Scaling limit of stationary multi-type distributions in the KPZ class

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Banff
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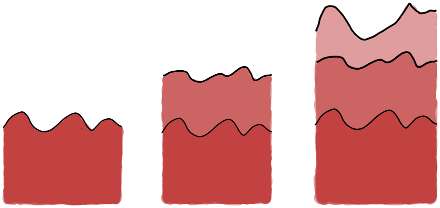
The KPZ class

Random growth interface

Consider a two-dimensional area growing in time. We are interested in dynamics of the one-dimensional interface.



t_1 t_2 t_3

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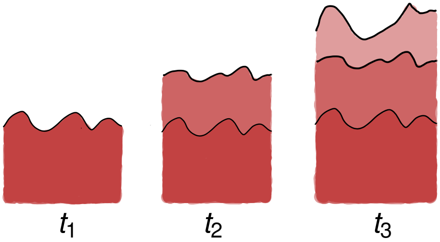
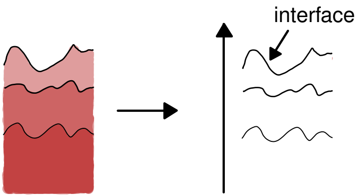
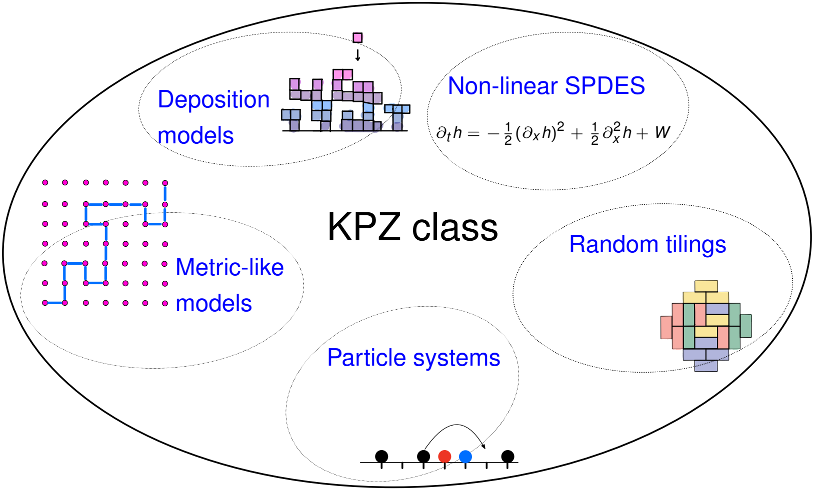


A random growth interface h is a random map from time to the space of functions i.e. $h : \mathbb{R}_+ \to \mathbb{R}^{\mathbb{R}}$



The KPZ class



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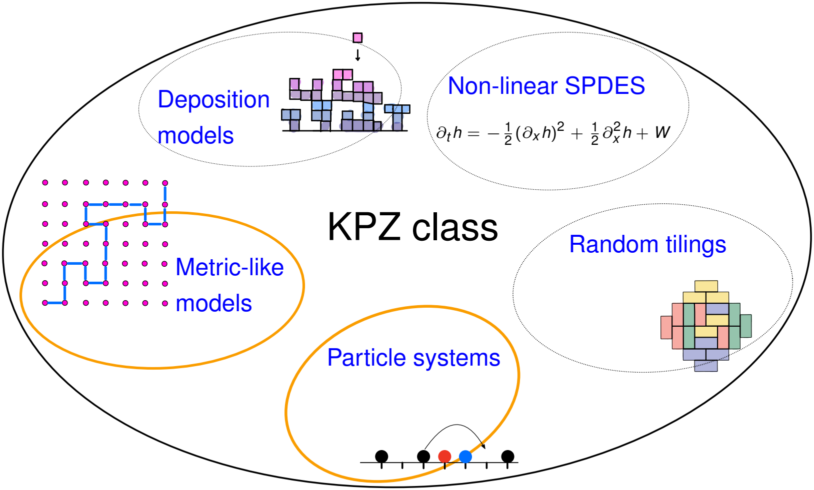
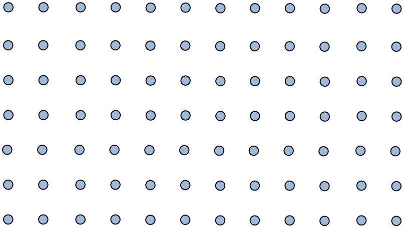


SMTD in metric-like models

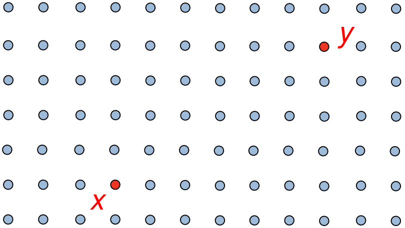
Metric-like models - last passage percolation (LPP)

Assume Exp(1) distributed and independent weights on the lattice.



Metric-like models - last passage percolation (LPP)

For any two ordered points x, y on the lattice, we let $L_{x,y}$ be the maximal weight that can be collected by an up-right path connecting x and y. The unique maximal path is called the *geodesic* from x to y.



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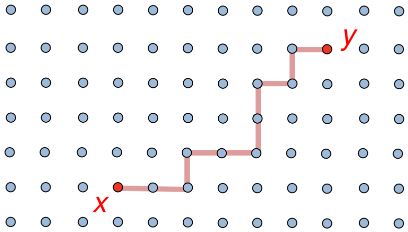
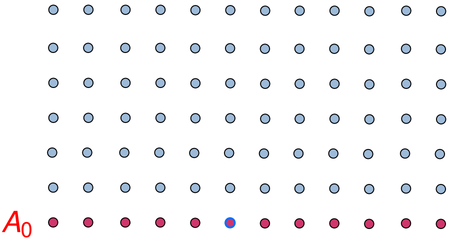


Metric-like model as growth models

One can think of metric-like models as growth models or infection models. For each $t > 0$, we color a point 'red' if the distance from the bottom line to the point is smaller of equal to t.



Assign weights $\{h(z)\}_{z \in A_0}$ to the points in A_0. Then, for $t > 0$ we color the site x 'red' if and only if

$$d_{LPP}(A_0, x) := \sup_{z \in A_0} h(z) + L(z, x) \leq t$$

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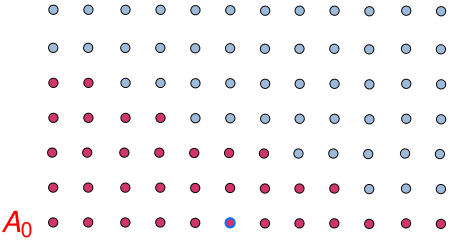

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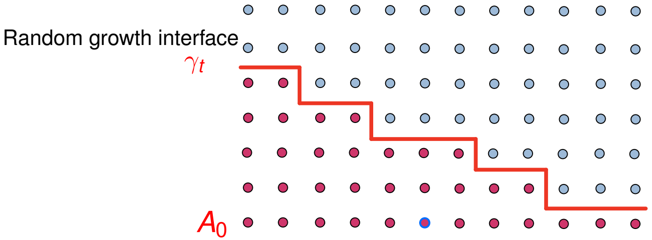

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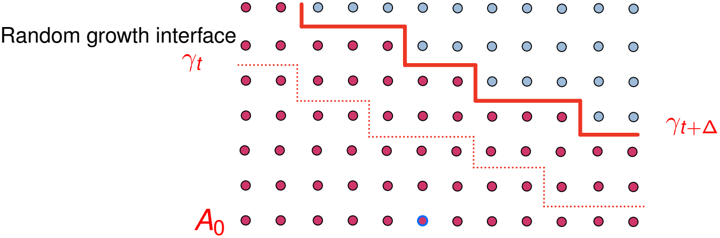

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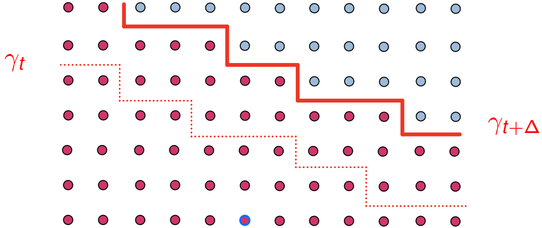
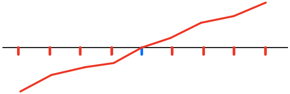
One-type stationary measures

Question: Is there a distribution on *h* such that up to some random translation $\gamma_t \sim \gamma_{t+\Delta}$?

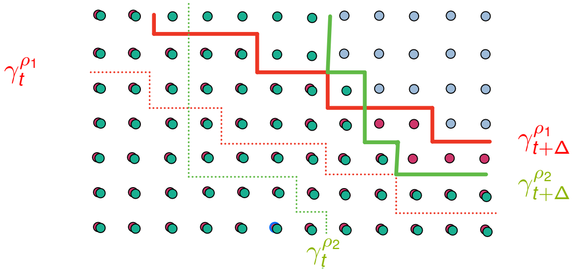


Answer: Yes, for any $\rho \in (0, 1)$ there exists a bi-infinite i.i.d. random walk h^ρ such that $\gamma_\cdot$ is stationary and its slope depends on ρ.

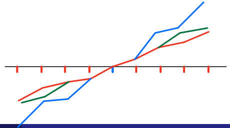


Two-type stationary measures

Question: Fix $0 < \rho_1 < \rho_2 < 1$. Is there a distribution on (h^{ρ_1}, h^{ρ_2}) such that such that up to random translation $(\gamma_t^{\rho_1}, \gamma_t^{\rho_2}) \sim (\gamma_{t+\Delta}^{\rho_1}, \gamma_{t+\Delta}^{\rho_2})$?



Answer: Yes, the stationary multi-type distributions (SMTD) $(h^{\rho_1}, \ldots, h^{\rho_k})$ for $0 < \rho_1 < \rho_2 < \ldots < \rho_k < 1$ exists(Fan and Sepäläinen 18').



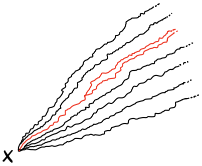
Stationary measures for exponential LPP

Coupling all densities at once, the Busemann process

- It is possible to couple all $\{h^\rho\}_{\rho\in(0,1)}$ on one probability space such that for any $k \in \mathbb{Z}$ and $0 < \rho_1 < \ldots < \rho_k < 1$ the distribution $(h^{\rho_1}, \ldots, h^{\rho_k})$ is stationary with respect to the LPP dynamics.
- Under this coupling, the process $\rho \mapsto h^\rho$ is the Busemann process of the exponential LPP.
- Busemann process is instrumental in studying infinite geodesics in metric-like models .

FPP: Newman 90's, Hoffman 02', Damron, Hanson, 14',

LPP: Georgiou, Rassoul-Agha, Seppäläinen 17'-19'
Janjigian, Shen , Balazs, B., Seppäläinen 19'.

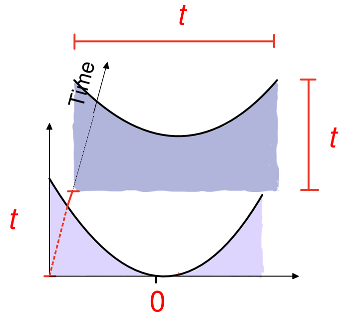


Positive temp.: Georgiou, Janjigian, Rassoul-Agha, Seppäläinen, Yilmaz 15'-19'

The KPZ scaling and the KPZ fixed-point

It is believed that the long time behavior of models in the KPZ class is universal.

The scaling is often referred to as the ***KPZ scaling*** (1:2:3) and the universal limit h_{KPZ} is called the ***KPZ fixed-point***.



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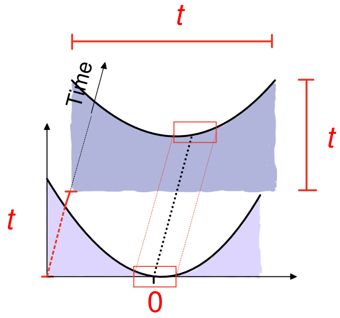

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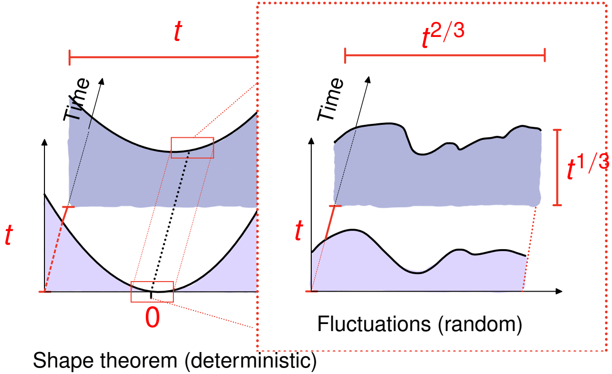

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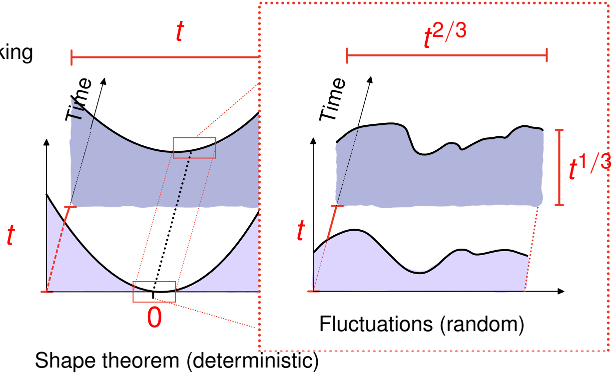
h_{KPZ} is a Markov process taking values in the space of continuous functions i.e.

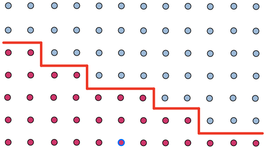
$$t \mapsto h_{\mathrm{KPZ}}(t, \cdot) \in UC$$

(Matetski, Quastel, Remenik 16')



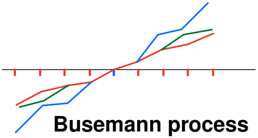
Stationary multi type distribution for the KPZ fixed point

Exponential LPP



1:2:3 scaling $\longrightarrow$ h_{KPZ}

SMTD $\Updownarrow$? $\Updownarrow$



Busemann process

$\overset{?}{\longrightarrow}$?

Scaling limit of (exponential LPP) Busemann process

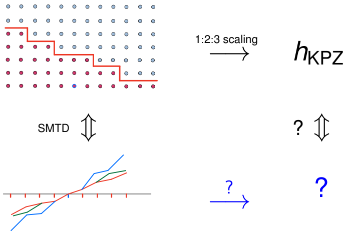
Theorem (B. 21')

Let $\{h^\rho\}_{\rho\in(0,1)}$ be the exponential LPP Busemann process. Let

$$G^N_\mu(x) = N^{-1/3}\left[h^{1/2+\mu N^{-1/3}}(2xN^{2/3})\right].$$

Then $G^N \to G$ in the Skorohord space $D(\mathbb{R}, C(\mathbb{R}))$.

We refer to the limit G as the *Stationary Horizon*.



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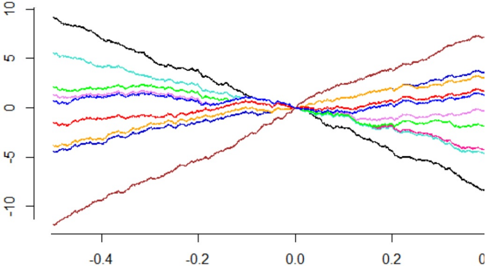
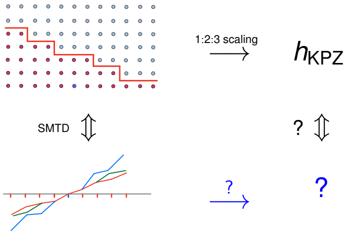


Simulation by Evan Sorensen



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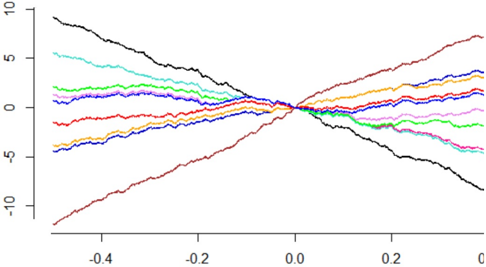

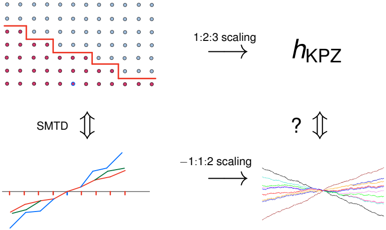

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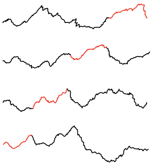
Scaling limit of (exponential LPP) Busemann process

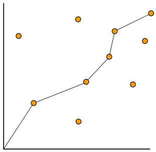
Conjecture (B. 21')

The stationary horizon is the unique scaling limit of Busemann processes of metric-like models in KPZ class.

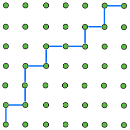
Brownain LPP



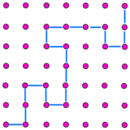
Poissonian LPP



Lattice LPP



First Passage Percolation



Scaling limit of (exponential LPP) Busemann process

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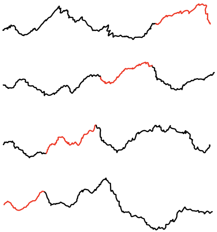
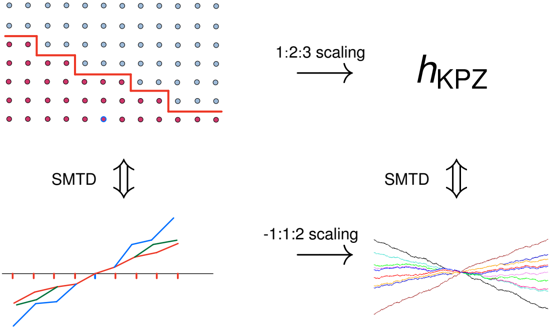


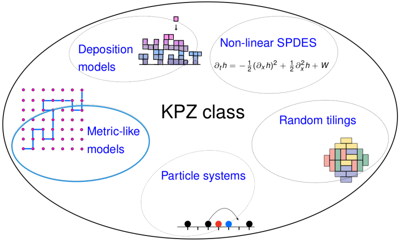
The stationary horizon was independently discovered by Seppäläinen and Sorensen 21' as the **macroscopic** Busemann process of the Brownian LPP.

Theorem (B., Sorensen, Sepäläinen 22')

The stationary horizon is the unique stationary multi-type distribution of the KPZ fixed-point.



How universal is the stationary horizon?



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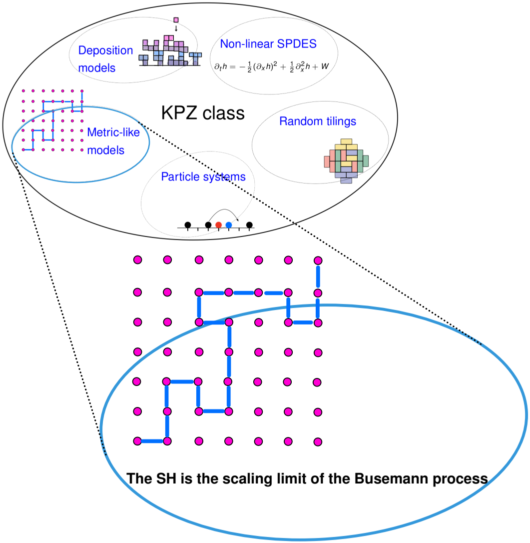

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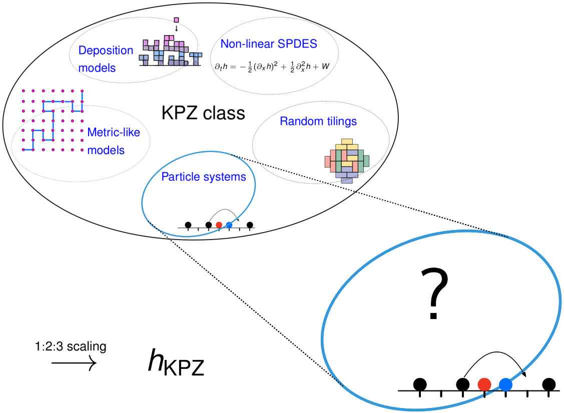

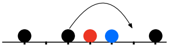

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SMTD $\Updownarrow$ SMTD $\Updownarrow$

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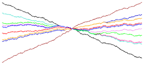

SMTD in particle systems

Totally asymmetric simple exclusion process (TASEP)

Consider a configuration of particles on $\mathbb{Z}$. Assume that each site of $\mathbb{Z}$ is equipped with a Poisson clock such that all clocks are independent. If the site x is occupied by a particle, at the ring of the clock at site x, the particle attempts a jump to site $x+1$, which is executed only if there is no particle at site $x+1$.

Introduced by Spitzer in 1970.

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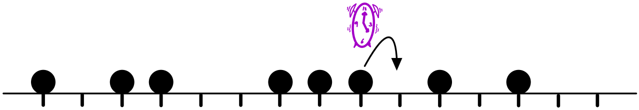

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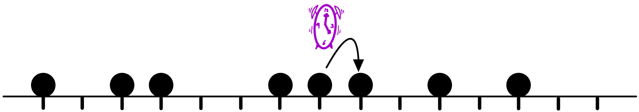

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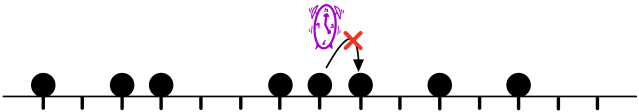

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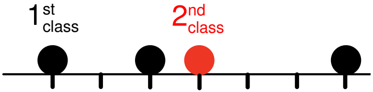
Particles with classes

One can introduce particles with priority. First class particles treat second class particle as holes, while second class particles see the first class particles as particles.



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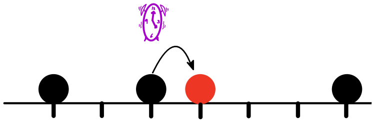

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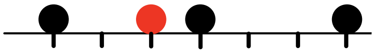

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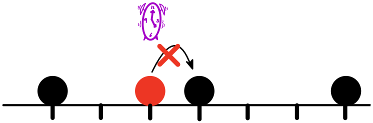
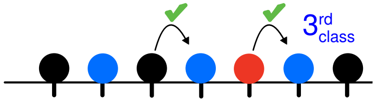


Particles with classes - more than two classes

It is possible to study configurations with more than 2 classes (the number of classes can be infinite!).



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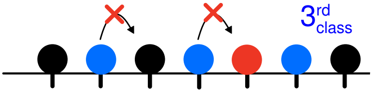
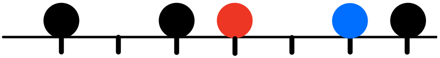


Multi-type stationary measures

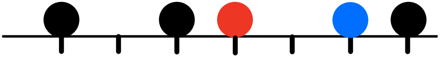
Are there stationary measures for the multi-type exclusion process?



Multi-type stationary measures

Are there stationary measures for the multi-type exclusion process?

Yes! For any $k \in \mathbb{N}$ and $\rho_1 < \ldots < \rho_k$ there exists a unique stationary and translation invariant measure $\mu^{k,(\rho_1,\ldots,\rho_k)}$ such that particles of class $i \in \{1, \ldots, k\}$ have density ρ_i. (Ferrari, Martin 05')



The TASEP speed process

Is there a random object that couples all $\mu^{k,(\rho_1,\ldots,\rho_k)}$ on one probability space?

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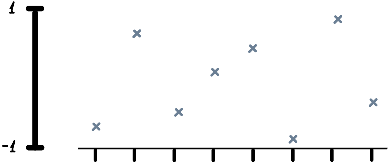
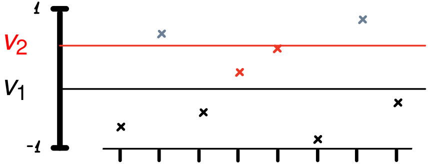
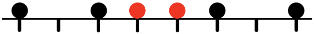
Yes! the *TASEP speed process* $U := \{U_i\}_{i\in\mathbb{Z}} \in [0,1]^{\mathbb{Z}}$ (Amir, Angel, Valko 08')



Theorem:(Amir, Angel, Valko 08') The unique stationary and translation invariant measures of the TASEP dynamics can be read off from the TASEP speed process.

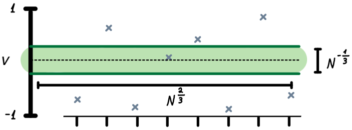


$$\mu^{2,\left(\frac{v_1+1}{2},\frac{v_2+1}{2}\right)}$$



Scaling the TASEP speed process

We fix a speed v, and consider all the particles sitting within $N^{2/3}$ away from the origin whose speed is of order $v + O(N^{-1/3})$.



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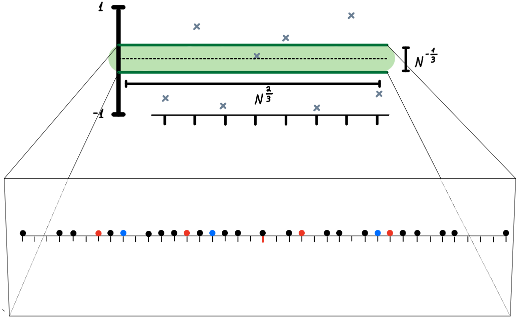
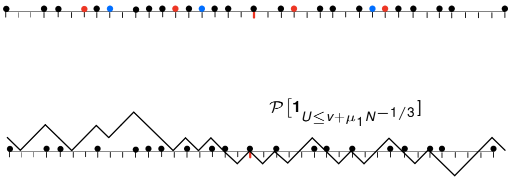


Scaling the TASEP speed process

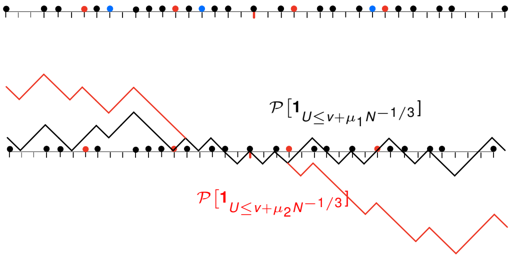
We can turn the first particles in the configuration into an interface by going up(down) upon seeing a particle(hole).



Scaling the TASEP speed process

We can turn the first particles in the configuration $\mathcal{P}$ into an interface by going up(down) upon seeing a particle(hole).

Add the second class particles and construct a new interface from the new configuration



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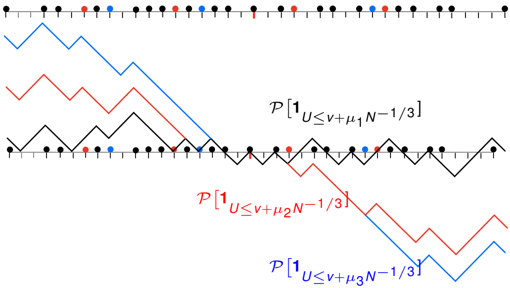
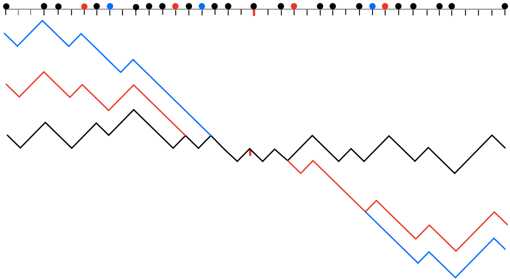
Add the third class particles to construct...



Scaling the TASEP speed process

We have obtained an ensemble of random walks that 'stick' to one another around the origin.



Theorem (B., Sepäläinen, Sorensen 22')

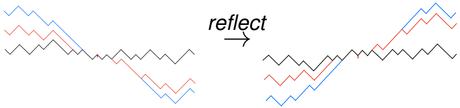
Let U be the TASEP speed process. Fix $v \in (-1,1)$ and let

$$G^N(x) := -N^{-1/3}\mathcal{P}\left[\mathbf{1}_{U<v+\mu N^{-1/3}}\right](2N^{2/3}x).$$

Then

$$G^N \xrightarrow{N\to\infty} G,$$

where G is the stationary horizon, and the convergence is in $D(\mathbb{R}, C(\mathbb{R}))$.



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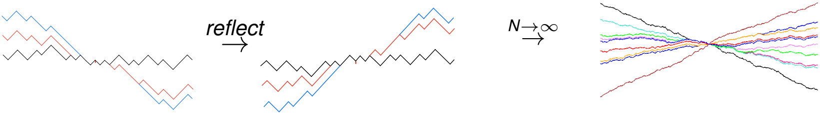

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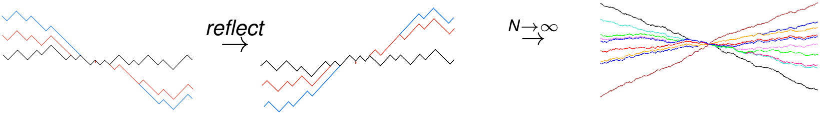

Happy birthday Timo!