Artin L-functions and the proof of the Chebotarev Density Theorem

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This lecture is based on material found in

- J. Lagarias and A. Odlyzko. Effective versions of the Chebotarev density theorem. In Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), pages 409–464. Academic Press, London, 1977
- M. R. Murty and V. K. Murty. Non-vanishing of L-functions and applications, volume 157 of Progress in Mathematics. Birkhauser Verlag, Basel, 1997.
- A. Zaman, Analytic estimates for the Chebotarev Density Theorem and their applications, Ph.D. thesis, University of Toronto, 2017

Algebraic Setup

$$\begin{array}{ccc} \mathcal{P} & O_L & L \\ & & | \\ P & O_K & K \\ & & | \\ p & & \mathbb{Q} \end{array}$$

Let L/K be a Galois extension of number fields with Galois group $G = Gal(L/K)$:

- $n_L = [L : \mathbb{Q}]$ and $n_K = [K : \mathbb{Q}]$
- $\Delta_L = |disc(L/\mathbb{Q})|$ and $\Delta_K = |disc(K/\mathbb{Q})|$
- $N := N_{\mathbb{Q}}^{K}$

We use P to denote prime ideals in O_K and $\mathcal{P}$ to denote prime ideals in O_L. Almost always in these notes, P and $\mathcal{P}$ will be related as follows

$$\mathcal{P} | PO_L \iff \mathcal{P} \cap O_K = P.$$

$$PO_L = \mathcal{P}_1^e \mathcal{P}_2^e \cdots \mathcal{P}_r^e$$

Associated with a prime ideal $\mathcal{P}$ in O_L we recall the subgroups of G given by

$$D_{\mathcal{P}} = \{\sigma \in G : \mathcal{P}^{\sigma} = \mathcal{P}\}$$

$$I_{\mathcal{P}} = \{\sigma \in G : \sigma(x) \equiv x \mod \mathcal{P} \text{ for all } x \in L\}$$

It is known that $I_{\mathcal{P}}$ is a normal subgroup of $D_{\mathcal{P}}$ and the quotient group $D_{\mathcal{P}}/I_{\mathcal{P}}$ is cyclic generated by the Frobenius automorphism $\sigma_{\mathcal{P}}$ that satisfies

$$\sigma_{\mathcal{P}}(x) \equiv x^{N(P)} \mod \mathcal{P}$$

$\sigma_{\mathcal{P}}$ is well-defined modulo $I_{\mathcal{P}}$

if $\mathcal{P}_1$, $\mathcal{P}_2$ lie above P, then $\sigma_{\mathcal{P}_1}$ and $\sigma_{\mathcal{P}_2}$ are conjugates in G.

The Artin Symbol

Notice that $|I_{\mathfrak{p}}| = e$

so if P is unramified in L then $e = 1$

and $I_{\mathfrak{p}} = 1$ and so $\sigma_{\mathfrak{p}}$ is well-defined.

We will denote by σ_P the conjugacy class

of $\sigma_{\mathfrak{p}}$ in G.

We use the symbol $\left[\frac{L/K}{P}\right] = \sigma_P$

Chebotarev Density Theorem (CDT)

Let C be a conjugacy class of G. Set

$$\pi_C(x; L/K) = \left| \left\{ P \subset O_K : P \text{ unramified in } L,\ \left[\frac{L/K}{P}\right] = C,\ N(P) \le x \right\} \right|.$$

Theorem (Chebotarev 1926)

As $x \to \infty$, we have

$$\pi_C(x; L/K) \sim \frac{|C|}{|G|} Li(x),$$

where $Li(x) = \int_2^x \frac{1}{\log t}\, dt.$ $\sim \frac{x}{\log x}$

let $L = K$

$$\pi(x) = |\{ \mathfrak{p} \subset \mathcal{O}_K \ ; \ N(\mathfrak{p}) \leq x \}|$$

$$\sim \mathrm{Li}(x)$$

PNT for number fields

$L = \mathbb{Q}\left(e^{\frac{2\pi i}{q}}\right)$ $\qquad K = \mathbb{Q}.$

$(a, q) = 1$

$$\text{CDT} \Longrightarrow |\{ p \equiv a \bmod q \ ; \ N(p) \leq x \}|$$

$$\sim \frac{1}{\varphi(q)} \mathrm{Li}(x).$$

Effective CDT under GRH

$L/K \qquad G = \mathrm{Gal}(L/K)$

Theorem (Lagarias-Odlyzko 1977)

Assume GRH for $\zeta_L(s)$. There exists effectively computable positive absolute constant c_1 such that for every $x > 2$ we have

$$\left|\pi_C(x; L/K) - \frac{|C|}{|G|} Li(x)\right| \le c_1 \left(\frac{|C|}{|G|} x^{\frac{1}{2}} \log(\Delta_L x^{n_L}) + \log \cancel{\Delta_L} \right).$$

Refinement by Serre 1984

Least Prime Ideal in CDT

Corollary (Lagarias-Odlyzko 1977)

Assume GRH for $\zeta_L(s)$. For every conjugacy class C of G, there exists a prime ideal $P \subset O_K$ unramified in L such that $\left[\frac{L/K}{P}\right] = C$ *and*

can be removed

$$N(P) \leq c_2(\log \Delta_L)^2 \cancel{(\log\log \Delta_L)^4}$$

for some effectively computable positive absolute constant c_2.

Theorem (Fiori 2019)

There exist infinitely many number fields L Galois over $\mathbb{Q}$ for which the smallest prime $p \in \mathbb{Z}$ which splits completely in L satisfies $p \geq (\log \Delta_L)^{2+o(1)}$.

Unconditional Effective CDT

Theorem

There exists effectively computable absolute positive constants c_3, c_4 *such that if* $\log x > 10 n_L (\log \Delta_L)^2$*, then*

$$\left| \pi_C(x; L/K) - \frac{|C|}{|G|} Li(x) \right| \leq \frac{|C|}{|G|} Li(x^\beta) + c_3 x \exp\left(-c_4 \sqrt{\frac{\log x}{n_L}} \right).$$

Here β *is the possible exceptional zero of* $\zeta_L(s)$ *in the region*

$$1 - \frac{1}{4 \log \Delta_L} \leq \Re(s) \leq 1 \qquad |\Im(s)| \leq \frac{1}{4 \log \Delta_L}.$$

Idea of Lagarias-Odlyzko's Proof of Effective CDT

$$\pi_C(x; L/K) = \sum_{\substack{P \subset O_K \\ P \text{ unramified in } L \\ \left[\frac{L/K}{P}\right] = C}} 1$$

Seek asymptotic for the prime counting function $\pi_C(x; L/K)$.

We instead consider the prime power counting function

$$\psi_C(x; L/K) = \sum_{\substack{m,\ P \subset O_K \\ P \text{ unramified} \\ \left[\frac{L/K}{P}\right]^m = C}} \log N(P)$$

Relate $\psi_C(x; L/K)$ to a contour integral involving the Dirichlet series

$$\sum_{m=0}^{\infty} \sum_{\substack{P \subset O_K \\ P \text{ unramified in } L}} \theta(P^m) \frac{\log N(P)}{N(P)^{ms}}$$

$$\theta(P^m) = \begin{cases} 1 & \left[\frac{L/K}{P}\right]^m = C \\ 0 & \text{otherwise.} \end{cases}$$

Classical PNT

$$\psi(x) = \sum_{n \le x} \Lambda(n)$$

$$= \sum_{p^m \le x} \log p$$

$$-\frac{\zeta'}{\zeta}(s)$$

$$\sigma > 1 \qquad \psi(x) = \int_{\sigma - iT}^{\sigma + iT} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \frac{x^s}{s}\, ds$$

$$+ R(x, T)$$

$$\boxed{g \in C.}$$

Define

$$F_C(s) = \sum_{m \geq \text{[redacted]}} \sum_{P \subset O_K} \theta(P^m) \log N(P) N(P)^{-ms}$$

with $|\theta(P^m)| \leq 1$ for ramified primes P.

Step 1:

$$\psi_C(x, L/K) = \underbrace{\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} F_C(s) \frac{x^s}{s}\, ds}_{I_C(x,T)} + R_1(x,T)$$

Step 2: Use Deuring reduction to express $F_C(s)$ as a linear combination of logarithmic derivatives of Hecke L-functions:

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \frac{L'}{L}(s, \chi)$$

1-dimensional characters

Hecke L-function

Step 3: consider

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{L'}{L}(s,\chi)\,\frac{x^s}{s}\,ds.$$

move the line of integration to the left, apply Cauchy's residue theorem and collect residues of the integrand at poles contained within some contour. This will give rise to a sum

$$S_\chi(x,T) = \sum_{\substack{\rho \\ |\gamma|<T}} \frac{x^\rho}{\rho} + \sum_{|\rho|<\frac{1}{2}} \frac{1}{\rho}.$$

Step 4: Estimate $S_\chi(x,T)$ with or without GRH.

Step 5: Choose T to minimize error term

Step 6: partial summation to go from

ψ_C to π_C.

Hecke L-functions

Consider the finite extension $K/\mathbb{Q}$. Let $\mathfrak{f}$ be an integral ideal in O_K.

Let $I(\mathfrak{f})$ be the group of fractional ideals of K relatively prime to $\mathfrak{f}$. Let $P(\mathfrak{f})$ be the subgroup of principal ideals (α) in $I(\mathfrak{f})$ such that $\alpha \equiv 1 \mod \mathfrak{f}$ and α is totally positive.

We set $Cl(\mathfrak{f}) = I(\mathfrak{f})/P(\mathfrak{f})$, the ray class group of K modulo $\mathfrak{f}$.

Hecke characters are characters of $Cl(\mathfrak{f})$. For a subgroup A of $I(\mathfrak{f})$ containing $P(\mathfrak{f})$, we use the notation $\chi \mod A$ for a character χ of $Cl(\mathfrak{f})$ such that $\chi(A) = 1$.

View χ as a function on $I(O_K)$ by setting $\chi(\mathfrak{a}) = 0$ for all $(\mathfrak{a}, \mathfrak{f}) \neq 1$.

The L-function associated with a primitive character $\chi \mod \mathfrak{f}$ is given by

$$L(s,\chi,K) = \sum_{\substack{\mathfrak{a}\subset O_K \\ \mathfrak{a}\neq 0}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} \qquad \Re(s) > 1.$$

We set

$$\gamma(s,\chi) = \left(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)^{a(\chi)} \left(\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right)^{b(\chi)},$$

where $a(\chi), b(\chi)$ are non-negative integers with $a(\chi) + b(\chi) = n_K$. We also set

$$\begin{aligned}\Lambda(s,\chi,K) &= (s(s-1))^{\delta(\chi)}(\Delta_K N(\mathfrak{f}))^{\frac{s}{2}}\gamma(s,\chi)L(s,\chi,K) \\ &= e^{A(\chi)+B(\chi)s}\prod_{\rho}(1-\frac{s}{\rho})e^{\frac{s}{\rho}}\end{aligned}$$

$\Lambda(s,\chi,K)$ is entire and satisfies

$$\Lambda(s,\chi,K) = W(\chi)\Lambda(1-s,\overline{\chi},K)$$

If χ primitive principal character, then $L(s, \chi, K) = \zeta_K(s)$.

The trivial zeros of $L(s, \chi, K)$ are denoted by ω with

$$ord_{s=\omega} L(s, \chi, K) = \begin{cases} a(\chi) - \delta(\chi) & \omega = 0 \\ b(\chi) & \omega = -1, -3, -5, \cdots \\ a(\chi) & \omega = -2, -4, -6, \cdots \end{cases}$$

Finally, we have

$$-\frac{L'}{L}(s, \chi, K) = \delta(\chi)\left(\frac{1}{s} - \frac{1}{s-1}\right) + \frac{1}{2}\log\left(\Delta_K N(\mathfrak{f})\right)$$
$$+ \frac{\gamma'}{\gamma}(s, \chi) - B(\chi) - \sum_\rho \frac{1}{s-\rho} - \frac{1}{\rho}.$$

$$\begin{cases} 1 & \chi = \chi_0 \leftarrow \text{principal}. \\ 0 & \text{otherwise} \end{cases}$$

Artin L-functions

Let L/K be a Galois extension with Galois group $G = Gal(L/K)$.
Let

$$\phi : G \to GL_n(\mathbb{C})$$

underlying vector space is V.

$\chi(g) = Tr(\phi(g))$

be a representation of G whose character we denote by χ.
The Artin L-function associated to ϕ is given by

$$L(s, \phi, L/K) = \prod_{P \subset O_K} L_P(s, \phi, L/K)$$

with

$$L_P(s, \phi, L/K) = \begin{cases} det\left(I_n - \phi(\sigma_P)N(P)^{-s}\right)^{-1} & P \text{ is unramified in } L \\ det\left(I_n - \phi(\sigma_{\mathcal{P}})_{|_{V^{I_{\mathcal{P}}}}} N(P)^{-s}\right)^{-1} & P \text{ is ramified in } L \end{cases}$$

exercise well defined.

Artin Showed that

- $L(s, \phi_1 \oplus \phi_2, L/K) = L(s, \phi_1, L/K)L(s, \phi_2, L/K)$
- If H is a subgroup of G and τ is representation of H, then

$$L(s, Ind_H^G \tau, L/K) = L(s, \tau, L/L^H)$$

Theorem (Brauer 1947)

If χ is an irreducible character of G, then there exists subgroups H_i of G, $m_i \in \mathbb{Z}$ and 1-dimensional characters ψ_i of H_i such that $\chi = \sum_i m_i Ind_{H_i}^G \psi_i$

Brauer's induction and Artin's reciprocity theorems give

Corollary

$L(s, \chi, L/K)$ *admits a meromorphic continuation.*

$$\chi = \sum_i m_i \, \mathrm{Ind}_{H_i}^{G}(\psi_i)$$

$$L(s, \chi, L/K) = \prod_i L(s, \mathrm{Ind}_{H_i}^{G}(\psi_i), L/K)^{m_i}$$

$$= \prod_i L(s, \psi_i, L/L^H)^{m_i}$$

$$= \prod_i L(s, \pi_{\psi_i}, L^H)^{m_i}$$

Hecke L-function.

Artin Conjecture: ϕ irreducible. If ϕ is not not the trivial representation then $L(s, \phi, L/K)$ is entire.

Artin's Reciprocity

Let H be an Abelian subgroup of $G = Gal(L/K)$, and let χ is a 1-dimensional character of H. Let $E = L^H$.

There exist an integral ideal $\mathfrak{f}$ attached to the extension L/E and a subgroup A of $I(\mathfrak{f})$ such that $I(\mathfrak{f})/A \cong Gal(L/E)$. We get $\cong Gal(L/L^H) = H$.

$$L(s, \chi, L/E) = L(s, \tilde{\chi}^*, E)$$

for some primitive Hecke L-function $L(s, \tilde{\chi}^*, E)$.

Step 1 : L/K $\quad G = Gal(L/K)$

ϕ irreducible rep of G with character χ

Define $\chi_K(P^m) = \frac{1}{|I_P|} \sum_{\alpha \in I_P} \chi(\sigma_P^m \alpha)$

if P is unramified in L, then $I_P = 1$,

$$\chi_K(P^m) = \chi(\sigma_P^m)$$

$$L_{\text{unramified}}(s, \chi, L/K) = \prod_{P \text{ unramified}} \det(I_n - \phi(\sigma_P) N(P)^{-s})^{-1}$$

$$\log L_{\text{unramified}} = -\sum_{P \text{ unramified}} \log(\det(I_n - \phi(\sigma_P) N(P)^{-s}))$$

$$= -\sum_{P \text{ unramified}} \log(\det(I_n - \sigma \phi(\sigma_P) \sigma^{-1} N(P)^{-s}))$$

$$= -\sum_{P \text{ unramified}} \log\left((1-\lambda_1 N(P)^{-s}) \cdots (1-\lambda_n N(P)^{-s})\right)$$

$$= -\sum_{P \text{ unramified}} \sum_i \underbrace{\log(1-\lambda_i N(P)^{-s})}$$

$$= \sum_{P \text{ unramified}} \sum_i \sum_{m=1}^{\infty} \frac{\lambda_i^m N(P)^{-ms}}{m}$$

$$= \sum_{p \text{ unramified}} \sum_{m=1}^{\infty} \frac{1}{m} N(p)^{-ms} \operatorname{Tr}(\rho(\sigma_p^m))$$

$$= \sum_{p \text{ unramified}} \sum_{m=1}^{\infty} \frac{1}{m} N(p)^{-ms} \chi(\sigma_p^m)$$

$$= \sum_{p \text{ unramified}} \sum_{m=1}^{\infty} \frac{1}{m} N(p)^{-ms} \chi_K(p^m)$$

be more careful with ramified primes to get

$$\log L(s, \chi, L/K) = \sum_{p} \sum_{m=1}^{\infty} \chi_K(p^m) \frac{\log N(p)}{N(p)^{ms}}$$

$$-\frac{L'}{L}(s,\chi,L/K) = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \chi_K(\mathfrak{p}^m) \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{ms}}$$

Let C be a conjugacy class and let $g \in C$.

$$f_C = \sum_{\chi \in \mathrm{Irr}(G)} \overline{\chi}(g)\, \chi \qquad f_C : G \to \mathbb{C}$$

$$f_C(y) = \begin{cases} \frac{|G|}{|C|} & \text{if } y \in C \\ 0 & \text{otherwise.} \end{cases}$$

Consider $F_C(s) = \frac{-|C|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \overline{\chi}(g) \frac{L'}{L}(s,\chi,L/K)$

$$F_C(s) = \frac{-|C|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \bar{\chi}(g) \frac{L'}{L}(s, \chi, K/F)$$

$$= \frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \sum_{p, m} \chi_K(p^m) \frac{\log N(p)}{N(p)^{ms}}$$

$$= \frac{|C|}{|G|} \sum_{p, m} \frac{\log N(p)}{N(p)^{ms}} \sum_{\chi} \bar{\chi}(g) \chi_K(p^m)$$

$$= \boxed{\frac{|C|}{|G|}} \sum_{p, m} \frac{\log N(p)}{N(p)^{ms}} \frac{1}{|I_p|} \underbrace{\sum_{\alpha \in I_p} \sum_{\chi} \bar{\chi}(g) \chi(\sigma_p^m \alpha)}$$

$$= \sum_{p, m} \frac{\log N(p)}{N(p)^{ms}} \Theta(p^m)^{f_C(\sigma_p^m \alpha)}$$

Punramfie $\left[\frac{K/K}{\sigma_p}\right]^m \in C$

$$\Theta(p^m) = \begin{cases} 1 & \\ 0 & p \quad '' \quad \notin C \\ \leq 1 \text{ in abs val} & p \text{ ramified} \end{cases}$$

~~Perron's Formula~~

$\sigma_0 > 1, \quad x \geq 2.$

Define $I_C(x,T) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} F_C(s) \frac{x^s}{s} \, ds$

Perron $\Rightarrow$

$$\left| I_C(x,T) - \sum_{\substack{p,m \\ N(p^m) \leq x}} \theta(p^m) \log N(p) \right|$$

$$\leq R_0(x,T)$$

$$\left| \sum_{\substack{p \text{ ramified} \\ N(p^m) \leq x}} \theta(p^m) \log N(p) \right| \leq \sum_{\substack{p \text{ ramified} \\ N(p^m) \leq x}} \log N(p)$$

$$\leq \log x \log D.$$

Step 2:

C conjugay of G

$g \in C$.

$H = \langle g \rangle \subset G$

$E = L^H$ $\qquad$ H is cyclic $\Rightarrow$ its irred. rep are 1-dim

Claim $\displaystyle\sum_{\chi \in Irr(G)} \bar{\chi}(g)\, \chi = \sum_{\psi \in Irr(H)} \bar{\psi}(g)\, \psi^*$

where $\psi^* = Ind_H^G \psi$

Excercise: verify this.

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\chi \in Irr(G)} \bar{\chi}(g) \frac{L'}{L}(s, \chi, L/K)$$

$$F_C(s) = -\frac{|C|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \bar{\chi}(g) \frac{L'}{L}(s, \chi, L/K)$$

$$= \frac{|C|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \bar{\chi}(g) \sum_{\mathfrak{p}, m} \frac{\log N(\mathfrak{p})\, \chi(\mathfrak{p}^m)_K}{N(\mathfrak{p})^{ms}}$$

$$= \frac{|C|}{|G|} \sum_{\mathfrak{p}, m} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{ms}} \frac{1}{|I_{\mathfrak{P}}|} \sum_{\alpha \in I_{\mathfrak{P}}} \sum_{\chi} \bar{\chi}(g)\, \chi(\sigma_{\mathfrak{P}_\alpha}^m)$$

$$= \frac{|C|}{|G|} \sum_{\mathfrak{p}, m} \frac{\log N(\mathfrak{P})}{N(\mathfrak{P})^{ms}} \frac{1}{|I_{\mathfrak{P}}|} \sum_{\alpha \in I_{\mathfrak{P}}} \sum_{\psi \in \mathrm{Irr}(H)} \bar{\psi}(g)\, \psi^*(\sigma_{\mathfrak{P}_\alpha}^m)$$

$$= -\frac{|C|}{|G|} \sum_{\psi \in \mathrm{Irr}(H)} \bar{\psi}(g) \frac{L'}{L}(s, \psi^*, L/K)$$

$$= -\frac{|C|}{|G|} \sum_{\psi \in \mathrm{Irr}(H)} \bar{\psi}(g) \frac{L'}{L}(s, \psi, L/E)$$