The periodic PNG model is solvable

Eric Cator

Joint work with Pablo Ferrari

Radboud Universiteit Nijmegen

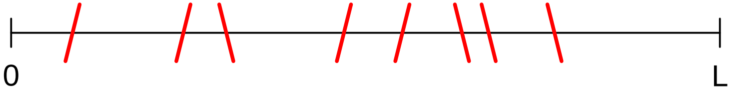
Periodic PNG model

State space

State space is given by

$$E = \{(X, Y) \mid X, Y \subset [0, L], |X| < +\infty, |Y| < +\infty\}.$$

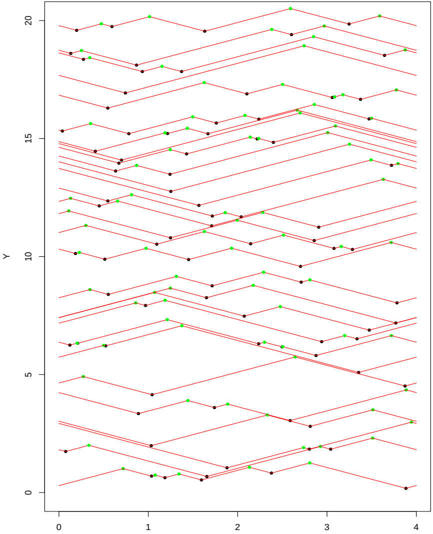
Topology induced by $\Delta_n = \{(x_1, \ldots, x_n) \in [0, L]^n \mid x_1 < \ldots < x_n\}$.



Dynamics

Positive particles move at unit speed to the right, negative move to the left, annihilate each other when they meet. Pairs "born" according to Poisson process on $[0, L] \times [0, \infty)$ with intensity 2.

Periodic PNG model



Stationary measure

Theorem: Independent Poisson is stationary

X is realisation of Poisson process on $[0, L]$ with intensity λ, Y with intensity $1/\lambda$, independent. This distribution is stationary for the periodic PNG.

Proof:

$$\theta_h X = \{x + h \mid x \in X\},$$

where we calculate modulo L.

$$\mathbb{P}(X_h = x, Y_h = y) = \mathbb{P}(X_0 = \theta_{-h} x, Y_0 = \theta_h y)(1 - 2hL) + \frac{1}{L^2} \int_0^L \int_u^{u+2h} \mathbb{P}(X_0 = \theta_{-h} x \cup u, Y_0 = \theta_h y \cup v) dvdu + o(h).$$

Stationary measure

Proof

Suppose $|x| = m$ and $|y| = n$. We see that

$$\mathbb{P}(X_0 = \theta_{-h}x, Y_0 = \theta_h y) = \mathbb{P}(X_0 = x, Y_0 = y)$$

and

$$\frac{1}{L^2}\int_0^L \int_u^{u+2h} \mathbb{P}(X_0 = \theta_{-h}x \cup u, Y_0 = \theta_h y \cup v)dvdu =$$

$$\frac{2hL}{L^2}\mathbb{P}(X_0 = x, Y_0 = y)(m+1)(n+1)\frac{\lambda L}{m+1}\frac{L/\lambda}{n+1}.$$

So the $O(h)$ terms cancel.

Stationary measure

Ergodic components

Clearly the dynamic preserves the difference $|X| - |Y|$. Therefore, if we condition the independent Poisson processes so that $|X| - |Y| = k$ (for some $k \in \mathbb{Z}$), that will also be a stationary measure. These conditioned measures do not depend on λ!

Poisson-squared distribution

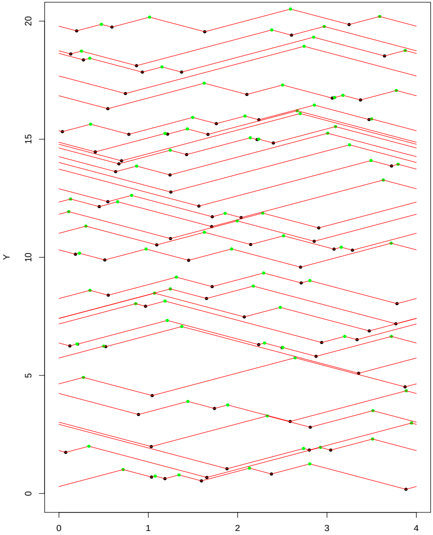
Suppose $X, Y \sim \mathrm{Pois}(L)$ independent. Define

$$p_L(k) = \mathbb{P}(X = k \mid X = Y).$$

So for integer $k \geq 0$

$$p_L(k) = \frac{L^{2k}}{(k!)^2} \cdot \frac{1}{Z_L}, \qquad Z_L = \sum_{k=0}^{\infty} \frac{L^{2k}}{(k!)^2}.$$

Dual points



Dual points

Theorem: Reversibility

(X_0, Y_0) independent Poisson, rate λL and L/λ resp. The dual points in $[0, L] \times [0, T]$ form a Poisson process of intensity 2, independent of (X_T, Y_T).

Proof:

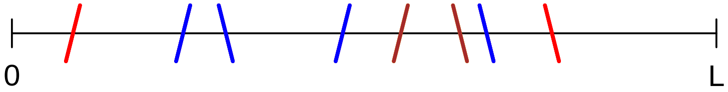
Choose $(x, y) = ((x_1, \ldots, x_m), (y_1, \ldots, y_n))$. Then

$$\mathbb{P}(\text{dual in } [a, b] \times [T-h, T] \mid (X_T, Y_T) = (x, y)) = \frac{\mathbb{P}((X_{T-h}, Y_{T-h}) = (\theta_{-h}x, \theta_h y))}{\mathbb{P}((X_T, Y_T) = (x, y))} \cdot \frac{\lambda L}{m+1} \frac{L/\lambda}{n+1} \cdot (m+1)(n+1) \frac{b-a}{L} \frac{2h}{L} + o(h) = 2(b-a)h + o(h). \square$$

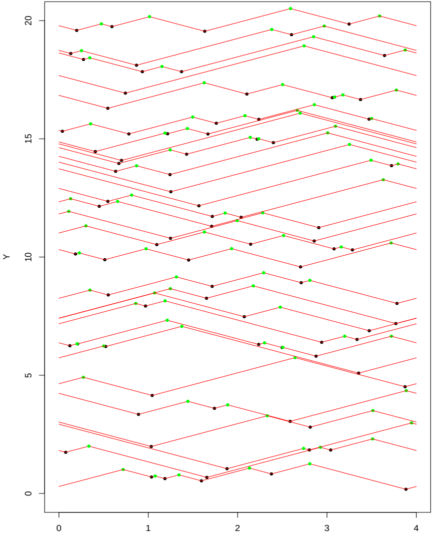
Same number of negative and positive particles

Paths are rings

Consider particles at fixed time. Each positive particle is linked to a specific negative particle to the right, and each negative particle is linked to a specific positive particle to the right. In this way, the paths form closed rings.



Number of rings



Poisson-squared distribution

Moments

Modified Bessel function of the first kind:

$$I_\alpha(x) = \sum_{n=0}^{\infty} \frac{1}{n! \cdot \Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha}.$$

Suppose $W \sim \mathrm{Pois}^2(L)$. Then $Z_L = I_0(2L)$ and

$$\mathbb{E}(W) = L \cdot I_{-1}(2L)/Z_L.$$
$$\mathbb{E}(W^2) = L^2.$$

Last equation is relatively simple because of the factor $(k!)^2$.

Number of rings

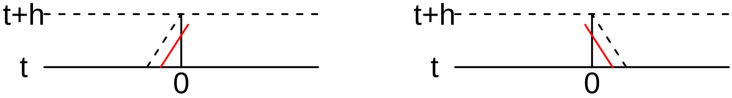
Expected number of rings

Stationary process: for small h

$$\text{expected number of rings through } \{0\} \times [t, t+h] =$$
$$\#\text{+particles in } [L-h, L] \times \{t\} + \#\text{−particles in } [0, h] \times \{t\}$$

If $W \sim \mathrm{Pois}^2(L)$,

$$\text{Expected numer of rings in } [0, T] = \frac{2T\mathbb{E}(W)}{L} = 2T\frac{I_{-1}(2L)}{I_0(2L)}.$$



Distribution of the rings

Number of minima in a ring

Define N as the number of minima in random ring in the stationary case. These correspond to the Poisson points. Therefore:

$$\mathbb{E}(N) = \frac{\text{Poisson points}}{\text{rings}} = \frac{L^2}{\mathbb{E}(W)} = \frac{\mathbb{E}(W^2)}{\mathbb{E}(W)}.$$

Distribution of the rings

Number of minima in a ring

Define N as the number of minima in random ring in the stationary case. These correspond to the Poisson points. Therefore:

$$\mathbb{E}(N) = \frac{\text{Poisson points}}{\text{rings}} = \frac{L^2}{\mathbb{E}(W)} = \frac{\mathbb{E}(W^2)}{\mathbb{E}(W)}.$$

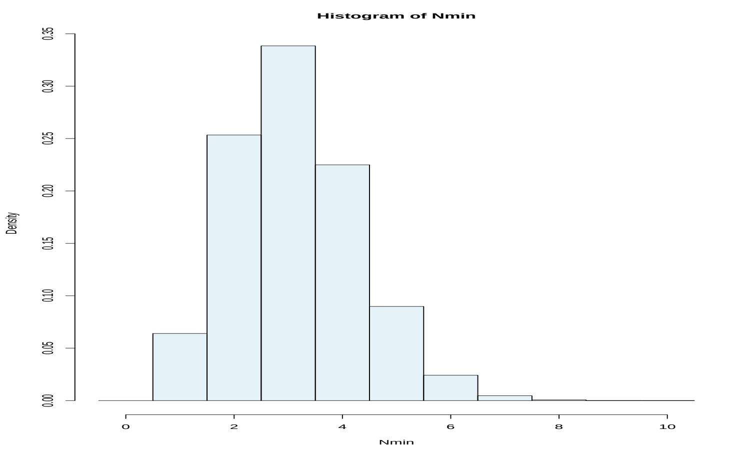
Size biased squared Poisson

The number of minima in a random ring, N, has a size-biased squared Poisson distribution with parameter L.

$$\mathbb{P}(N = k) = \frac{kL^{2k}}{(k!)^2} \cdot \frac{1}{L \cdot I_{-1}(2L)} =: p_L^*(k).$$

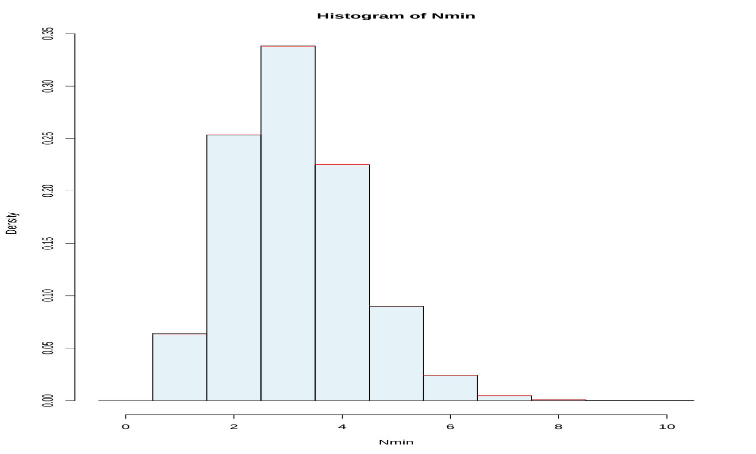
Distribution of rings

Simulation results



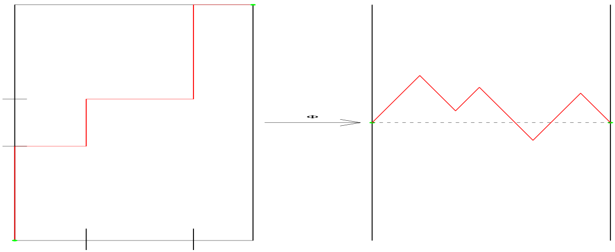
Distribution of rings

Simulation results



Distribution of rings

Statespace of rings: E



Distribution of rings

Statespace of rings: E

A path $\sigma \in E$ has three features: $\sigma_N \geq 1$, the number of minima, and if $\sigma_N \geq 2$

$$\sigma_x = (\sigma_{x,1}, \ldots, \sigma_{x,\sigma_N - 1}) \text{ and } \sigma_y = (\sigma_{y,1}, \ldots, \sigma_{y,\sigma_N - 1}).$$

$\Phi(\sigma)$ is the corresponding ring in the cylinder, starting with a minimum at $(0,0)$. The other minima (if any) satisfy

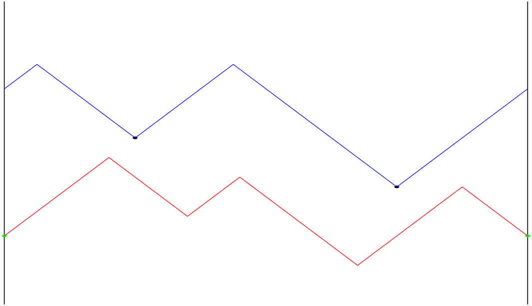
$$P_i = \frac{1}{2}(\sigma_{x,i} + \sigma_{y,i}, \sigma_{y,i} - \sigma_{x,i}).$$

Note Φ has Jacobian equal to $2^{-(\sigma_N - 1)}$.

Distribution of rings

Markov chain on E

We define a Markov chain on E: start with $\sigma_0 \in E$. Consider $R_0 = \Phi(\sigma_0)$ and Poisson process above $\Phi(\sigma)$: find next ring R_1. To go back, pick a minimum of R_1 at random, translate it to $(0,0)$ and apply Φ^{-1}: $\sigma_1 = \Phi^*(R_1)$.



Distribution of rings

Transition kernel

Dominating measure on $E = \sqcup_{n=0}^{\infty}(\Delta_n \times \Delta_n)$ is ν, with

$$\nu = \sum_{i=0}^{\infty} \nu_n \quad \text{and} \quad \nu_n(dxdy) = L^{-2n}(n!)^2 dxdy.$$

Transition kernel $K : E \times E \to [0, \infty)$ satisfies

$$\forall \sigma \in E : \int_E K(\sigma, \tau)\nu(d\tau) = 1.$$

On the cylinder

For two rings in the cylinder $[0, L] \times \mathbb{R}$, $R_0 \leq R_1$, define $A(R_0, R_1)$ as the size of the area between the two rings. If $R_0 \not\leq R_1$, $A(R_0, R_1) = +\infty$. Fix R_0, denote the next random ring by R_1. Suppose ring Q has n minima.

Distribution of rings

Transition kernel

Choose $\sigma, \tau \in E$. $\tau_N = n$ and dz is neighbourhood of τ in $\Delta_{n-1} \times \Delta_{n-1}$. Define $R_0 = \Phi(\sigma)$ and R_1 as the next random ring.

$$\mathbb{P}(\Phi^*(R_1) \in \tau + dz \mid \sigma) = \frac{1}{\tau_N} \int \int e^{-2A(\Phi(\sigma),\Phi(\tau)+(x,t))} dxdt \cdot$$

$$\cdot 2^{\tau_n - 1} \cdot \left(\frac{1}{2}\right)^{\tau_N - 1} dx_1 \dots dx_{\tau_N - 1} dy_1 \dots dy_{\tau_N - 1}.$$

$$K(\sigma, \tau) = \int \int e^{-2A(\Phi(\sigma),\Phi(\tau)+(x,t))} dxdt \cdot \frac{1}{\tau_N} \cdot \frac{L^{2(\tau_N - 1)}}{((\tau_N - 1)!)^2}.$$

Distribution of rings

Stationary distribution

Suppose $\sigma \sim f$, with f a density on E with respect to ν. Find a function

$$K^* : E \times E \to [0, \infty)$$

such that

1. $\int_E K^*(\sigma, \tau)\nu(d\tau) = 1$
2. $\forall \sigma, \tau \in E : \; K(\sigma, \tau) f(\sigma) = K^*(\tau, \sigma) f(\tau)$.

Then f is the stationary measure for K (and for K^*).

Reverse process

Take K^* the transition kernel corresponding to the time-reversed process on the cylinder. Take

$$f(\sigma) = p_L^*(\sigma_N).$$

Distribution of rings

Reverse process

$$\begin{aligned} K^*(\tau,\sigma) &= \int\int e^{-2A(\Phi(\sigma)-(x,t),\Phi(\tau))}dxdt \cdot \frac{1}{\sigma_N} \cdot \frac{L^{2(\sigma_N-1)}}{((\sigma_N-1)!)^2} \\ &= \int\int e^{-2A(\Phi(\sigma),\Phi(\tau)+(x,t))}dxdt \cdot \frac{1}{\sigma_N} \cdot \frac{L^{2(\sigma_N-1)}}{((\sigma_N-1)!)^2} \\ &= \int\int e^{-2A(\Phi(\sigma),\Phi(\tau)+(x,t))}dxdt \cdot \sigma_N \cdot \frac{L^{2\sigma_N}}{(\sigma_N!)^2} \cdot L^{-2}. \end{aligned}$$

Detailed balance

$$\frac{K(\sigma,\tau)}{K^*(\tau,\sigma)} = \frac{p_L^*(\tau_N)L^{-2}}{p_L^*(\sigma_N)L^{-2}} = \frac{f(\tau)}{f(\sigma)}.$$