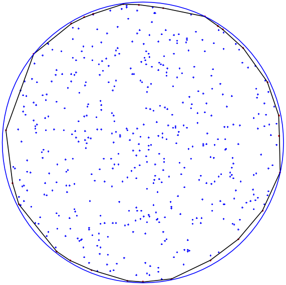
Fluctuations of Random Convex Hulls

Pierre Calka   Joe Yukich

Banff BIRS, May 28- June 2, 2023

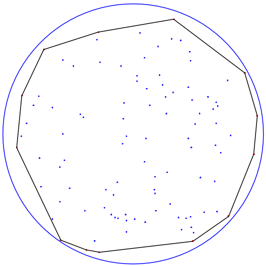
Introduction

- K: smooth convex body in $\mathbb{R}^d$.
- K_n: convex hull of n i.i.d. uniform points in K.
- For $K := \mathbb{B}^2, n = 500$, we have:



- How does the boundary of K_n fluctuate? Parabolic global constraints.

Introduction



- Boundary: ∂K_n . As n increases, new points appear, creating new facets which may subsume existing facets.

Goals

When $n \to \infty$ we seek:

- limit distribution of the area of a facet chosen at random; limit distribution of distance between boundary of K and a facet chosen at random,
- limit distribution of *maximal* facet distance and *maximal* facet area,

Goals

When $n \to \infty$ we seek:

- limit distribution of the area of a facet chosen at random; limit distribution of distance between boundary of K and a facet chosen at random,
- limit distribution of *maximal* facet distance and *maximal* facet area,
- distributional convergence of process of heights of convex hull boundary,
- distributional convergence of process of heights for dynamic two-parameter process.

Notation

- $K \subset \mathbb{R}^d$ smooth C^3 convex body; $\kappa(\cdot) :=$ Gauss curvature along ∂K; $\kappa > 0$
- Facets of K_n: Simplices a.s.
- $\mathcal{F}_n$: Facet chosen at random from the facets of K_n.
- $\mathrm{dist}(\mathcal{F}_n)$: distance between the hyperplane containing $\mathcal{F}_n$ and nearest supporting hyperplane.
- When K is the unit ball, $\mathrm{dist}(\mathcal{F}_n) := 1 - \mathrm{height}(\mathcal{F}_n)$.

Convergence in distribution of height/distance

- **Thm** As $n \to \infty$, we have $\mathbb{P}(n^{\frac{2}{d+1}} \mathrm{dist}(\mathcal{F}_n) \le t) \to 1 - F^{\text{height}}_{CH(K)}(t)$,

  where

$$F^{\text{height}}_{CH(K)}(t) = c_d \int_{\partial K} \kappa(z)^{\frac{1}{d+1}} \int_0^\infty e^{-v} \left(v + \frac{1}{\sqrt{\kappa(z)}} \frac{\kappa_{d-1}}{d+1} (2t)^{(d+1)/2} \right)^{\frac{d(d-1)}{d+1}} \cdot \exp\left(-\frac{\kappa_{d-1}}{d+1} \frac{1}{\sqrt{\kappa(z)}} (2t)^{(d+1)/2} \right) \mathrm{d}v \mathrm{d}z.$$

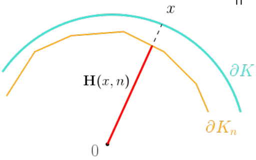
- Particular case $K = \mathbb{B}^2$:

$$\mathbb{P}\left(\frac{n(1 - \mathrm{height}(\mathcal{F}_n))}{n^{\frac{1}{3}}} \ge t \right) \sim Ct \exp(-\frac{4\sqrt{2}}{3} t^{\frac{3}{2}}) \quad \text{when } t \to \infty.$$

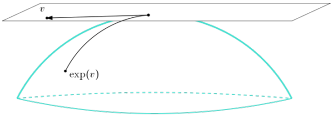
- $\mathbb{P}(n^{\frac{d-1}{d+1}} \mathrm{Vol}_{d-1}(\mathcal{F}_n) \le t) \to 1 - F^{\text{vol}}_{CH(K)}(t)$.

Notation $K = \mathbb{B}^d$

$\cdot \mathbf{H}(\mathbf{x}, \mathbf{n}) :=$
height of K_n in the direction $x \in \mathbb{S}^{d-1}$;
$\cdot \mathbf{H}(\mathbf{x}, \mathbf{n}) = \|\mathbf{x}\| \mathbf{H}(\frac{\mathbf{x}}{\|\mathbf{x}\|}, \mathbf{n}), \quad \mathbf{x} \in \mathbb{R}^{\mathbf{d}}$



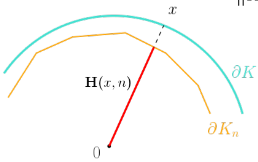
$\exp_{r\mathbb{S}^{d-1}} :=$
exponential map
at the north pole of $r\mathbb{S}^{d-1}$



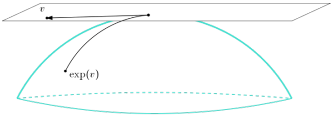
Process convergence of height function; unit ball $\mathbb{B}^d$

Notation $K = \mathbb{B}^d$

$\cdot \mathbf{H}(\mathbf{x}, \mathbf{n}) :=$
height of K_n in the direction $x \in \mathbb{S}^{d-1}$;
$\cdot \mathbf{H}(\mathbf{x}, \mathbf{n}) = \|\mathbf{x}\| \mathbf{H}(\frac{\mathbf{x}}{\|\mathbf{x}\|}, \mathbf{n}), \quad \mathbf{x} \in \mathbb{R}^{\mathbf{d}}$



$\exp_{r\mathbb{S}^{d-1}} :=$
exponential map
at the north pole of $r\mathbb{S}^{d-1}$

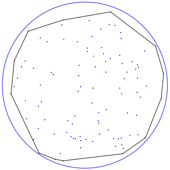


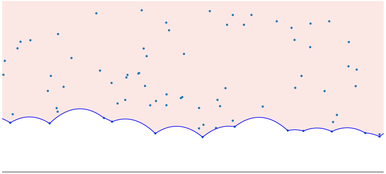
Theorem. As $n \to \infty$

$$\left\{ \frac{n - \mathbf{H}(n^{d/(d+1)} \exp_{n^{1/(d+1)}\mathbb{S}^{d-1}}(v), n)}{n^{(d-1)/(d+1)}} \right\}_{|v| \le n^{1/d}} \xrightarrow{\mathcal{D}} \text{Burgers' festoon.}$$

$\cdot \quad d = 2: \quad \frac{1}{3} \quad \frac{2}{3}$ scaling

Convergence of the height function in dimension $1+1$





- Down paraboloid with apex at $(x_0, h_0) \in \mathbb{R} \times \mathbb{R}^+$:

$$\Pi^{\downarrow}(x_0, h_0) := \{(x, h) \in \mathbb{R} \times \mathbb{R},\ h - h_0 \leq -\frac{|x - x_0|^2}{2}\}.$$

- $\mathcal{P}$: Poisson pt process on $\mathbb{R} \times \mathbb{R}^+$. Burgers' festoon process Φ is

$$\Phi(x) := \sup_{(x_0,h_0)\in\mathbb{R}\times\mathbb{R}^+,\ \Pi^{\downarrow}(x_0,h_0)\cap\mathcal{P}=\emptyset} \left(h_0 - \frac{|x - x_0|^2}{2}\right).$$

- The parabolic faces in Φ are the re-scaled asymptotic images of the facets of $K_n, n \to \infty$.

Comparison between the convex hull interface and KPZ

- $d = 2 : \frac{1}{3},\ \frac{2}{3}$ scaling, but the limit process (Burgers' festoon) contains no Airy process.

- The marginal radial fluctuations converge to a limit distribution which has right-sided Tracy-Widom like tails.

- A time coordinate is missing (we return to this later).

Extreme Values: Radial Fluctuations

- **Maximal radial fluctuation** $= MRF(K_n) =$ maximal facet distance.
- **Theorem** $MRF(K_n)$ asymptotically follows a Gumbel law, i.e., there are constants $a_i := a_i(K)$, $i \in \{0,1,2,3\}$, such that if

$$t_n(\tau, K) := n^{-\frac{2}{d+1}} [a_0(a_1 \log n + a_2 \log(\log n) + a_3 + \tau)]^{\frac{2}{d+1}},$$

then as $n \to \infty$ we have

$$\mathbb{P}\left(MRF(K_n) \le t_n(\tau, K)\right) \to \exp(-e^{-\tau}),\ \tau \in (-\infty, \infty).$$

- $d = 2$: Bräker, Hsing, Bingham (1998).

Extreme Values: Facet Volumes

· $MFV(K_n) :=$ maximal volume of facets in ∂K_n

· **Theorem.** $MFV(K_n)$ asymptotically follows a Gumbel law, i.e., there are constants $b_i := b_i(K)$, $i \in \{0, 1, 2, 3\}$, such that if

$$t_n(\tau, K) := n^{-\frac{d-1}{d+1}} [b_0(b_1 \log n + b_2 \log(\log n) + b_3 + \tau)]^{\frac{d-1}{d+1}},$$

then as $n \to \infty$ we have

$$\mathbb{P}(MFV(K_n) \leq t_n(\tau, K)) \to \exp(-e^{-\tau}),\ \tau \in (-\infty, \infty).$$

Growth of fluctuations; $d \geq 2$

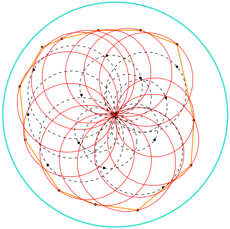
- K_n: convex hull of an i.i.d. uniform sample in K of size n.
- **Corollary** (growth of fluctuations, with log precision in $d = 2$).

$$MRF(nK_n) \stackrel{P}{=} \Theta(n^{1/3}(\log n)^{2/3}); \;\; MFV(nK_n) \stackrel{P}{=} \Theta(n^{2/3}(\log n)^{1/3}).$$

- $\frac{1}{3}, \frac{2}{3}$ scaling.
- Hammond: Convex hull boundary belongs to the **'baby KPZ class'**. Global parabolic constraints, but no local Brownian fluctuations.
- $MRF(nK_n) \stackrel{P}{=} \Theta(n^{\chi(d)}(\log n)^{\frac{2}{d+1}}), \;\; MFD(nK_n) \stackrel{P}{=} \Theta(n^{\xi(d)}(\log n)^{\frac{1}{d+1}})$, where $\chi(d) := (d-1)/(d+1)$ and $\xi(d) := d/(d+1)$ satisfy $\chi = 2\xi - 1$.
- Is there a two parameter space-time process?

Two parameter process: dynamic flower

- $X_i, 1 \leq i \leq n$, i.i.d. uniform in $\mathbb{B}^2$. **Their flower is** $\bigcup_{i=1}^n B(\frac{X_i}{2}, \frac{|X_i|}{2})$:



Support function of convex hull is boundary of flower, i.e.,

$$\max_{i \leq n} |X_i| \cos\left(|\theta - \theta_{X_i}|\right), \theta \in [0, 2\pi].$$

- Rescale radially by n, longitudinally by $\frac{1}{\sqrt{t}}$.
- 'Height of boundary of rescaled flower' at spatial coordinate θ at time $t > 0$:

$$h_n(\theta, t) = \max_{i \leq n} n|X_i| \cos\left(\frac{|\theta - \theta_{X_i}|}{\sqrt{t}\sqrt{n}}\right)$$

- For fixed n and large t the petals have nearly slope dependent growth.

Two parameter process: dynamic flower

- Re-scale space by $n^{2/3}$ and time by n:

$$h_n(n^{2/3}\theta, nt) = \max_{i \le n} n|X_i| \cos\left(\frac{|\theta - \theta_{X_i}|}{\sqrt{t} \cdot n^{1/3}}\right).$$

- Define two parameter process with $1:2:3$ scaling:

$$H_n(\theta, t) = \frac{h_n(n^{2/3}\theta, nt) - n}{n^{1/3}}, \quad \theta \in \mathbb{R}, \ t > 0.$$

- **Theorem.** Fix $t, L \ge 0$. As $n \to \infty$

$$\{H_n(\theta, t)\}_{|\theta| \le L} \xrightarrow{\mathcal{D}} \{H(\theta, t)\}_{|\theta| \le L}.$$

- Limit process given by variational formula (Burgers' festoon)

$$H(\theta, t) := \sup_{(v_0, h_0) \in \mathcal{P}} (h_0 - \frac{|\theta - v_0|^2}{2t}), \quad \theta \in \mathbb{R},$$

with $\mathcal{P}$ a PPP in $\mathbb{R} \times \mathbb{R}_-$.

* Convergence in the space of cont. fcts on $|\theta| \le L$ w. sup norm metric.

Two parameter process: dynamic flower

- **Theorem.** Fix $t, L \geq 0$. As $n \to \infty$

$$\{H_n(\theta, t)\}_{|\theta| \leq L} \xrightarrow{\mathcal{D}} \{H(\theta, t)\}_{|\theta| \leq L},$$

  with

$$H(\theta, t) := \sup_{(v_0, h_0) \in \mathcal{P}} (h_0 - \frac{|\theta - v_0|^2}{2t}), \quad \theta \in \mathbb{R},$$

  with $\mathcal{P}$ a PPP in $\mathbb{R} \times \mathbb{R}_-$
- $\mathbb{P}(H_n(\theta_o, 1) \geq s) \to \exp(-\frac{4\sqrt{2}}{3} s^{3/2}),\ n \to \infty.$
- Shape profile is a pattern of coarsening paraboloids.
- $H_n(\theta, t)$ is an example of a process which satisfies 1:2:3 scaling, but does not belong to the KPZ fixed point universality class of Matetski, Quastel and Remenik (no Airy process).

Proof ideas: $d = 2$

- Fix n, t. Define the parabolic scaling transform $T^{(n)} : n\mathbb{B}^2 \to \mathbb{R} \times \mathbb{R}_-$

$$T^{(n)}(x) = \left(n^{-2/3} \exp_{n\mathbb{S}}^{-1}(\frac{x}{|x|}),\ n^{-1/3}(|x| - n) \right), \quad x \in n\mathbb{B}^2.$$

- $T^{(n)}$ maps $n\mathbb{S}^1$ to $[-\pi n^{1/3}, \pi n^{1/3}]$ and maps boundary of flower to a piecewise quasi-parabolic process $H^{(n)}$ in $[-\pi n^{1/3}, \pi n^{1/3}] \times [-n^{2/3}, 0]$.
- the shape of the quasi-parabolas constituting $H^{(n)}$ depends on n through via $T^{(n)}$; their apices are the image of an i.i.d. sample in $n\mathbb{B}^2$ under $T^{(n)}$, here denoted $\mathcal{P}^{(n)}$.
- the quasi-parabolas in the finite-area rectangle $[-L, L] \times [-\ell, 0]$, converge uniformly to parabolas in $[-L, L] \times [-\ell, 0]$, $n \to \infty$.
- couple $\mathcal{P}^{(n)}$ with a rate one Poisson point process on $\mathbb{R} \times \mathbb{R}_-$ such that with high probability they coincide on $[-L, L] \times [-\ell, 0]$.

Summary

- Limit distribution of height function of convex hull boundary has right-sided Tracy-Widom like tails in $d = 2$,
- Limit distributions of *maximal* facet distance and *maximal* facet area are of Gumbel type,
- Height process of convex hull boundary converges to Burgers' festoon,
- Height process of dynamic two-parameter flower converges to dynamic Burgers' festoon, with $1:2:3$ scaling and right-sided Tracy-Widom like tails in $d = 2$.

Thank you for your attention!