fields.)

Some results for semisimple groups: Inner forms of A

Theorem 8.

Suppose K is a finitely generated field, V a divisorial set of places, and $n \geqslant 2$ integer prime to $\mathrm{char}\, K$. **Then** *number of K-isomorphism classes of groups of the form $\mathrm{SL}_{1,A}$, with A central simple K-algebra of degree n, having good reduction at all $v \in V$, is finite.*

Theorem 9.

(1) *Let D be a central division algebra of exponent 2 over $K = k(x_1, \ldots, x_r)$ where k is a number field or a finite field of characteristic $\neq 2$. Then for $G = \mathrm{SL}_{m,D}$ $(m \geqslant 1)$, we have $|\mathbf{gen}_K(G)| = 1$.*

(2) *Let $G = \mathrm{SL}_{m,D}$, where D is a central division algebra over a finitely generated field K. Then $\mathbf{gen}_K(G)$ is finite.*

Following Kato, we say K is a 2-dimensional global field if

- $K = k(C)$, with C smooth geometrically integral curve over number field k; or
- $K = \mathbb{F}_q(S)$, with S smooth geometrically integral surface over finite field $\mathbb{F}_q$.

Theorem 10.

Let K be a 2-dimensional global field of char. $\neq 2$, and V divisorial set of places. Fix $n \geqslant 5$.

Then *set of K-isomorphism classes of* $\mathrm{Spin}_n(q)$ *with good reduction at all $v \in V$ is finite.*

- Similar results for some special unitary groups of types A_n, C_n and groups of type G_2.
- **More recently:** similar result for $\widetilde{\mathrm{SU}}_n(D,h)$, with D a quaternion division algebra over $K = k(C)$ and h a skew-hermitian form over D (I.R. — work in progress)

Theorem 11.

Let $K = k(C)$ be a 2-dimensional global field of char. 0 and set $G = \mathrm{Spin}_n(q)$. If $n \geqslant 5$, then $\mathbf{gen}_K(G)$ is finite.

Theorem 12.

Let G be a simple algebraic group of type G_2.

(1) *If $K = k(x)$, where k is a number field, then $|\mathbf{gen}_K(G)| = 1$;*

(2) *If $K = k(x_1, \ldots, x_r)$, where k is a number field, or K is a 2-dimensional global field of char. $\neq 2$, then $\mathbf{gen}_K(G)$ is finite.*

- Some further finiteness results over function fields of rational surfaces and certain Severi-Brauer varieties over number fields.

Some results on Properness Conjecture

Consider the global-to-local map

$$\theta_{G,V}\colon H^1(K,G) \to \prod_{v\in V} H^1(K_v,G).$$

Several cases where we have established properness of $\theta_{G,V}$:

- $\mathrm{PSL}_{1,A}$ over arbitrary finitely generated fields.

- K a 2-dimensional global field and
  - $G = \mathrm{SO}_n(q)$ $(n \geqslant 5)$;
  - G of type G_2;
  - $G = \mathrm{SU}_n(L/K,h)$, L/K quadratic extension, h nondegenerate hermitian form of $\dim \geqslant 2$;
  - $G = \mathrm{SL}_{1,A}$, A a c.s.a$/K$ of square-free degree.

- K a purely transcendental extension or function field of Severi-Brauer variety over number field and G of type G_2.

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