Ergodicity and Synchronization of the Kardar-Parisi-Zhang Equation

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Based on joint works with
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Outline

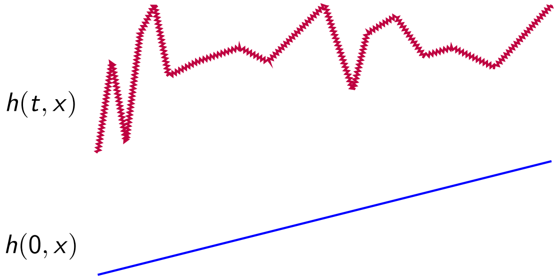
1. Introduce the model and the questions:
   a. Hopf-Cole solutions.
   b. Why stationary mod + c distributions?
   c. Synchronization and 1F1S.
2. Results
   a. Regularity of the solution semi-group and sharp characterization of non-explosive initial conditions. (AJRS)
   b. Ergodicity and synchronization (JRS)
3. Tools:
   a. Busemann process.
   b. Continuum directed polymer.
   c. Gibbs-DLR measures.
   d. Martingales.

The **KPZ equation** (Kardar-Parisi-Zhang 1986) is a prototypical example of **random planar growth**:

$$\partial_t h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}\left(\partial_x h\right)^2 + W$$

W is space-time white noise, "$W(t,x) \sim N(0, \delta_{t,x})$"

$$\text{``}\mathbb{E}[W(t,x)W(s,y)] = \delta_{t=s}\delta_{x=y}\text{''}.$$



KPZ/SHE

$$\partial_t Z = \frac{1}{2}\partial_{xx} Z + ZW, \qquad \text{(SHE/PAM)}$$

Formally, $h = \ln Z$ solves

$$\partial_t h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}(\partial_x h)^2 + W \qquad \text{(KPZ)}$$

This is the **Hopf-Cole** definition of solutions to KPZ.

Fact: (Bertini-Giacomin '97,Alberts-Khanin-Quastel '14, Hairer-Quastel '16,...) The Hopf-Cole solutions where Z is the *mild solution* to (SHE/PAM), is a scaling limit of many models $\implies$ physical.

"Theorem." (Hairer '14 ($\mathbb{T}$), Perkowski, Rosati '19 ($\mathbb{R}$)) The KPZ equation is well-posed for nice initial data. This solution agrees with the Hopf-Cole(*) solution $h(t,x) = \ln Z(t,x)$.

Stationary mod +c distributions:

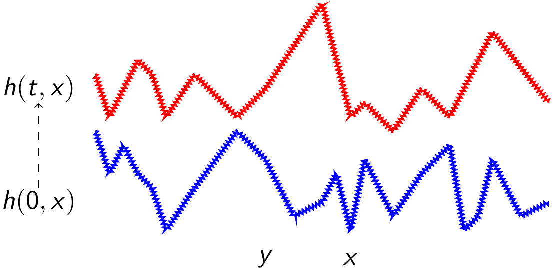
Theorem: (e.g. Amir, Corwin, Quastel '11) With "narrow-wedge" IC, in probability,

$$\lim_{t\to\infty} \frac{1}{t} h(t,0|0,0) = -\frac{1}{24} \implies \text{transience}$$

A (random) initial condition h_0 is **stationary mod + c** if

$$\partial_t h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}(\partial_x h)^2 + W, \qquad h(0,x) = h_0(x)$$

$$(h(t,x) \text{ mod } + \text{c})_{x\in\mathbb{R}} \overset{d}{=} (h(0,x) \text{ mod } + \text{c})_{x\in\mathbb{R}}$$



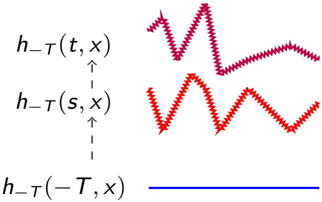
Synchronization/1F1S:

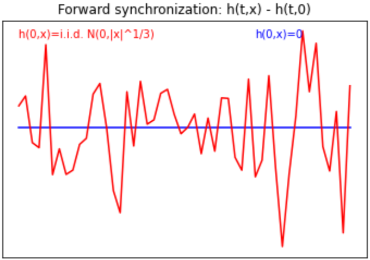
Theorem. (Bertini, Giacomin '97) For $\lambda \in \mathbb{R}$, Brownian motion with drift, $(B(x) + \lambda x)_{x \in \mathbb{R}}$, is stationary mod + c for KPZ.

Conjecture: (1F1S, ex. Bakhtin, Khanin '18) If $h_0(x) = \lambda x + \psi(x)$, $\psi(x)$ sublinear

$$\begin{cases} \partial_t h_{-T} = \frac{1}{2}\partial_{xx} h_{-T} + \frac{1}{2}(\partial_x h_{-T})^2 + W \\ h_{-T}(-T, x) = h_0(x) \end{cases}$$

$$\lim_{T \to \infty} (h_{-T}(t,x) - h_{-T}(t,y)) = b^\lambda(t,x,t,y) \overset{d}{=} B(x) - B(y) + \lambda(x - y).$$





Mild solutions:

$$\partial_t Z = \frac{1}{2}\partial_{xx} Z + ZW, \qquad \rho(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} 1_{(0,\infty)}(t)$$

$$\mu \in \mathcal{M}_{\text{HE}} = \{\mu \in \mathcal{M}_+(\mathbb{R}) : \forall a > 0, \int_{\mathbb{R}} e^{-ax^2} \mu(dx) < \infty\}$$

Chen-Dalang '14/'15 show $\exists!$ solution for fixed $s \in \mathbb{R}, \mu \in \mathcal{M}_{\text{HE}}$ of

$$\begin{aligned} Z(t,x|s;\mu) = &\int_{\mathbb{R}} \rho(t-s, x-y)\mu(dy) \\ &+ \int_s^t \int_{-\infty}^{\infty} \rho(t-u, x-z) Z(u,z|s;\mu) W(dudz) \end{aligned}$$

The Green's function takes $\mu = \delta_y$:

$$Z(t,x|s,y) = Z(t,x|s;\delta_y)$$

Other existence/uniqueness results: Bertini and Cancrini '95, Bertini and Giacomin '97.

Renormalized Green's function:

$$\mathcal{Z}(t,x|s,y) = \begin{cases} \frac{Z(t,x|s,y)}{\rho(t-s,x-y)} & s < t \\ 1 & s = t \end{cases} \text{“} = \mathbf{E}^{BB}_{(s,y),(t,x)}\left[: e^{\int_s^t W(u,B_u)} : \right]\text{”}$$

Theorem. (AJRS 22+) $\exists$ a modification $\mathcal{Z}(t,x|s,y) \in \mathcal{C}(\mathbb{R}^4,\mathbb{R})$. A.s. $\exists C = C(T,\omega)$: if $-T \leqslant s \leqslant t \leqslant T$,

$$C^{-1}(1+|x|^4+|y|^4)^{-1} \leqslant \mathcal{Z}(t,x|s,y) \leqslant C(1+|x|^4+|y|^4).$$

We **define**

$$Z(t,x|s,y) = \mathcal{Z}(t,x|s,y)\rho(t-s,x-y)$$
$$Z(t,x|s;\mu) = \int_{\mathbb{R}} Z(t,x|s,y)\mu(dy) = \int_{\mathbb{R}} \mathcal{Z}(t,x|s,y)\rho(t-s,x-y)\mu(dy).$$

Previous work: Alberts-Khanin-Quastel '14 constructed $Z(t,x|s,y)$ for $s < t$ previously.

KPZ solution semi-group:
For f Borel, call

$$h(t,x|s;f) := \ln \int Z(t,x|s,y)e^{f(y)}dy, \qquad h(s,x) = f(x).$$

$$\mathcal{M}_{\mathrm{HE}} = \{\mu \in \mathcal{M}_+(\mathbb{R}) : \forall a > 0, \int_{\mathbb{R}} e^{-ax^2}\mu(dx) < \infty\}$$

$$\mathcal{C}_{\mathrm{KPZ}} = \left\{f \in \mathcal{C}(\mathbb{R},\mathbb{R}) : \int_{\mathbb{R}} e^{f(x)-ax^2}dx \text{ for all } a > 0\right\}.$$

Theorem. (AJRS '22+)

1. $e^f \in \mathcal{M}_{\mathrm{HE}} \implies h(t,\cdot|s;f) \in \mathcal{C}_{\mathrm{KPZ}}$, and
   $e^f \notin \mathcal{M}_{\mathrm{HE}}, t \gg s \implies h(t,\cdot|s;f) \equiv \infty$.
2. $e^f \in \mathcal{M}_{\mathrm{HE}}$, $s \in \mathbb{R} \implies h(\cdot,\cdot|s;f)$ is the Hopf-Cole solution, $\mathbb{P}$ a.s.
3. (Quenched continuous DS) $(f,s,t) \mapsto h(t,\cdot|s;f)$ is continuous $\mathcal{C}_{\mathrm{KPZ}} \times \mathbb{R}^2_{s\leqslant t} \to \mathcal{C}_{\mathrm{KPZ}}$.
4. (Conservation law) $\lim_{x\to\pm\infty} f(x)/x = \lim_{x\to\pm\infty} h(t,x|s;f)/x$

Ergodic distributions.

$$f \sim g \text{ if } f(\cdot) = g(\cdot) + c, \qquad [f] \in \widetilde{\mathcal{C}}_{\mathrm{KPZ}} = \mathcal{C}_{\mathrm{KPZ}} / \sim$$

Corollary. (AJRS 22+) $[h(t,\cdot|s;f)] : \mathbb{R}^2_{s\leqslant t} \times \widetilde{\mathcal{C}}_{\mathrm{KPZ}} \to \widetilde{\mathcal{C}}_{\mathrm{KPZ}}$ is continuous $\implies$ Feller.

Theorem. (JRS 22+) The distribution of $[B(\cdot) + \lambda\cdot]$ is (totally) ergodic. If P is *any* ergodic distribution, then either

1. There exists $\lambda \in \mathbb{R}$ such that P is the distribution of $[B(\cdot) + \lambda\cdot]$.
2. There exists $\lambda > 0$ such that P is supported on equivalence classes $[f] \in \widetilde{\mathcal{C}}_{\mathrm{KPZ}}$ with

$$-\lambda = \lim_{x\to-\infty} \frac{f(x)}{x} \qquad \text{and} \qquad \lim_{x\to\infty} \frac{f(x)}{x} = \lambda$$

for all $f \in [f]$.

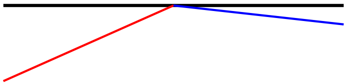
Open problem: Do ergodic measures of the second type exist?

Synchronization:

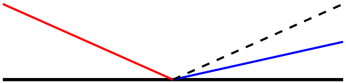
(Simplified) **Theorem.** (JRS 22+) There exists a random countable Λ^ω with $\lambda \in \mathbb{R}$, $\mathbb{P}(\lambda \in \Lambda) = 0$ so that for $\lambda \notin \Lambda$, if ψ is sublinear and

$$h_{-T}(-T, x) = (\mu x)1_{(-\infty,0)}(x) + (\eta x)1_{(0,\infty)}(x) + \psi(x),$$

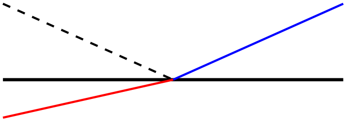
$\mu \geqslant 0, \quad \eta \leqslant 0 \implies \lambda = 0$



$\mu < 0, \quad \eta < |\mu| \implies \lambda = \mu$



$\eta > 0, \quad -\mu < \eta \implies \lambda = \eta$



$$\lim_{T\to\infty} (h_{-T}(t, x) - h_{-T}(s, y)) = b^\lambda(s, y, t, x)$$

Busemann process:

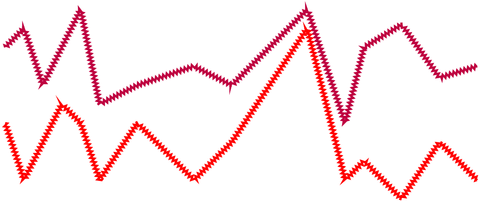
Theorem. (JRS 22+) $\exists\{b^{\lambda\pm}(s,y,t,x) : \lambda, s, x, t, y \in \mathbb{R}\}$, and a random countable Λ^ω satisfying

1. For all $\lambda \in \mathbb{R}$, $\mathbb{P}(\lambda \in \Lambda) = 0$.
2. (Equal off Λ) If $\lambda \notin \Lambda$, then $b^{\lambda+} = b^{\lambda-}(:= b^{\lambda})$.
3. (Continuous) For all λ, $b^{\lambda\pm}(\cdot,\cdot,\cdot,\cdot) \in \mathcal{C}(\mathbb{R}^4, \mathbb{R})$
4. (Brownian) For all $t, \lambda \in \mathbb{R}$, $b^{\lambda}(t,y,t,x) \stackrel{d}{=} B(x) - B(y) + \lambda(x-y)$.
5. (Cocycle) $b^{\lambda\pm}(s,y,r,z) + b^{\lambda\pm}(r,z,t,x) = b^{\lambda\pm}(s,y,t,x)$.
6. (Eternal solution of KPZ) For all s, t, y, x, λ and $r < t$,

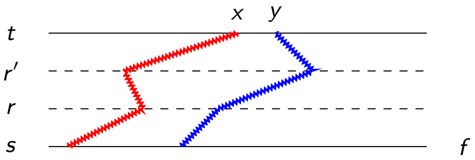
$$b^{\lambda\pm}(s,y,t,x) = \ln \int Z(t,x|r,u)e^{b^{\lambda\pm}(s,y,r,u)}du = h(t,x|r;b^{\lambda\pm}(s,y,r,\cdot))$$

Time t: $b^{\lambda\pm}(s,y,t,\cdot)$

Time $r < t$: $b^{\lambda\pm}(s,y,r,\cdot)$



Continuum Directed Random Polymer:



The CDRP is the measure $Q_{(s;f),(t,x)}$ on $\mathcal{C}([s,t],\mathbb{R})$ with transitions

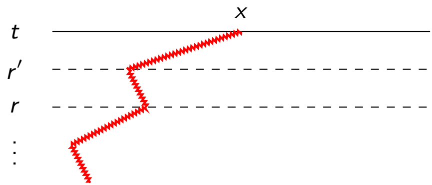
$$\pi_{(s;f),(t,x)}(r,du|r',u') = Z(r',u'|r,u) \underbrace{\frac{Z(r,u|s;f)}{Z(r',u'|s;f)}}_{e^{h(r,u|s;\ln f)-h(r',u'|s;\ln f)}} du'$$

Originally introduced by Alberts-Khanin-Quastel '14 for fixed t, x, s, f.

Theorem.(AJRS '22+) A regular (continuous, monotone, etc.) coupling of all $Q_{(s;\mu),(t,x)}$ exists.

Infinite volume (Gibbs) polymers and eternal cocycles:
$b^{\lambda\pm}$ is an eternal (cocycle) KPZ solution: with $f(y) = b^{\lambda\pm}(s,y,t,x)$,

$$\pi_{(s;e^{b^{\lambda\pm}(s,\cdot,t,x)}),(t,x)}(r',du'|r,du) = Z(r',u'|r,u)\frac{Z(r,u|s;f)}{Z(r',u'|s;f)}du'$$
$$= \underbrace{Z(r',u'|r,u)e^{b^{\lambda\pm}(r,u,r',u')}du'}_{\text{Does not depend on s}}$$

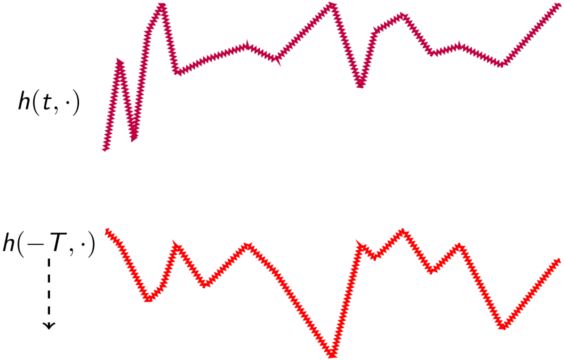


which defines $Q^{\lambda\pm}_{(t,x)}$ on $\mathcal{C}((-\infty,t],\mathbb{R})$.

Ergodicity, eternal cocycle solutions, and Gibbs polymers:

If P is ergodic take $[h(-T,\cdot)] \sim$ P. Extension $\implies$

$\exists$ global cocycle sol $b^P(s,x,t,y)$ (on an extended space)



Global cocycle $\implies$ $\exists$ semi-infinite polymer $Q^P_{t,x}$ with transitions

$$\pi^P_{t,x}(r',du'|r,du) = Z(r',u'|r,u)e^{b^P(r,u,r',u')}du'.$$

Busemann limits and Gibbs martingales:

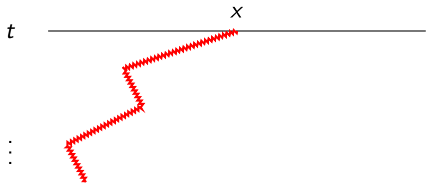
The Busemann limit says that if $\lambda \notin \Lambda$, $z_r/r \to -\lambda$ as $r \to -\infty$

$$\frac{Z(s,y|r,z_r)}{Z(t,x|r,z_r)} \to e^{b^\lambda(s,y,t,x)}.$$

On the other hand, because $Q^P_{t,x}$ is Gibbs,

$$M_r^{s,y,t,x} = \frac{Z(s,y|r,X_r)}{Z(t,x|r,X_r)} \text{ is a } Q^P_{t,x} \text{ bmg, } \mathbf{E}^{Q^P_{t,x}}\left[\frac{Z(s,y|r,X_r)}{Z(t,x|r,X_r)}\right] = e^{b^P(s,y,t,x)},$$

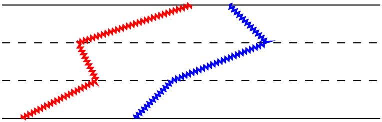
$$\chi = \lim_{r\to-\infty} \frac{X_r}{r} \text{ exists} \qquad Q^P_{t,x} - a.s. \qquad (\chi \in [-\infty,\infty] \text{ is } Q^P_{t,x} \text{ random}).$$



LLN/slope duality:

By monotonicity, for large $-r$ and $x < 0$, $Q^P_{t,x}$ a.s.

$$\frac{Z(t,x|r,-(\lambda+\varepsilon)r)}{Z(t,0|r,-(\lambda+\varepsilon)r)} \cdot 1_{\{\chi \geqslant -\lambda\}} \leqslant \frac{Z(t,x|r,X_r)}{Z(t,0|r,X_r)} 1_{\{\chi \geqslant -\lambda\}} \implies$$
$$e^{b^{\lambda+\epsilon}(t,0,t,x)} Q^P_{t,x}(\chi \geqslant -\lambda) \leqslant e^{b^P(t,0,t,x)}.$$



$$Q^P_{0,0}(\chi \geqslant -\lambda) > 0 \implies \varlimsup_{x\to-\infty} \frac{b^P(t,0,t,x)}{x} \leqslant \lambda$$

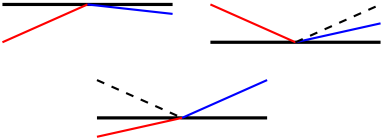
$Q^P_{0,0}(\chi \geqslant -\lambda) > 0 \iff Q^P_{t,x}(\chi \geqslant -\lambda) > 0 \implies \{Q^P_{0,0}(\chi \geqslant -\lambda) > 0\}$ is translation invariant.

LLN/slope duality continued:
If P is ergodic, then since $[b^P(t,0,t,\cdot)] \sim P$

$$\mathbb{P}(Q^P_{0,0}(\chi \geqslant -\lambda) > 0) > 0 \implies \mathrm{P}(\overline{\lim_{x\to-\infty}} f(x)/x \leqslant \lambda) = 1$$

Similar statements for $\overline{\lim}_{x\to\pm-\infty}$ and $\underline{\lim}_{x\to\pm-\infty}$ and a finiteness lemma $\implies \exists$ finite $\overline{\lambda}, \underline{\lambda}$:

$$\mathrm{P}\Big(\lim_{x\to-\infty} f(x)/x = \underline{\lambda}\Big) = \mathrm{P}\Big(\lim_{x\to\infty} f(x)/x = \overline{\lambda}\Big) = 1.$$



Consistency with the Busemann limits now says that if we do not have $\overline{\chi} > 0$ and $\underline{\chi} = -\overline{\chi}$, then $b^P = b^\lambda$ for $\lambda = \overline{\lambda} = \underline{\lambda}$.

Thank you!