Short intervals containing primes

Akshaa Vatwani

Indian Institute of Technology, Gandhinagar

July 6, 2023

BIRS at University of British Columbia–Okanagan Kelowna, BC, Canada

Introduction

- The prime number theorem (PNT):

$$\pi(x) := \#\{\text{primes } p \leq x\} \sim \frac{x}{\log x},$$

  as $x \to \infty$.

- Expectation: For some $0 < \theta < 1$, we would like to have

$$\#\text{primes in } (x, x+x^\theta] = \pi(x + x^\theta) - \pi(x) \sim \frac{x + x^\theta}{\log(x + x^\theta)} - \frac{x}{\log x}$$

- Since

$$\frac{x + x^\theta}{\log(x + x^\theta)} - \frac{x}{\log x} = (1 + o(1))\frac{x^\theta}{\log x},$$

  Question: How small can we make θ so that

$$\pi(x + x^\theta) - \pi(x) \sim \frac{x^\theta}{\log x} \tag{1}$$

  holds as $x \to \infty$?

Goal

- Let p_n denote the nth prime. If (1) holds for some θ, then we also have

$$p_{n+1} - p_n \ll p_n^{\theta}, \tag{2}$$

as $n \to \infty$.

- Hoheisel was the first to prove the existence of a $\theta < 1$ such that (1) (and hence (2)) holds.
- Hoheisel: $\theta = 32999/33000$, Heilbronn: $\theta = 249/250$, Tchudakoff: $\theta = \frac{3}{4} + \epsilon$
- Our goal today is to prove Ingham's result [1]:

Theorem 1 (Ingham)

If there exists $c > 0$ such that $\zeta(\frac{1}{2} + it) \ll t^c$ as $t \to \infty$, then (1) holds for any θ satisfying

$$\frac{4c+1}{4c+2} < \theta < 1.$$

[1] On the difference between consecutive primes, *The Quarterly Journal of Mathematics*, 1937.

Some Remarks

- Even the classical value $c = \frac{1}{4} + \epsilon$ reduces θ to $\frac{2}{3} + \epsilon$.
- The Hardy-Littlewood value $c = \frac{1}{6} + \epsilon$ gives $\theta = \frac{5}{8} + \epsilon$.
- The Lindelöf hypothesis conjectures that $\zeta(\frac{1}{2} + it) \ll t^{\epsilon}$ for any $\epsilon > 0$. This would give $\theta = \frac{1}{2} + \epsilon$.
  This is comparable to Cramer's result that

$$p_{n+1} - p_n \ll p_n^{1/2} \log p_n,$$

  under the Riemann hypothesis.

Step 1: Connecting θ to zeros of $\zeta(s)$

Consider the following hypotheses.

(ZF) "Zero free region": $\zeta(s)$ has no zeros in a region of the type

$$\sigma > 1 - A\frac{\log\log t}{\log t}, \quad t > t_0,$$

where $A > 0, t_0 > 3$ are some parameters.

(ZD) "Zero-density result":
$N(\sigma, T) := \#\{\text{zeros } \rho = \beta + i\gamma \text{ of } \zeta(s) : \beta \geq \sigma, 0 < \gamma \leq T\}$ satisfies

$$N(\sigma, T) \ll T^{b(1-\sigma)}(\log T)^B$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ as $T \to \infty$ for some parameters $b > 0, B \geq 0$.

Lemma

Suppose (ZF), (ZD) hold. Then (1) holds for any θ satisfying

$$1 - \frac{1}{b + A^{-1}B} < \theta < 1.$$

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Proof of Lemma

Let $\Psi(x) = \sum_{n \le x} \Lambda(n)$. We use a truncated version of the Riemann-von Mangoldt explicit formula which connects Ψ to non-trivial zeros ρ of $\zeta(s)$:

$$\Psi(x) = x - \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|\le T}} \frac{x^\rho}{\rho} + O\left(\frac{x}{T}(\log x)^2\right),$$

uniformly for $3 \le T \le x$ as $x \to \infty$.
This gives for $0 < h \le x$,

$$\Psi(x+h) - \Psi(x) = h - \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|\le T}} \frac{(x+h)^\rho - x^\rho}{\rho} + O\left(\frac{x}{T}(\log x)^2\right).$$

$$\left|\frac{(x+h)^\rho - x^\rho}{\rho}\right| = \left|\int_x^{x+h} u^{\rho-1}\,du\right| \le \int_x^{x+h} u^{\overset{<0}{\beta-1}}\,du \ll h\,x^{\beta-1}$$

We have obtained

$$\frac{\Psi(x+h)-\Psi(x)}{h} = 1 + O\Bigg(\sum_{|\gamma|\le T} x^{\beta-1}\Bigg) + O\left(\frac{x}{Th}(\log x)^2\right). \qquad (*)$$

Goal: To show RHS ~ 1 for $h = x^{\theta}$, with $1-(b+A^{-1}B)^{-1} < \theta < 1$ and T chosen suitably.

Next step: Connecting to $N(\sigma, T)$: We write

$$\sum_{\substack{\rho=\beta+it\\ |\gamma|\le T}} x^{\beta-1} = -2\int_0^1 x^{\sigma-1} d_\sigma N(\sigma, T) \qquad (3)$$

$$= 2\sum_{0<\beta<1} x^{\beta-1} \sum_{\substack{\rho:\, \mathrm{Re}\,\rho=\beta\\ 0\le \mathrm{Im}\,\rho\le T}} 1$$

$$\approx \lim_{\epsilon\to 0} N(\beta,T)-N(\beta+\epsilon,T) \approx -d_\sigma N(\sigma,T)\Big|_{\sigma=\beta}$$

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(Intg by parts) $\underbrace{-2\, x^{\sigma-1} N(\sigma,T)\Big|_0^1}_{2x^{-1}N(0,T)} + 2\int_0^1 N(\sigma,T)\, x^{\sigma-1}(\log x)\, d\sigma$

We have obtained

$$\sum_{|\gamma|\leq T} x^{\beta-1} = 2x^{-1}N(0,T) + 2\int_0^1 N(\sigma,T)x^{\sigma-1}\log x\,d\sigma.$$

We use:

1. The known estimate $N(0,T) \ll T\log T$.
2. Hypothesis (ZF): $\zeta(s) \neq 0$ for $\sigma > 1 - A\frac{\log\log t}{\log t}, \quad t > t_0 > 3$, which means that $\exists T_0 > 3$ such that

$$N(\sigma,T) = 0 \text{ for } \sigma > 1-\eta(T),\ T \geq T_0,$$

where $\eta(T) = A(\log\log T)/\log T$.

This gives, uniformly for $x \geq T \geq T_0$,

$$\sum_{|\gamma|\leq T} x^{\beta-1} \ll \frac{T\log T}{x} + \int_0^{1-\eta(T)} N(\sigma,T)x^{\sigma-1}\log x\,d\sigma.$$

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This gives, uniformly for $x \ge T \ge T_0$,

$$\sum_{|\gamma|\le T} x^{\beta-1} \ll \frac{T\log T}{x} + \int_0^{1-\eta(T)} N(\sigma,T)x^{\sigma-1}\log x\,d\sigma.$$

Using Hypothesis (ZD), i.e. $N(\sigma, T) \ll T^{b(1-\sigma)}(\log T)^B$, we have

$$\sum_{|\gamma| \leq T} x^{\beta-1} \ll \frac{T \log T}{x} + \int_0^{1-\eta(T)} \left(\frac{T^b}{x}\right)^{1-\sigma} (\log T)^B \log x \, d\sigma.$$

Take $T = x^\alpha$, $\alpha < 1$.

$$\ll x^{\alpha-1} \log x + (\log x)^B \left[x^{(\alpha b-1)(1-\sigma)}\right]_0^{1-\eta(x^\alpha)}$$

$$\ll (\log x)^{-\delta}$$

with $\delta = A(\alpha^{-1} - b) - B$.
To ensure $\delta > 0$, we take $\alpha < \frac{1}{b+BA^{-1}}$.

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$$\ll x^{\overset{<0}{\alpha-1}} \log x + (\log x)^B \Big[x^{(\alpha b-1)(1-\sigma)} \Big]_0^{1-\eta(x^\alpha)}$$

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To ensure $\delta > 0$, we take $\alpha < \frac{1}{b+BA^{-1}}$.

Putting this into $(*)$, we have

$$\frac{\Psi(x+h)-\Psi(x)}{h} = 1 + O\left((\log x)^{-\delta}\right) + O\left(\frac{x}{Th}(\log x)^2\right). \qquad (*)$$

$x^{1-\theta-\alpha}(\log x)^2$

Put $h = x^{\theta}$, $T = x^{\alpha}$ where α can be any number satisfying

$$0 < \alpha < \frac{1}{b - BA^{-1}}.$$

Then

$$\Psi(x + x^{\theta}) - \Psi(x) \sim x^{\theta}, \qquad (4)$$

provided $\theta > 1-\alpha$, that is, for any for θ satisfying

$$1 - \frac{1}{b - BA^{-1}} < \theta < 1.$$

(4) implies $\pi(x+x^{\theta}) - \pi(x) \sim \frac{x^{\theta}}{\log x}$. □

Putting this into $(*)$, we have

$$\frac{\Psi(x+h)-\Psi(x)}{h} = 1 + O\big((\log x)^{-\delta}\big) + O\left(\frac{x}{Th}(\log x)^2\right). \qquad (*)$$

Put $h = x^\theta$, $T = x^\alpha$ where α can be any number satisfying

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Then

$$\Psi(x + x^\theta) - \Psi(x) \sim x^\theta, \qquad (4)$$

provided $\theta > 1 - \alpha$, that is, for any for θ satisfying

$$1 - \frac{1}{b - BA^{-1}} < \theta < 1.$$

(4) implies $\pi(x + x^\theta) - \pi(x) \sim \frac{x^\theta}{\log x}$. □

Step 2: Improving the value of b in (ZD)

- Recall (ZD): $N(\sigma, T) \ll T^{b(1-\sigma)}(\log T)^B$, uniformly for $\frac{1}{2} \leq \sigma \leq 1$.
- Previously known values of b?
- Hoheisel: $b = 4\sigma$.
- Titchmarsh: $b = 4/(3 - 2\sigma)$.

Ingham proves the following.

Theorem 2

If

$$\zeta\left(\frac{1}{2} + it\right) \ll t^c \tag{†}$$

for some absolute constant $c > 0$, then

$$N(\sigma, T) \ll T^{2(1+2c)(1-\sigma)}(\log T)^5,$$

as $T \to \infty$, uniformly for $\frac{1}{2} \leq \sigma \leq 1$.

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$$N(\sigma, T) \ll T^{2(1+2c)(1-\sigma)}(\log T)^5,$$

as $T \to \infty$, uniformly for $\frac{1}{2} \le \sigma \le 1$.

Obtaining an improved range of θ from Theorem 2

Recall:

Lemma

Suppose (ZF), (ZD) hold. Then (1) holds for any θ satisfying

$$1 - \frac{1}{b + A^{-1}B} < \theta < 1.$$

Theorem 2 gives (ZD) with $b = 4c + 2$.
The zero-free region hypothesis (ZF) $\zeta(s) \neq 0$ in $\sigma > 1 - A\frac{\log\log t}{\log t}$ for $t > t_0$ is known with A arbitrarily large.
Taking $A \to \infty$ and $b = 4c + 2$ in the Lemma, we obtain

$$\frac{4c+1}{4c+2} < \theta < 1$$

as needed.

Strategy of the proof of Theorem 2

Mollified function $= 1-(3M_X-1)^2$

Zero detection Method : Logarithmic.

Let $f_X(s) = \zeta(s)M_X(s) - 1$ where $M_X(s) = \sum_{n<X} \frac{\mu(n)}{n^s}$.

$$= \sum_{n=1}^{\infty} \frac{1}{n^s}\Big(\sum_{\substack{d|n \\ d<X}} \mu(d)\Big) - 1. \quad (\text{for } \mathrm{Re}(s)>1)$$

Observations about $f_X(s)$:

- For $\sigma > 1$, we have

$$f_X(s) = \sum_{n \geq X} \frac{a_X(n)}{n^s},$$

$a_X(1)=0$

$a_X(n)=0$ for $1<n<X$

with $a_X(n) = \sum_{\substack{d|n \\ d<X}} \mu(d)$. , $|a_X(n)| \leq d(n)$

- For $\sigma \geq 2$,

$$|f_X(s)|^2 \leq \left(\sum_{n \geq X} \frac{d(n)}{n^2}\right)^2 \ll \frac{1}{X}$$

as $X \to \infty$.

Strategy of the proof of Theorem 2

$h(s) = 1 - (3M_X - 1)^{2k}$
$= \zeta \cdot g$ for
some g.

- To 'pull out' a ζ from f_X, we consider

$$\begin{aligned} h(s) &= 1 - f_X^2(s) \\ &= (1 - f_X(s))(1 + f_X(s)) \\ &= \zeta(s)g(s), \end{aligned}$$

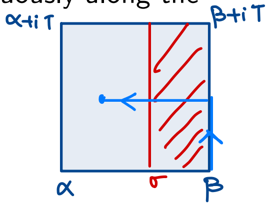
where $g = M_X(2 - \zeta M_X)$.

- Observe that $N_\zeta(\sigma, T) \leq N_h(\sigma, T)$.

- We use a result of Littlewood which relates $N_h(\sigma, T)$ for $\alpha \le \sigma \le \beta$, to integrals of the form $\int_0^T \log|h(\alpha + it)|$, $\quad \int_0^T \log|h(\beta + it)|$. More precisely:

$$2\pi \int_\alpha^\beta N_h(\sigma, T) d\sigma = \int_\alpha^\beta \big(\arg h(\sigma + iT) - \arg h(\sigma)\big) d\sigma + \int_0^T \big(\log|h(\alpha + it)| - \log|h(\beta + it)|\big) dt$$

where $\arg h(s) = 0$ at $s = \beta$ and varies continuously along the segments $[\beta, \beta + iT]$ and $[\beta + iT, \alpha + iT]$.



Strategy continued

$h = 1 - f_X^2$

- Since $\log|h| \le \log(1+|f_X|^2) \le |f_X|^2$,
  we get an upper bound for $N_\zeta(\sigma, T)$ in terms of second moments of f_X, more precisely in terms of the integrals

$$\int_1^T |f_X(\alpha+it)|^2 dt, \qquad \int_1^T |f_X(\beta+it)|^2 dt.$$

$1/2 \le \alpha \le 1$.

$\beta > 1$. Use Dirichlet series

- To deal with the second moments, we use

Claim

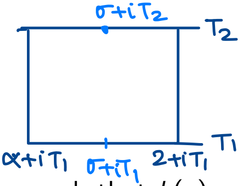
If $\zeta(\frac{1}{2}+it) \ll t^c$ for some absolute constant $c > 0$, then

$$\int_1^T |f_X(\sigma+it)|^2\, dt \ll \frac{T^{4c(1-\sigma)}}{X^{2\sigma-1}}(T+X)(\log(T+X))^4$$

uniformly for $\frac{1}{2} \le \sigma \le 1$, $T > 1$, $X > 1$.

For now, we assume this Claim.

Proof of Theorem 2



Define $h = 1 - f_X^2 = \zeta g$.

Let $\alpha \in [\frac{1}{2}, 1], \beta = 2$. Let $T_1 \in (3, 4)$, $T_2 \in (T, T+1)$ be such that $h(s)$ has no zeros on the segments $[\alpha + iT_j, \beta + iT_j], \forall j = 1, 2$.

Writing $N_h(\sigma; T_1, T_2) = N_h(\sigma, T_2) - N_h(\sigma, T_1)$, from the previous exercise, we get

$$\begin{aligned} 2\pi \int_\alpha^\beta N_h(\sigma; T_1, T_2) d\sigma &= \int_\alpha^\beta \big(\arg h(\sigma + iT_2) - \arg h(\sigma + iT_1) \big) d\sigma \\ &+ \int_{T_1}^{T_2} \big(\log |h(\alpha + it)| - \log |h(\beta + it)| \big) dt \\ &= I_1 + I_2 \qquad \text{(say)} \end{aligned}$$

Proof (contd.): Upper bound for I_2

$|f_X(2+it)|^2 \le \left(\sum_{n>X}^{\infty} \frac{d(n)}{n^2}\right)^2$

$\ll \frac{1}{X}$

We use $\log|h(s)| \le \log(1+|f_X(s)|^2) \le |f_X(s)|^2$ for $s = \alpha + it, \beta + it = 2 + it$.
For the latter, the second observation on $f_X(s)$ yields $\log|h(2+it)| \ll \frac{1}{X}$.
Hence,

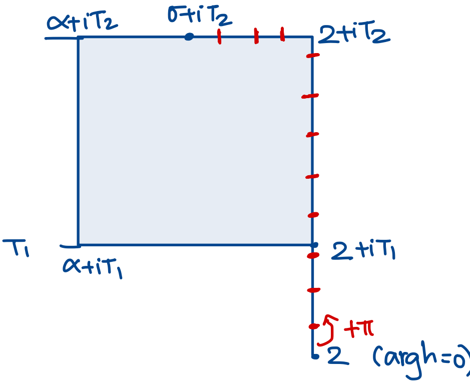
$$|I_2| \ll \int_{T_1}^{T_2} |f_X(\alpha+it)|^2 dt + \int_{T_1}^{T_2} \frac{1}{X} dt$$
$$\ll \frac{T^{4c(1-\alpha)}}{X^{2\alpha-1}} (T+X)(\log(T+X))^4 + \frac{T}{X}$$

using claim. yet to be proved

Proof (contd.): Upper bound for arg $h(\sigma + iT_j)$

Claim

For any $\sigma \in [\alpha, \beta]$, and $j = 1, 2$, $\arg h(\sigma + iT_j) \leq (m_j + 1)\pi$, where m_j is the number of points at which h is purely imaginary on $[2, 2 + iT_j] \cup [2 + iT_j, \alpha + iT_j]$.



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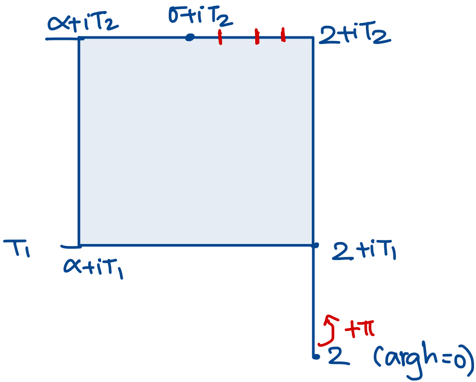
On $\sigma = 2$ line

$$\operatorname{Re} h(s) = \operatorname{Re}\left(1 - f_x^2(s)\right)$$
$$\geqslant 1 - \left(\sum_{n>x} \frac{d(n)}{n^2}\right)^2$$
$$> \frac{1}{2} \text{ for all } x \text{ suff large}$$



But

$$
\begin{aligned}
m_j &= \#\{\sigma \in [\alpha, 2] : \operatorname{Re} h(\sigma + iT_j) = 0\} \\
&\le \#\Big\{\sigma \in [\tfrac{1}{2}, 2] : \frac{1}{2}\Big(h(\sigma + iT_j) + h(\sigma - iT_j)\Big) = 0\Big\}
\end{aligned}
$$

Writing $H_j(s) = \frac{1}{2}\Big(h(\sigma + iT_j) + h(\sigma - iT_j)\Big)$, we see that

$$
m_j \le \#\{\text{zeros of } H_j(s) \text{ in the disc } |s-2| \le \tfrac{3}{2}\}
$$

We use an application of Jensen's formula: *If $f(z)$ is analytic on the open disc $D = \{z \in \mathbb{C} : |z - z_0| < R\}$ and $|f(z)| \le M$ on the boundary of D, then the number of zeros of f in $|z - z_0| < r$ is at most*

$\frac{3.5}{2}$ $\quad z_0 = 2 \quad r = 3/2$

$$
\frac{1}{\log(R/r)} \log\left(\frac{M}{|f(z_0)|}\right)
$$

$M \ll \max_{\substack{\sigma \ge 1/2 \\ 1 \le t \le T}} |h(s)|$

$H_j(2) \ge 1/2$

$$h = 1 - f_x^2$$

$$f_x = 3M_x - 1$$

We obtain

$$m_j \ll \log\left(\max_{\sigma\ge\frac{1}{2},1\le t\le T} |h(s)|\right) \ll \log(T+X),$$

using known bounds on $\zeta(s)$.
Thus

$$|I_1| := \left|\int_\alpha^\beta \big(\arg h(\sigma+iT_2) - \arg h(\sigma+iT_1)\big)\, d\sigma\right|$$
$$\ll (\beta-\alpha)\pi(m_2+m_1) \ll \log(T+X).$$

Proof (contd.): An upper bound for $\int_\alpha^\beta N_h(\sigma; T_1, T_2)d\sigma$

Putting together the upper bounds for $|I_1|$ and $|I_2|$, we have obtained for any $\alpha \in [\frac{1}{2}, 1]$,

$$\int_\alpha^2 N_h(\sigma; T_1, T_2)d\sigma \ll \frac{T^{4c(1-\alpha)}}{X^{2\alpha-1}}(T+X)(\log(T+X))^4 + \frac{T\cancel{(\log X)^2}}{X} + \log(T+X)$$

$$\ll \frac{T^{4c(1-\alpha)}}{X^{2\alpha-1}}(T+X)(\log(T+X))^4$$

Now, for any $0 < \delta < 1$, we have

$$\int_\alpha^2 N_h(\sigma; T_1, T_2)d\sigma \geq \int_\alpha^{\alpha+\delta} N_\zeta(\sigma; T_1, T_2)d\sigma \gg \delta N_\zeta(\alpha+\delta; T),$$

since $T_1 \asymp 1$, $T_2 \asymp T$.

Proof (contd.): An upper bound for $\int_\alpha^\beta N_h(\sigma; T_1, T_2)d\sigma$

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$$\ll \frac{T^{4c(1-\alpha)}}{X^{2\alpha-1}}(T+X)(\log(T+X))^4$$

Now, for any $0 < \delta < 1$, we have $\left(\delta = \frac{1}{\log T}\right)$

$$\int_\alpha^2 N_h(\sigma; T_1, T_2)d\sigma \geq \int_\alpha^{\alpha+\delta} N_\zeta(\sigma; T_1, T_2)d\sigma \gg \delta N_\zeta(\alpha+\delta; T),$$

since $T_1 \asymp 1, T_2 \asymp T$.

Proof (contd.): An upper bound for $N(\sigma; T)$

Putting $\alpha + \delta = \sigma$, we have obtained for $\sigma \in [\frac{1}{2} + \delta, 1]$,

$$\begin{aligned} N_\zeta(\sigma, T) &\ll \frac{1}{\delta} \frac{T^{4c(1-\sigma+\delta)}}{X^{2\sigma-1-2\delta}} (T + X)(\log(T + X))^4 \\ &\ll T^{4c(1-\sigma)+2(1-\sigma)} (\log T)^5, \end{aligned}$$

taking $T = X$ and $\delta = (\log T)^{-1}$. For the 'missing' region $\sigma \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{\log T}]$, we use the known bound

$$\begin{aligned} N(\sigma, T) &\ll T \log T \\ &\ll T^{2(1-\sigma)} (\log T)^5, \end{aligned}$$

to complete the proof. $\square$

Recall that we used an estimate for the second moment of f_X: $f_X = 3M_X - 1$

Claim

If $\zeta(\frac{1}{2} + it) \ll t^c$ for some absolute constant $c > 0$, then

$$\int_1^T |f_X(\sigma + it)|^2 \, dt \ll \frac{T^{4c(1-\sigma)}}{X^{2\sigma-1}} (T + X)(\log(T + X))^4$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$, $T > 1$, $X > 1$.

Ideas to prove this:

- Get an estimate for the moment when $\sigma = 1 + \delta$ where $0 < \delta < 1$:

$$\int_0^T |f_X(1 + \delta + it)|^2 dt \ll \left(\frac{T}{X} + 1\right) \frac{1}{\delta^4}$$

- Using $\zeta(\frac{1}{2} + it) \ll t^c$, obtain an estimate when $\sigma = 1/2$:

$$\int_0^T |f_X(\tfrac{1}{2} + it)|^2 dt \ll T^{2c}(T + X) \log X$$

Bound for $\int_0^T |f_X(1+\delta+it)|^2 dt$

$f_X(s) = \sum \frac{a_X(n)}{n^s}$

$Re(s) > 1$

$$\int_0^T |f_X(1+\delta+it)|^2 dt = \sum_{n,m \geq X} \frac{a_X(n) a_X(m)}{(nm)^{1+\delta}} \int_0^T (m/n)^{it} dt$$

$$\leq T \sum_{m=n \geq X} \frac{d(n)^2}{n^{2+2\delta}} + 4 \sum_{n>m \geq X} \frac{d(m)d(n)}{(nm)^{1+\delta} \log(n/m)}$$

Using the inequality $(\log \lambda)^{-1} < 1 + \lambda^{-1}(\log \lambda)^{-1} < 1 + \lambda^{-1/2}(\log \lambda)^{-1}$ for $\lambda > 1$ and the known bound

$$\sum_{m<n\leq t} \frac{d(m)d(n)}{\sqrt{mn}\log(n/m)} \ll t(\log t)^3,$$

one gets

$$\int_0^T |f_X(1+\delta+it)|^2 dt \ll \frac{1}{\delta^4}\left(1 + \frac{T}{X}\right)$$

Bound for $\int_0^T |f_X(\frac{1}{2}+it)|^2 dt$

$f_X = 3M_X - 1$

Using the inequality $(\log\lambda)^{-1} < \lambda(\lambda-1)^{-1} < 1+\sqrt{\lambda}(\lambda-1)^{-1}$ for $\lambda > 1$, one can obtain

$$\begin{aligned}\int_0^T |M_X(\tfrac{1}{2}+it)|^2 dt &\le T\sum_{n<X}\frac{\mu^2(n)}{n} + 4\sum_{m<n<X}\frac{|\mu(n)||\mu(m)|}{(mn)^{1/2}\log(n/m)} \\ &\ll T\log X + \sum_{m<n<X}\left(\frac{1}{\sqrt{mn}}+\frac{1}{n-m}\right) \\ &\ll (T+X)\log X\end{aligned}$$

Assuming $\zeta(\frac{1}{2}+it) \ll t^c$, one deduces that

$$\int_0^T |f_X(\tfrac{1}{2}+it)|^2 dt \ll T^{2c}(T+X)\log X$$

- Use a convexity result for integrals, by Hardy, Ingham and Polya [2]

Theorem

Suppose that in some strip $S : \alpha < \mathrm{Re}(s) < \beta$,

1. *$f(z)$ is analytic*
2. *$f(z) \ll \exp(e^{k|\operatorname{Im} z|})$, for some $0 < k < \pi/(\beta - \alpha)$, uniformly in S*
3. *$|f(z)|$ is continuous in any compact subset of the closed strip $\alpha \leq \mathrm{Re}(s) \leq \beta$*
4. *The integral $J(x) = \int_{-\infty}^{\infty} |f(x + iy)|^p dy$ is convergent when $x = \alpha$ or $x = \beta$. Then $\log J(x)$ is a convex function of x, so that*

$$J(x) \leq (J(\alpha))^{\frac{\beta - x}{\beta - \alpha}} (J(\beta))^{\frac{x - \alpha}{\beta - \alpha}}$$

Let $\Phi(s) = \dfrac{s-1}{s\cos(s/2\tau)} f_x(s)$ $(\tau > 3/\pi)$ Put

$$J(\sigma) = \int_{-\infty}^{\infty} |\Phi(\sigma + it)|^2 dt.$$

[2] Theorems concerning mean values of analytic functions, Proc. Royal Soc. A, 1927

Primes between consecutive large powers

- Legendre conjectured that there exists a prime between every pair of consecutive squares n^2 and $(n+1)^2$. This is unresolved even under RH.
- An easier question: Does there exist a prime between every pair of consecutive cubes?
  This is easier because the interval $(x^3, (x+1)^3)$ contains the interval $(y^2, (y+1)^2)$ if we take $y = x^{3/2}$.
- In general, the existence of primes between consecutive m-th powers implies the existence of primes between consecutive $(m+1)$th powers.
- To obtain a prime between n^m and $(n+1)^m$ for all sufficiently large n, it is sufficient to show that there exists a prime p in the interval

$$(x, x + mx^{\frac{m-1}{m}}) \text{ for all } x \text{ sufficiently large}$$

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$x = n^m$

$$(x, (x^{1/m}+1)^m)$$

$$(x, x + m x^{\frac{m-1}{m}} + \cdots + m x^{1/m} + 1)$$

$$(x, x + mx^{\frac{m-1}{m}}) \text{ for all } x \text{ sufficiently large}$$

To get primes between consecutive cubes n^3 and $(n+1)^3$ with n sufficiently large, we need

$$\pi(x + 3x^{2/3}) - \pi(x) > 0$$

for all x sufficiently large.
Ingham's result gives

$$\zeta(\tfrac{1}{2} + it) \ll t^c \implies \pi(x + x^{\theta}) - \pi(x) > 0$$

for all x sufficiently large, with $\theta = \frac{4c+1}{4c+2}$.
Let's use the known exponent $c = \frac{1}{6} + \epsilon$ to get $\theta = \frac{5}{8} + \epsilon$.
Since $(x, x + 3x^{2/3}] \subseteq (x, x + x^{5/8+\epsilon}]$, this gives primes between consecutive cubes *for all sufficiently large cubes.*

Explicit short-interval results

- Dudek (2016): There exists at least one prime between n^3 and $(n+1)^3$ for all $n \geq \exp(e^{33.3})$.
  There is at least one prime between n^m and $(n+1)^m$ for all $n \geq 1$ with $m = 5 \cdot 10^9$.
- Cully-Hugill (2023): There exists at least one prime between n^3 and $(n+1)^3$ for all $n \geq \exp(e^{32.537})$.
- Cully-Hugill and Johnston (2023): There is at least one prime between n^{140} and $(n+1)^{140}$ for all $n \geq 1$.

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Thank You