Octonions and spinors

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May 29th, 2023 – Banff International Research Station

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Outline

1. The octonions.
2. Clifford algebras and spinors.
3. Octonions as spinors.

Octonions

The normed division algebras

Theorem (Hurwitz 1898)

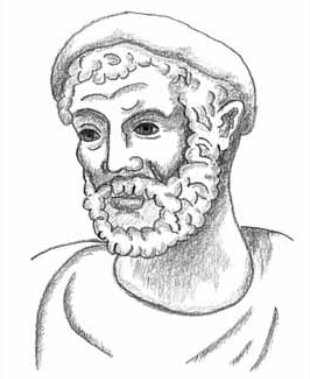
There are precisely four normed division algebras:

- $\mathbb{R}$, the real numbers;
- $\mathbb{C}$, the complex numbers;
- $\mathbb{H}$, the quaternions;
- $\mathbb{O}$, the octonions.

Each of these is **normed**:

$$|xy| = |x||y|, \text{ for all } x, y \in \mathbb{A}.$$

The real and complex numbers



Eudoxus of Cnidus
(A. Strick, MacTutor)

The real numbers:

$$\mathbb{R} = \mathrm{span}_{\mathbb{R}}\{1\},$$

the “dependable breadwinner” of number systems.

Guiseppe Cardano
(A. Strick, MacTutor)

The complex numbers:

$$\mathbb{C} = \mathrm{span}_{\mathbb{R}}\{1, i\}, \quad \text{where } i^2 = -1,$$

the flashy younger brother.

Quaternions and octonions

William Rowan Hamilton
(A. Strick, MacTutor)

The quaternions:

$$\mathbb{H} = \text{span}_{\mathbb{R}}\{1, i, j, k\},$$

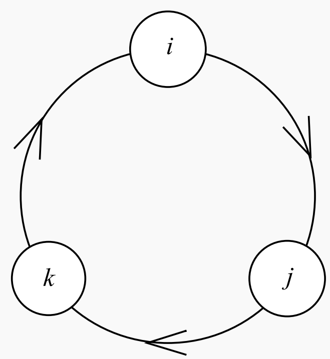
where $i^2 = j^2 = k^2 = ijk = -1$; the eccentric cousin.

$ijk = -1$ by $-i$

$jk = i$

$i = -kj$

Mnemonic for multiplying quaternions:



E.g., $ij = k = -ji$.

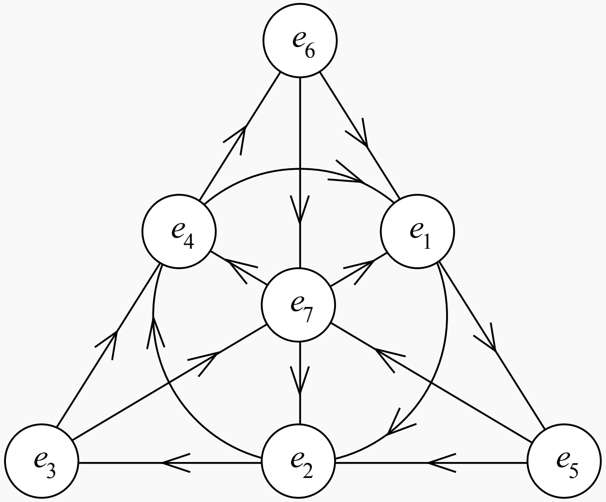
John T. Graves
(MacTutor)

The octonions

$$\mathbb{O} = \text{span}_{\mathbb{R}}\{1,\ e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7\},$$

where $e_i^2 = -1$; the *crazy old uncle!*

Multiplying octonions with the **Fano plane**, $\mathbb{F}_2\mathbb{P}^2$:

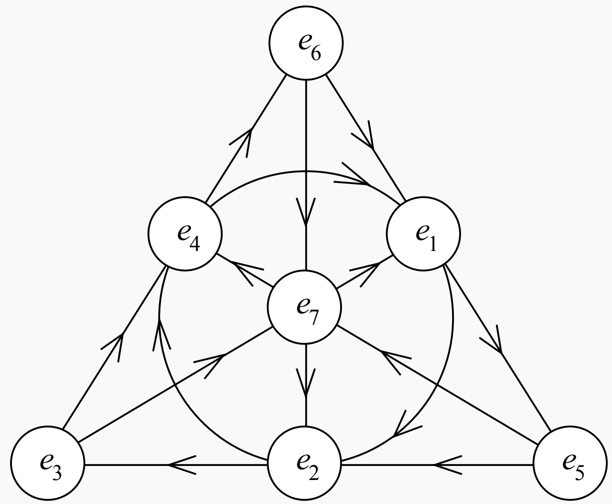


E.g., $e_7 e_1 = e_3 = -e_1 e_7$.

In the family of real algebras:

> The real numbers are the dependable breadwinner of the family. . . . The complex numbers are a slightly flashier but still respectable younger brother. . . . The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings. **But the octonions are the crazy old uncle nobody lets out of the attic: they are *nonassociative*.**
>
> ---
>
> John Baez
> The Octonions

Nonassociative, but alternative.



$\mathbb{O}$ is not associative:

$$(e_1 e_2)e_3 = -e_1(e_2 e_3).$$

$e_4 e_3$ $e_1 e_5$

$-e_6$ e_6

Nonassociative, but alternative.

But $\mathbb{O}$ is **alternative**:

$$\begin{aligned}(xx)y &= x(xy),\\ (xy)x &= x(yx),\\ (yx)x &= y(xx),\end{aligned}$$

for any $x, y \in \mathbb{O}$.

Just enough associativity!

The Cayley–Dickson construction

- Just as $\mathbb{C} = \mathbb{R}^2$, we can define the octonions as pairs of quaternions:

$$\mathbb{O} = \mathbb{H}^2, \text{ where } (a, b)(c, d) = (ac - d\overline{b}, da + b\overline{c}).$$

$q_1 + q_2 \ell, \quad \ell^2 = -1, \quad \ell q \ell^{-1} = \overline{q}$

The Cayley–Dickson construction

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- This works for any $*$-algebra! It's called the **Cayley–Dickson construction**.
- Iterating the Cayley–Dickson construction gives:

$$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \mathbb{S} = \mathbb{O}^2, \mathbb{S}^2, \ldots$$

an infinite sequence of $*$-algebras!

There are only four normed division algebras . . .

A **normed division algebra** $\mathbb{A}$ is a possibly nonassociative real algebra with unit, equipped with a positive-definite quadratic form $|\cdot|^2 \colon \mathbb{A} \to \mathbb{R}$ satisfying

$$|xy| = |x||y|, \text{ for all } x, y \in \mathbb{A}.$$

division
$xy = 0$
$\Rightarrow x = 0$ or
$y = 0$

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Theorem (Hurwitz 1898)

There are only four normed division algebras:

$$\mathbb{R}, \mathbb{C}, \mathbb{H}, \text{ and } \mathbb{O}.$$

The proof goes through Clifford algebras!

Clifford algebras: definition.

William Kingdon Clifford
(D. Chisholm)

The **Clifford algebra** $\mathrm{C}\ell(V, g)$ on the real inner product space (V, g) is the real associative algebra generated by V satisfying the **Clifford relation**:

$$v^2 = g(v, v), \text{ for } v \in V.$$

- g is nondegenerate, but not necessarily positive definite!
- The Clifford relation is equivalent to:

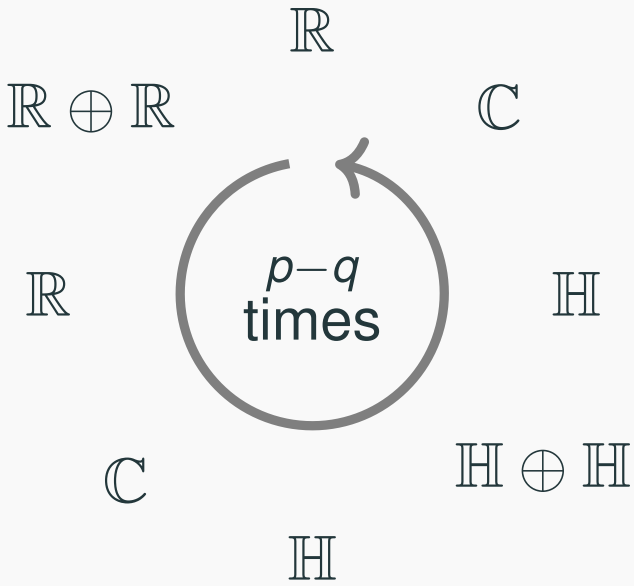
$$vw + wv = 2g(v, w), \text{ for } v, w \in V.$$

Clifford algebras: classification.

Write $\mathrm{C}\ell(p, q)$ for the Clifford algebra of $\mathbb{R}^{p,q}$.

Theorem

- As a real algebra, $\mathrm{C}\ell(p, q) \cong M_n(\mathbb{K})$, $n \times n$ matrices $/\mathbb{K}$;
- The size n is fixed by $\dim \mathrm{C}\ell(p, q) = 2^{p+q}$;
- The algebra of coefficients is fixed by the **Clifford algebra clock**:



Clifford algebras: examples.

$Cl(0,0) = \mathbb{R}$

$Cl(0,1) = \mathbb{C}$

$Cl(0,2) = \mathbb{H}$

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ — i, j, $k = ij$

$Cl(0,3) = \mathbb{H} \oplus \mathbb{H} \sim \mathbb{H}_2$, $\mathbb{H}_2$

$(q_1, q_2) \cdot s_2 = q_1 s_2$

$Cl(0,7) = M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$

$2^7 = 2 \cdot 2^3 \times 2^3$

$Cl(8,0) = M_{16}(\mathbb{R}) = Cl(0,8)$

Sketch of Hurwitz's theorem

Let $\mathbb{A}$ be a normed division algebra, and define $\operatorname{Im}\mathbb{A} := 1^\perp$.

Claim: $\mathbb{A}$ is a module for the Clifford algebra $\mathrm{C}\ell(\operatorname{Im}\mathbb{A})$.

Example

When $\mathbb{A} = \mathbb{O}$, we define a homomorphism:

$$\begin{aligned} \gamma_L \colon \mathrm{C}\ell(\operatorname{Im}\mathbb{O}) &\to \operatorname{End}(\mathbb{O}) \\ x \in \operatorname{Im}\mathbb{O} &\mapsto x_L, \end{aligned}$$

where $x_L \colon \mathbb{O} \to \mathbb{O}$ denotes left multiplication, $x_L(y) = xy$.

$x_L^2 = -|x|^2$

$x_L x_L(y) = x(xy) = (xx)y = -|x|^2 y$

$C\ell(\operatorname{Im}\mathbb{O}, -|\cdot|^2) \longrightarrow \operatorname{End}(\mathbb{O})$

Sketch of Hurwitz's theorem

Let $\mathbb{A}$ be a normed division algebra, and define $\mathrm{Im}\,\mathbb{A} := 1^{\perp}$.

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Thus given $\mathbb{A}$ a normed division algebra, we get a Clifford algebra $\mathrm{C}\ell(V)$ and a module M such that:

$$\dim M = \dim V + 1.$$

Sketch of Hurwitz's theorem

Let $\mathbb{A}$ be a normed division algebra, and define $\operatorname{Im}\mathbb{A} := 1^\perp$.

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Thus given $\mathbb{A}$ a normed division algebra, we get a Clifford algebra $\mathrm{C}\ell(V)$ and a module M such that:

$$\dim M = \dim V + 1.$$

That's rare!

- $\mathrm{C}\ell(0,0) = \mathbb{R}$, and has module $\mathbb{R}$;
- $\mathrm{C}\ell(0,1) = \mathbb{C}$, and has module $\mathbb{C}$;
- $\mathrm{C}\ell(0,3) = \mathbb{H} \oplus \mathbb{H}$, and has modules $\mathbb{H}_L$, $\mathbb{H}_R$;
- $\mathrm{C}\ell(0,7) = M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$, and has modules $\mathbb{R}^8_L$, $\mathbb{R}^8_R$.

Spinors

Geometry and the Clifford relation

Question

Why are Clifford algebras related to geometry?

The fundamental calculation

Let $v \in V$ be a unit vector. Compute vwv^{-1}:

$$v^2 = 1$$

$$v w v^{-1} = -w v v^{-1} + 2g(v,w)v^{-1}$$

$$-v w v^{-1} = w - 2g(v,w)v$$

The fundamental calculation

Proposition

The negative conjugate by a unit vector v is reflection in the hyperplane $v^\perp$:

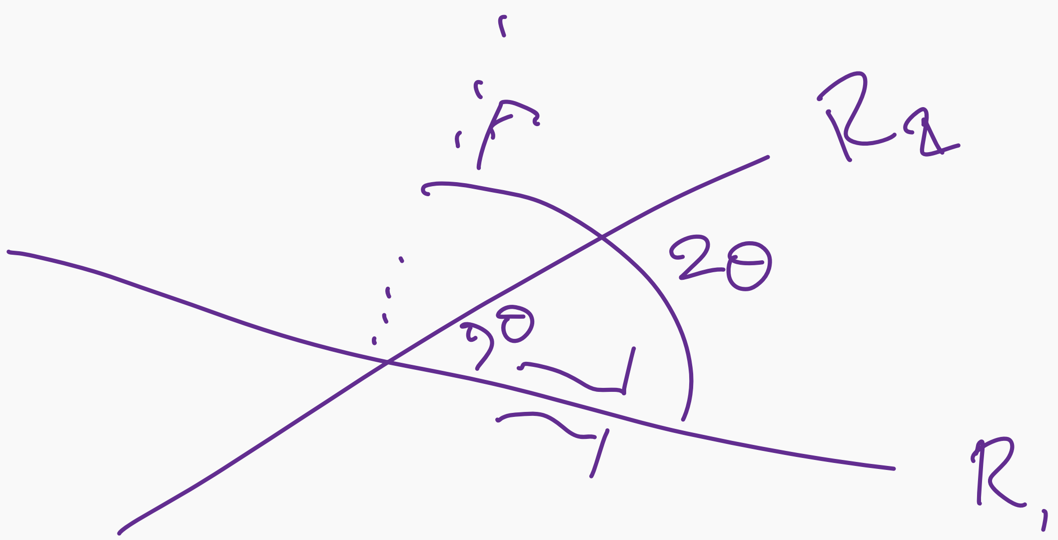
$$\mathbf{R}_v(w) = -vwv^{-1}.$$

Cartan's theorem

Theorem (Cartan)

Any rotation $g \in \mathrm{SO}(V)$ can be decomposed into an even number of reflections:

$$g = \mathbf{R}_{v_1} \mathbf{R}_{v_2} \cdots \mathbf{R}_{v_{2n}}.$$



Spin groups

Define the **spin group** to be:

$$\mathrm{Spin}(V) = \{v_1 v_2 \cdots v_{2n} \in \mathrm{C}\ell(V) \ : g(v_i, v_i) = \pm 1,\ n \in \mathbb{N}\}\,.$$

There's a 2-to-1 and onto homomorphism:

$$\begin{aligned} \rho\colon \mathrm{Spin}(V) &\to \mathrm{SO}(V) \\ v_1 v_2 \cdots v_{2n} &\mapsto \mathbf{R}_{v_1} \mathbf{R}_{v_2} \cdots \mathbf{R}_{v_{2n}}. \end{aligned}$$

Spin representations

- $\mathrm{Spin}(V)$ has more reps than $\mathrm{SO}(V)$!
- Since $\mathrm{Spin}(V) \subseteq \mathrm{C}\ell(V)$, $\mathrm{C}\ell(V)$-modules yield representations.
- To identify irreps note that $\mathrm{Spin}(V) \subseteq \mathrm{C}\ell(V)_+$, the **even part** of the $\mathbb{Z}_2$-graded $\mathrm{C}\ell(V)$.

Simple modules of $\mathrm{C}\ell(V)_+ \leftrightarrow$ **spin reps** of $\mathrm{Spin}(V)$.

Spin representations

Proposition

$\mathrm{C}\ell(p,q)_+ \cong \mathrm{C}\ell(p,q-1) \cong \mathrm{C}\ell(q,p-1)$.

Depending on dimension and signature, either:

- $\mathrm{C}\ell(V)_+ \cong M_n[\mathbb{K}] \Rightarrow$ one spin rep $S \cong \mathbb{K}^n$;
- $\mathrm{C}\ell(V)_+ \cong M_n[\mathbb{K}] \oplus M_n[\mathbb{K}] \Rightarrow$ two spin reps:

$$S_+ \cong \mathbb{K}^n_L, \text{ and } S_- \cong \mathbb{K}^n_R.$$

- Warning: Here $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, an *associative* normed division algebra.
- But for some special dimensions, $\mathbb{K} = \mathbb{O}$ makes more sense!

Octonions as spinors

Spin(7)

- We know that $\mathrm{C}\ell(0,7) \cong M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$, with modules $\mathbb{R}^8_L$ and $\mathbb{R}^8_R$.
- Secretly $\mathbb{R}^8 \cong \mathbb{O}$!
- Recall the homomorphism:

$$\begin{array}{rcl} \gamma_L : \mathrm{C}\ell(\mathrm{Im}\,\mathbb{O}) & \to & \mathrm{End}(\mathbb{O}) \\ x \in \mathrm{Im}\,\mathbb{O} & \mapsto & x_L, \end{array}$$

  where $x_L : \mathbb{O} \to \mathbb{O}$ denotes left multiplication, $x_L(y) = xy$.
- This works since $x_L x_L(s) = x(xs) = (xx)s = -|x|^2 s$.

Spin(7)

- In exactly the same way:

$$\begin{aligned} \gamma_R \colon \mathrm{C}\ell(\mathrm{Im}\,\mathbb{O}) &\to \mathrm{End}(\mathbb{O}) \\ x \in \mathrm{Im}\,\mathbb{O} &\mapsto x_R, \end{aligned}$$

  where $x_R \colon \mathbb{O} \to \mathbb{O}$ denotes *right* multiplication, $x_R(y) = yx$.
- In fact:

$$\mathrm{C}\ell(0,7) \cong \langle (x_L, x_R) \in \mathrm{End}(\mathbb{O}) \oplus \mathrm{End}(\mathbb{O}) \ : \ x \in \mathrm{Im}\,\mathbb{O} \rangle \,,$$

  and in turn:

$$\mathrm{Spin}(7) \cong \left\{ x_{1L} x_{2L} \cdots x_{2nL} \in \mathrm{End}(\mathbb{O}) \ : \ x_i^2 = -1, \ n \in \mathbb{N} \right\}.$$

- *This is the spin representation:*

$$x_{1L} x_{2L} \cdots x_{2nL}(s) = x_1(x_2(\cdots(x_{2n}s)\cdots)), \qquad \text{for } s \in \mathbb{O}.$$

Spin(7): summary

In dimension 7:

- Vectors are imaginary octonions:

$$V = \operatorname{Im} \mathbb{O}.$$

- Spinors are octonions:

$$S = \mathbb{O}.$$

- The action of $\mathrm{Spin}(7)$ on S is induced by left multiplication!

Spin(8)

- Let $V = \mathbb{O}$, $S_+ = \mathbb{O}$, and $S_- = \mathbb{O}$. *Triality!*
- Define

$$\begin{aligned} \gamma_+ : V \otimes S_+ &\to S_- \\ v \otimes s_+ &\mapsto v s_+ . \end{aligned}$$

$$\begin{aligned} \gamma_- : V \otimes S_- &\to S_+ \\ v \otimes s_- &\mapsto \overline{v} s_- . \end{aligned}$$

- $\mathrm{C}\ell(8) \cong \left\langle \begin{pmatrix} 0 & v_L \\ \overline{v}_L & 0 \end{pmatrix} \in \mathrm{End}(\mathbb{O}^2) \ : \ v \in \mathbb{O} \right\rangle$.

Spin(8)

- Multiplying pairs of unit vectors, we learn:

$$\mathrm{Spin}(8) \cong \langle (v_{1L}\overline{v_2}_L, \overline{v_1}_L v_{2L}) \in \mathrm{End}(\mathbb{O}) \oplus \mathrm{End}(\mathbb{O}) \ : \ |v_i|^2 = 1 \rangle.$$

- *These are the spin representations!*

$$v_1 v_2 \cdot s_+ = \overline{v_1}(v_2 s_+), \quad v_1 v_2 \cdot s_- = v_1(\overline{v_2} s_-).$$

for any generator $v_1 v_2 \in \mathrm{Spin}(8)$.

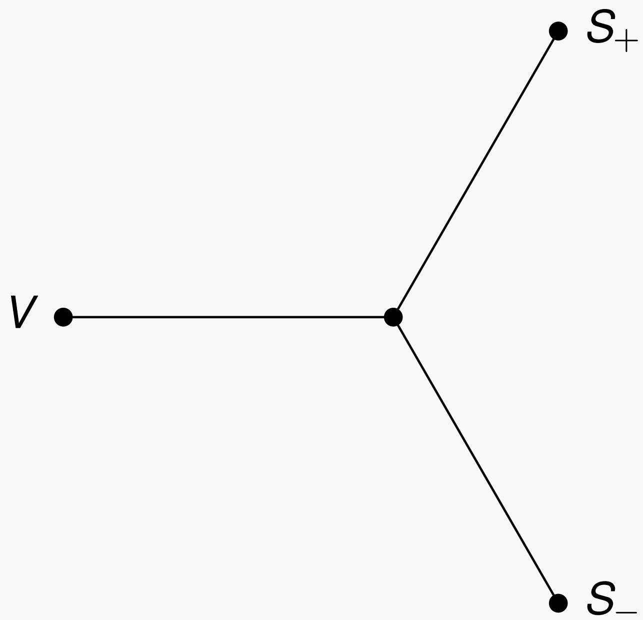
Spin(8): summary

In dimension 8:

- Vectors and both kinds of spinors are octonions:

$$V = \mathbb{O}, \quad S_+ = \mathbb{O}, \quad S_- = \mathbb{O}.$$

- Vectors act on S_+ by left multiplication, and on S_- by conjugate left multiplication, swapping S_+ and S_-.
- This induces the two spin reps of $\mathrm{Spin}(8)$.


S_+
V
S_-