High moments of the SHE & spacetime limit shapes of the KPZ equation

Random Growth Models and KPZ Universality, Banff

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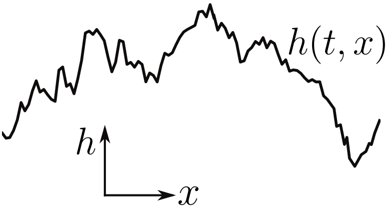
One-point upper-tail LDPs for models in the KPZ class

- Longest increasing subsequence: Deuschel, Seppäläinen, Zeitouni, . . .
- Last passage percolation: Ciech, Georgiou, Janjigian, Johansson, . . .
- Directed polymers: Georgiou, Janjigian, Seppäläinen, . . .
- Corner growth models: Emrah, Janjigian,. . .
- Random matrices and random operators: Baik, Buckingham, DiFranco, Deift, Dumaz, Its, Krasovsky, Ramirez, Rider, Viràg, Tracy, Widom, . . .
- ASEP height function: Damron, Das, Petrov, Sivakoff, Zhu, . . .
- Tagged particle in ASEP: Sethuraman, Varadhan,. . .
- KPZ equation: more on this later

Kardar–Parisi–Zhang (KPZ)

$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}(\partial_x h)^2 + \eta$$

η =spacetime white noise

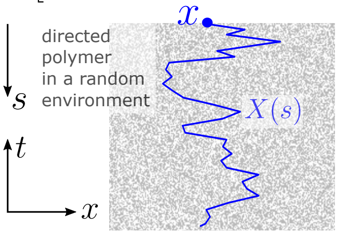


Stochastic Heat Equation (SHE)

$$\partial_t Z = \tfrac{1}{2}\partial_{xx} Z + \eta Z$$

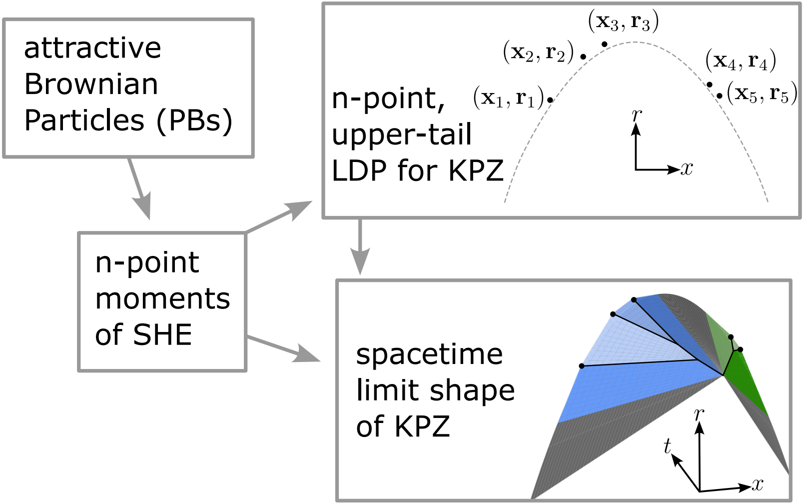
Feynman–Kac:

$$Z(T,x) = \mathbb{E}_{\mathrm{BM}}\Big[e^{\int_0^T \mathrm{d}s\, \eta(T-s, X(s))} Z(0, X(T))\Big]$$

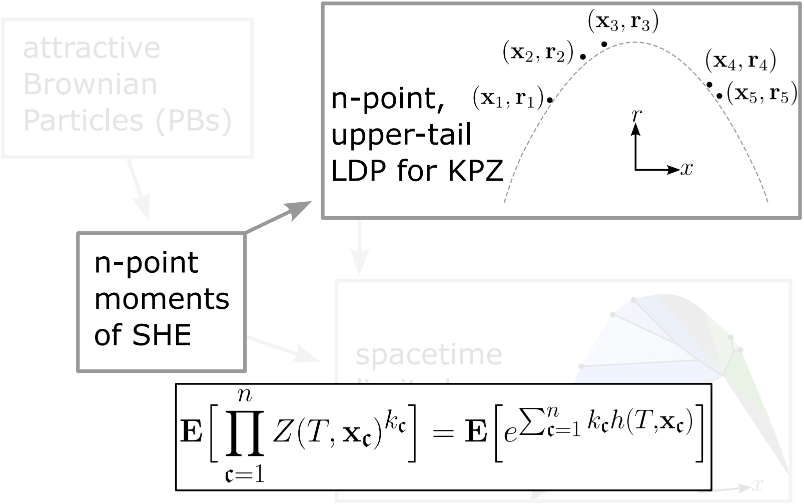


$$e^h = Z$$

Flow chart of this talk

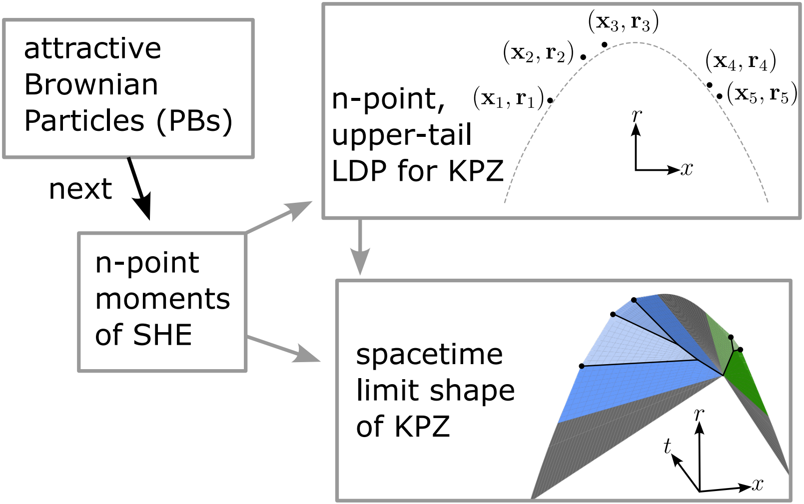


Flow chart of this talk



$$\mathbf{E}\Big[\prod_{\mathfrak{c}=1}^{n} Z(T,\mathbf{x}_{\mathfrak{c}})^{k_{\mathfrak{c}}}\Big] = \mathbf{E}\Big[e^{\sum_{\mathfrak{c}=1}^{n} k_{\mathfrak{c}} h(T,\mathbf{x}_{\mathfrak{c}})}\Big]$$

Flow chart of this talk



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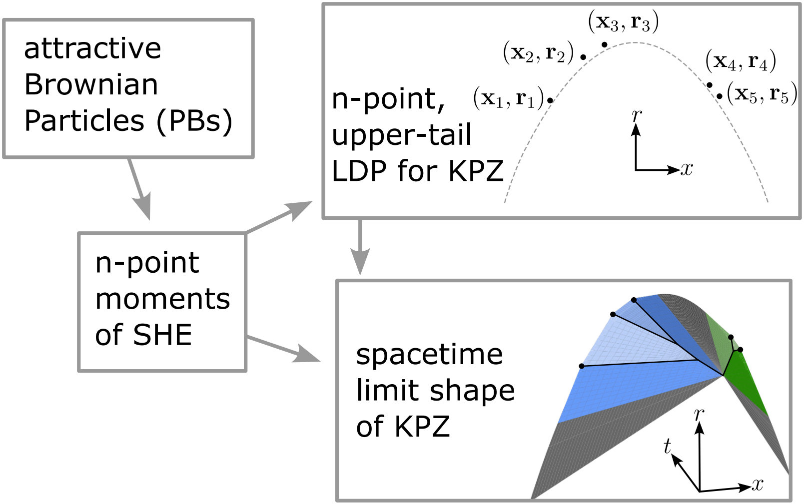
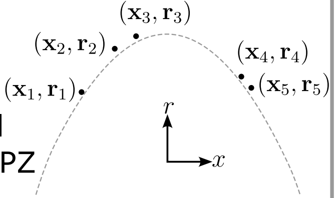


Flow chart of this talk

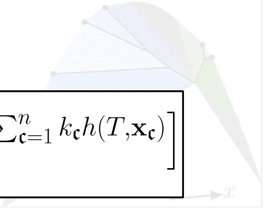
attractive Brownian Particles (PBs)

n-point, upper-tail LDP for KPZ



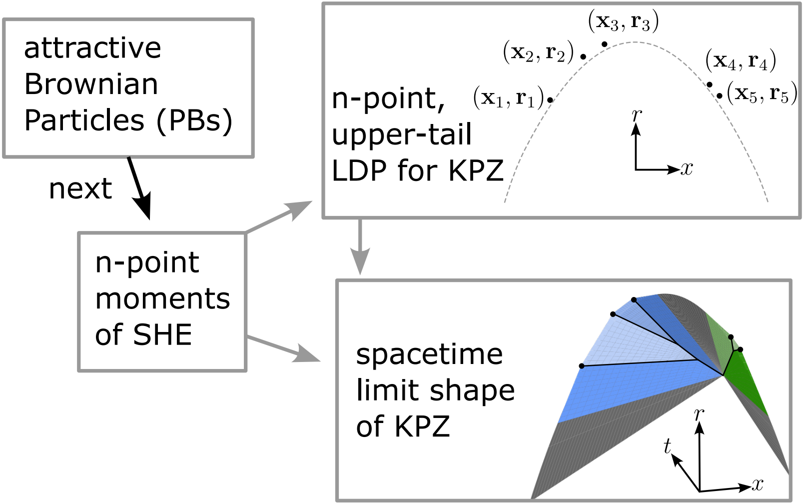
n-point moments of SHE

spacetime



$$\mathbf{E}\Big[\prod_{\mathfrak{c}=1}^{n} Z(T,\mathbf{x}_{\mathfrak{c}})^{k_{\mathfrak{c}}}\Big] = \mathbf{E}\Big[e^{\sum_{\mathfrak{c}=1}^{n} k_{\mathfrak{c}} h(T,\mathbf{x}_{\mathfrak{c}})}\Big]$$

Flow chart of this talk



SHE moments ⇔ attractive BPs

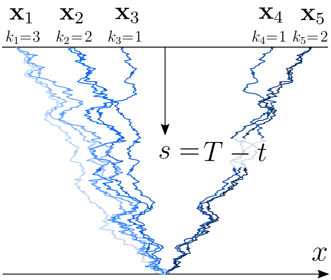
$k_1, \ldots, k_n \in \mathbb{Z}_{>0} \qquad k := k_1 + \ldots + k_n \qquad Z(0, \bullet) = \delta_0$

$$\mathbf{E}\Big[\prod_{\mathfrak{c}=1}^{n} Z(T, \mathbf{x}_{\mathfrak{c}})^{k_{\mathfrak{c}}} \Big]$$

(Feynman–Kac + Tanaka + Girsanov)

$$= e^{\frac{1}{24}T(k^3-k)+\frac{1}{2}\sum_{\mathfrak{c},\mathfrak{c}'=1}^{n} k_{\mathfrak{c}} k_{\mathfrak{c}'} |\mathbf{x}_{\mathfrak{c}}-\mathbf{x}_{\mathfrak{c}'}|}$$

$$\times \, \mathbb{E}_{\text{aBP}}\Big[\prod_{i=1}^{k} \delta_0(X_i(T)) \Big]$$



attractive Brownian Particles (BPs)

$$\mathrm{d}X_i(s) = \sum_{j=1}^{k} \frac{1}{2}\mathrm{sgn}(X_j - X_i)\mathrm{d}s + \mathrm{d}B_i(s)$$

A special case of rank-based diffusions: Banner, Banerjee, Budhiraja, Cabezas, Dembo, Fernholz, Ichiba, Jara, Karatzas, Olla, Pal, Pitman, Papathanakos, Sarantsev, Shkolnikov, Sidoravicius, Tsai, Varadhan, Zeitouni, . . .

Scaling

Particle numbers. $k_{\mathfrak{c}} = N\mathfrak{m}_{\mathfrak{c}}$; $\quad \mathfrak{m}_{\mathfrak{c}} \in (0,\infty)$; $\quad N \to \infty$ (1)

Time. Take any $T = T_N$ with $N^2T = N^2T_N \to \infty$ (2).
Allow $T \to 0$, $T \to 1$, $T \to \infty$ as long as (1)–(2) hold.

Space. $X_i^N(s) := \frac{1}{NT} X_i(Ts)$

$$\text{(moments)} \quad \mathbf{E}\Big[\prod_{\mathfrak{c}=1}^{n} Z(T, NT\mathbf{x}_{\mathfrak{c}})^{N\mathfrak{m}_{\mathfrak{c}}}\Big]$$

$$\text{(aBPs)} \quad \mathrm{d}X_i^N = \frac{1}{N}\sum_{j=1}^{N\mathfrak{m}} \frac{1}{2}\mathrm{sgn}(X_j^N - X_i^N)\,\mathrm{d}s + \frac{1}{\sqrt{N^2T}}\mathrm{d}B_i(s)$$

(1) $N\mathfrak{m}_{\mathfrak{c}} \in \mathbb{Z}_{>0} \Leftrightarrow \mathfrak{m}_{\mathfrak{c}} \in \frac{1}{N}\mathbb{Z}_{>0}$

(2) Drift dominates diffusive effect. Particles tend to cluster.

(Space) The scaling NT makes the total drift of order 1.

LDP for the attractive BPs

Theorem (T 23)

$$\mu_N(s) := \frac{1}{N}\sum_{i=1}^{N\mathfrak{m}} \delta_{X_i^N(s)}, \qquad \mu_N \in \mathscr{C}\big([0,1], \mathfrak{m}\mathscr{P}(\mathbb{R})\big)$$

As $N \to \infty$, the empirical measure μ_N satisfies an LDP on $\mathscr{C}([0,1], \mathfrak{m}\mathscr{P}(\mathbb{R}))$ with speed N^3T and an explicit rate function $\mathbb{I}$.

Remark

- Under the diffusive scaling, $N \to \infty$ and N^2T fixed, [Dembo–Shkolnikov–Varadhan–Zeitouni 16] proved the LDP for a general class of rank-based diffusions.
- The behavior under $N^2T \to \infty$ (considered here) is very different from that under the diffusion scaling (considered in [DSVZ 12]).

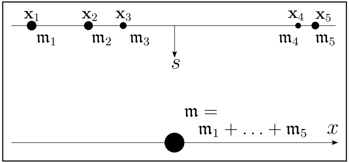
LDP for the attractive BPs

Theorem (T 23)

$\boldsymbol{\mu}_N$ satisfies an LDP with speed N^3T and an explicit rate function $\mathbb{I}$.

Corollary

Under $Z(0,\cdot) = \delta_0$, $\quad \mathbf{E}[\prod_{\mathfrak{c}=1}^{n} Z(T, NT\mathbf{x}_{\mathfrak{c}})^{N\mathfrak{m}_{\mathfrak{c}}}] \approx \exp(N^3T \cdot L_{\text{SHE}}(\vec{\mathfrak{m}}))$



$$L_{\text{SHE}}(\vec{\mathfrak{m}}) = L_{\text{SHE}}(\mathfrak{m}_1, \ldots, \mathfrak{m}_n)$$

$$:= \frac{\mathfrak{m}^3}{24} + \sum_{\mathfrak{c},\mathfrak{c}'=1}^{n} \frac{1}{2}\mathfrak{m}_{\mathfrak{c}}\mathfrak{m}_{\mathfrak{c}'}|\mathbf{x}_{\mathfrak{c}} - \mathbf{x}_{\mathfrak{c}'}| - \mathbb{I}_*$$

$$\mathbb{I}_* := \inf\Big\{\mathbb{I}(\mu) : \mu \in \mathscr{C}([0,1], \mathfrak{m}\mathscr{P}(\mathbb{R})), \mu(0) = \sum_{\mathfrak{c}=1}^{n} \mathfrak{m}_{\mathfrak{c}}\delta_{\mathbf{x}_{\mathfrak{c}}}, \mu(1) = \mathfrak{m}\delta_0\Big\}$$

LDP for the attractive BPs

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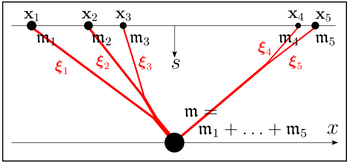

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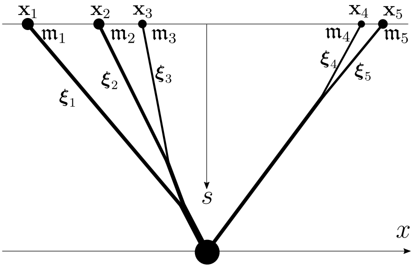
Theorem (T 23)

Unique minimizer of the infimum: $\boldsymbol{\xi} = \sum_{\mathfrak{c}=1}^{n} \mathfrak{m}_{\mathfrak{c}}\delta_{\boldsymbol{\xi}_{\mathfrak{c}}}$, *the* ***optimal deviation***.

Optimal deviation

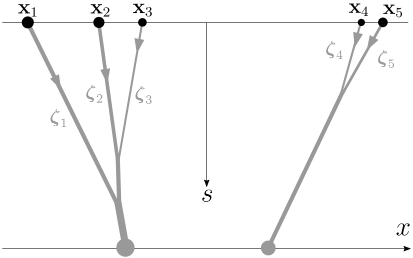


Optimal clusters $\xi_1, \ldots, \xi_n$ and optimal deviation ξ

- $\boldsymbol{\xi}_{\mathfrak{c}}(s) :=$ $\qquad\qquad$ $\boldsymbol{\xi}(s) = \sum_{\mathfrak{c}=1}^{n} \mathfrak{m}_{\mathfrak{c}} \delta_{\boldsymbol{\xi}_{\mathfrak{c}}(s)}$

Optimal deviation

$$\mathrm{d}X_i^N = \frac{1}{N}\sum_{j=1}^{N\mathfrak{m}} \frac{1}{2}\mathrm{sgn}(X_j^N - X_i^N)\,\mathrm{d}s + \frac{1}{\sqrt{N^2T}}\mathrm{d}B_i(s)$$



Inertia clusters, $\zeta_1, \ldots, \zeta_{\mathfrak{c}}$

- $\zeta_{\mathfrak{c}}$ has mass $\mathfrak{m}_{\mathfrak{c}}$.

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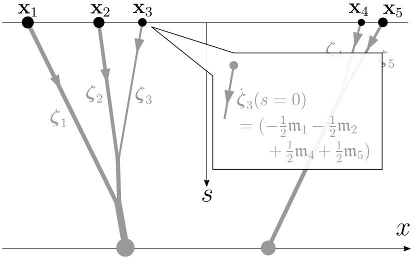

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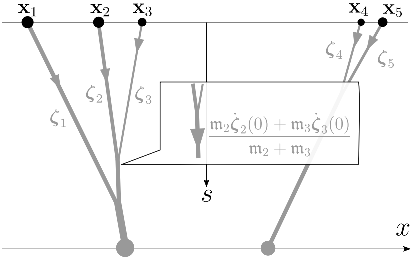
- Merge according to conservation of momentum.



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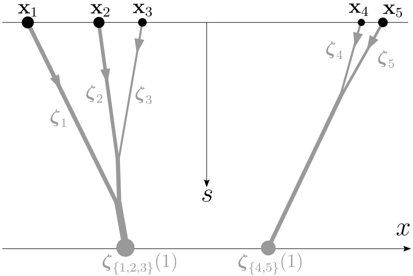

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Branches, $\mathfrak{b}$: $\mathfrak{c}, \mathfrak{c}' \in \mathfrak{b}$ if and only if $\zeta_{\mathfrak{c}}(1) = \zeta_{\mathfrak{c}'}(1)$

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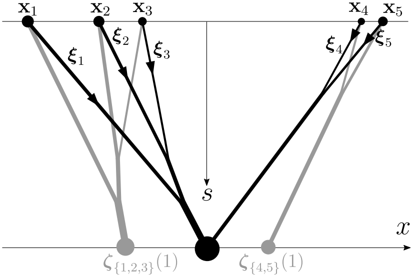

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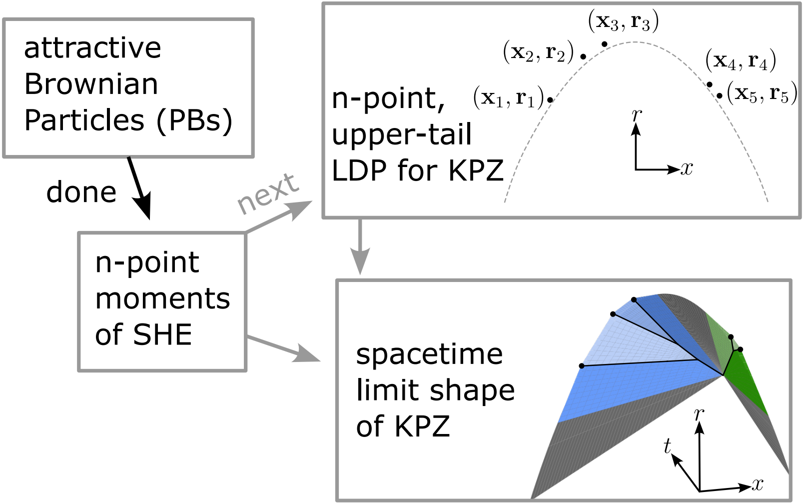
Optimal clusters $\xi_1, \ldots, \xi_n$ and optimal deviation ξ

- $\xi_\mathfrak{c}(s) := \zeta_\mathfrak{c}(s) + (-\zeta_\mathfrak{b}(1))\,s, \quad \mathfrak{c} \in \mathfrak{b}$ $\qquad \xi(s) = \sum_{\mathfrak{c}=1}^n \mathfrak{m}_\mathfrak{c}\delta_{\xi_\mathfrak{c}(s)}$

So far and what's next



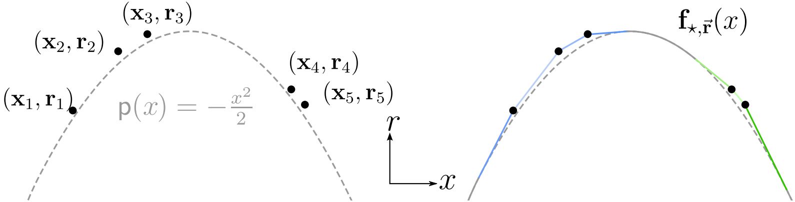
Moments of SHE → LDP for KPZ

Proposition (T 23)

Let $\mathscr{R}_{conc} := \{\vec{\mathbf{r}} : \mathsf{f}_{\star,\vec{\mathbf{r}}} \geq \mathsf{p},\ \mathsf{f}_{\star,\vec{\mathbf{r}}}$ *is concave*$\}$. *The functions*

$$L_{\text{SHE}}(\vec{\mathfrak{m}}) : [0,\infty)^n \to [0,\infty) \qquad I_{\text{KPZ}}(\vec{\mathbf{r}}) : \mathscr{R}_{conc} \to [0,\infty)$$

are strictly convex and the Legendre transform of each other.



$$I_{\text{KPZ}}(\vec{\mathbf{r}}) = I_{\text{KPZ}}(\mathbf{r}_1, \ldots, \mathbf{r}_n) := \int_{\mathbb{R}} \mathrm{d}x \left(\tfrac{1}{2}(\partial_x \mathsf{f}_{\star,\vec{\mathbf{r}}})^2 - \tfrac{1}{2}(\partial_x \mathsf{p})^2 \right)$$

Gibbs line ensembles [Corwin Hammond 14, 16] and [Ganguly–Hegde 22]. Our approach goes through moments and is different.

n-point, upper-tail LDP for the KPZ equation

$$h_N(t,x) := \tfrac{1}{N^2 T}\big(h(Tt, NTx) + \log\sqrt{T} + \tfrac{T}{24}\big)$$

Corollary (T 23 & Lin–T 23)

Under delta initial condition $Z(0,\bullet) = \delta_0$, for any $\vec{\mathbf{r}} \in \mathscr{R}^{\circ}_{\text{conc}}$,

$$\mathbf{P}\big[|h_N(1,\mathbf{x}_{\mathfrak{c}}) - \mathbf{r}_{\mathfrak{c}}| \le \delta, \mathfrak{c} = 1,\ldots,n\big] \approx e^{-N^3 T \cdot I_{\mathrm{KPZ}}(\vec{\mathbf{r}})}$$

$N \to \infty$ *and* $N^2 T = N^2 T_N \to \infty$ *first;* $\delta \to 0$ *later.*

Covered scaling regimes

- Short or unit-order time $T \to 0$ or $T \to 1$: any deviation $\gg 1$
- Long time $T \to \infty$: any deviation $\gg T$

Doesn't cover the hyperbolic scaling regime, $N = 1$ and $T \to \infty$, $h_T(t,x) := \frac{1}{T}(h(Tt, Tx) + \log\sqrt{T} + \frac{T}{24})$.

Related results

First, when $n = 1$ and $\mathbf{x}_1 = 0$, we recover $I_{\text{KPZ}}(\mathbf{r}) = \frac{4\sqrt{2}}{3}\mathbf{r}^{3/2}$.

One point, upper-tail LDPs

- Hyperbolic scaling regime

$$\mathbf{P}\Big[\tfrac{1}{T}h(Tt,0) \approx -\tfrac{1}{24} + \mathbf{r}\Big] \approx e^{-T\,\frac{4\sqrt{2}}{3}\mathbf{r}^{3/2}}, \quad T \to \infty, \mathbf{r} > 0$$

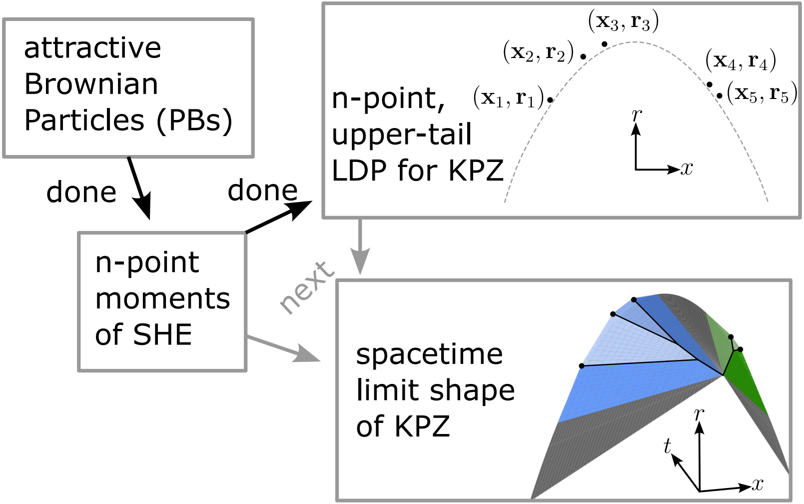
  - Predicted in [Le Doussal–Majumdar–Schehr 16]; proven in [Das–T 21].

- Other scaling regimes and/or other initial conditions
  - Physics: Asida, Hartman, Janas, Kolokolov, Korshunov, Katzav, Krajenbrink, Le Doussal, Majumdar, Livne, Meerson, Prolhac, Rosso, Sasorov, Schmidt, Smith, Vilenkin, . . .
  - Math rigorous: Corwin, Das, Gaudreau Lamarre, Ghosal, Lin, Tsai, . . .

n-point upper tails and terminal-time limit shape

- [Ganguly–Hegde 22]
  - Detailed and optimal n-point bounds that hold for all $t > t_0$.
  - When specialized onto the hyperbolic scaling regime: the n-point LDP and the terminal-time limit shape $\mathsf{f}_{\star,\mathbf{r}}$.

So far and what's next



Spacetime limit shape

$$\mathcal{E}_{N,\delta}(\vec{\mathbf{r}}) := \{|h_N(1,\mathbf{x}_{\mathfrak{c}}) - \mathbf{r}_{\mathfrak{c}}| \leq \delta, \mathfrak{c} = 1,\ldots,n\}$$

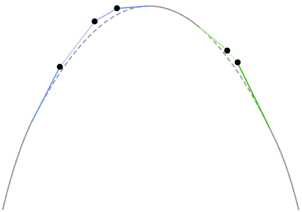
Theorem (Lin–T 23)

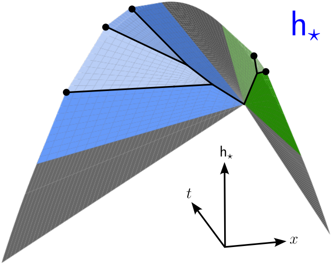
Under $Z(0,\bullet) = \delta_0$, for any $\vec{\mathbf{r}} \in \mathscr{R}^{\circ}_{\text{conc}}$ and $R < \infty$,

$$\mathbf{P}\Big[\|h_N - \mathsf{h}_\star\|_{\mathscr{L}^\infty([\frac{1}{R},1]\times[-R,R])} < \tfrac{1}{R} \,\Big|\, \mathcal{E}_{N,\delta}(\vec{\mathbf{r}})\Big] \longrightarrow 1$$

$N \to \infty$ and $N^2 T = N^2 T_N \to \infty$ first; $\delta \to 0$ later.

$$\mathsf{f}_{\star,\vec{\mathbf{r}}}(x) = \mathsf{h}_\star(1,x)$$

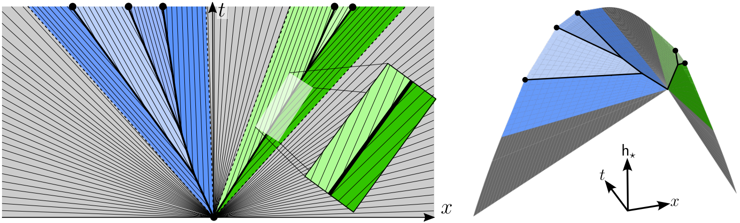




Spacetime limit shape

$\mathsf{h}_\star(1-s,x)$ is the entropy solution of the *backward* equation

$$-\partial_s \mathsf{h}_\star = \tfrac{1}{2}(\partial_x \mathsf{h}_\star)^2 \qquad \mathsf{h}_\star(1,x) = \mathsf{f}_{\star,\mathbf{r}}(x)$$



Remark. $\mathsf{h}_\star(t,x)$ does solve the forward equation $\partial_t \mathsf{h}_\star = \frac{1}{2}(\partial_x \mathsf{h}_\star)^2$, but is *non-entropic* for the forward equation. Consistent with [Jensen 00] and [Varadhan 04].

[Janjigian–Rassoul-Agha–Seppäläinen 22] The hydrodynamic limit h_0 is the entropy solution of $\partial_t \mathsf{h}_0 = \frac{1}{2}(\partial_x \mathsf{h}_0)^2$.

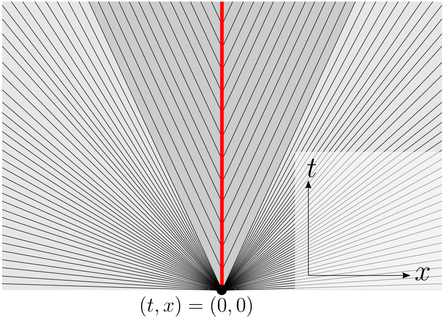
Mechanism of the deviations, noise-corridor effect

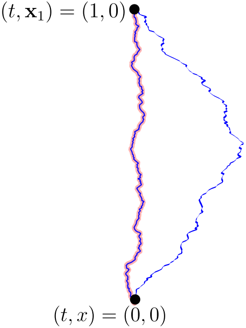
$$e^{h(T,NTx)} = Z(T,NTx) = \mathbb{E}_{\mathrm{BM}}\Big[e^{\int_0^T \mathrm{d}s\, \eta(T-s,X(s))}\delta_0(X(T))\Big]$$

Consider $n = 1$ and $\mathbf{x}_1 = 0$.

A known phenomenon. [Seppäläinen 98], [Deuschel–Zeitouni 99]

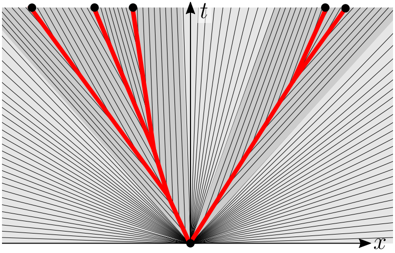
The noise η makes itself anomalously large only around $[0,1] \times \{0\}$. We call this the **noise-corridor effect**.





Mechanism of the deviations, noise-corridor effect

When $n > 1$, a similar noise-corridor effect occurs, with the noise-corridors being the shocks.



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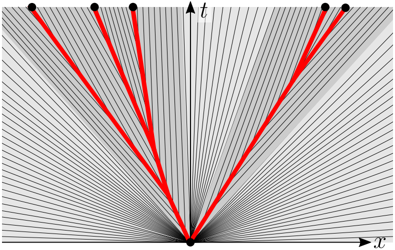

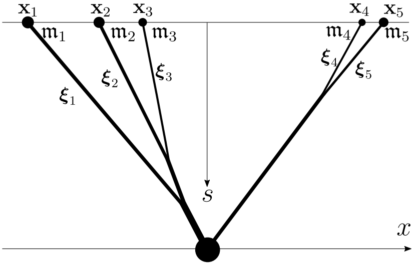

Proposition (T 23)

(Noise corridors in KPZ := shocks) = (optimal clusters in attractive BPs)

Summary and open problems

We utilize moments and a tree structure to obtain the spacetime limit shape. A crucial idea is to utilize the noise-corridor effect.

Conjecture. The same results should hold in the hyperbolic scaling regime ($h_T(t,x) := \frac{1}{T}(h(Tt,Tx) + \log\sqrt{T} + \frac{T}{24})$, $T \to \infty$) and for $\vec{\mathbf{r}} \in \mathscr{R}$ (all upper-tail deviations). The limit shape $\mathsf{h}_\star$ is still defined as the backward entropy solution, though the shocks are no longer piecewise linear when $\vec{\mathbf{r}} \notin \mathscr{R}_{\text{conc}}$.

More general initial conditions. One may seek to use the convolution formula as in [Corwin–Ghosal 20] and [Ghosal–Lin 23].

Possibility of symmetry breaking. Predicted in the weak-noise regime [Janas–Kamenev–Meerson 16], [Smith–Kamenev–Baruch Meerson 18], [Krajenbrink–Le Doussal 17, 19]; should hold here too.

Conjecture. Symmetry breaking under the two-delta initial condition $Z(0,\bullet) = \delta_{-NT} + \delta_{+NT}$. Very preliminary calculations in [Appendix B, Lin–T 23].

Thank you and cheers to Timo!