Tail bounds for KPZ models: a case study for ASEP

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Thursday, June 2nd, 2023

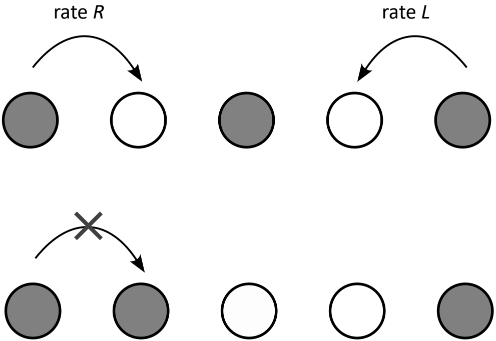
Goal of the talk

Goal 1: Report on what we learned from Timo's papers.

Goal 2: Complement one of Timo's breakthrough results, $T^{\frac{1}{3}}$ current fluctuations for ASEP (Balasz-Seppalainen, *Order of current variance and diffusivity in the asymmetric simple exclusion process*, Annals 2010).

The model: ASEP

Configuration $\eta(t) \in \{0,1\}^{\mathbb{Z}}$ of particles performing biased continous time walk with exclusion.

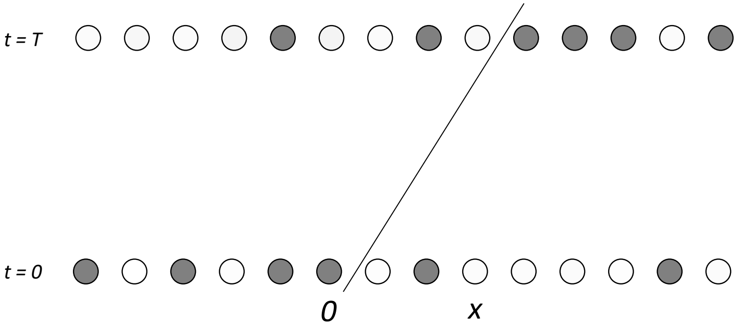


Current

Current: net number of particles that cross $[\frac{1}{2}, x + \frac{1}{2}]$ in time T.

$$J_T(x) = \underbrace{J_T(0)}_{\text{net flux across 0}} - \sum_{i=1}^{x} \eta_i(t).$$

Current



Current: Results

Balász-Seppäläinen, for the current in a characteristic direction $x_0 := (L-R)(2b-1)T$:

$$\mathrm{Var}(J_T(x_0)) \asymp T^{\frac{2}{3}}.$$

We have:

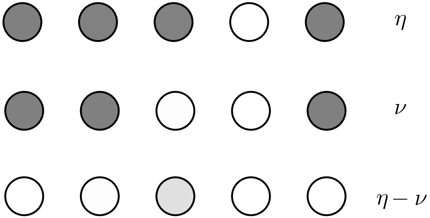
$$c\mathrm{e}^{-Cu^{3/2}T^{-1/2}} \leq \mathbb{P}\left[J_T(x_0) - \mathbb{E}[J_T(x_0)] > u\right] \leq C\mathrm{e}^{-cu^{3/2}T^{-1/2}}$$

Tail behavior consistent with Baik-Rains limit (Aggarwal, 2016).

2nd class particle

2nd class particle: ASEP dynamics, but 1st class particles take precedence (switch positions instead of exclusion).

Arise from considering discrepancy between coupled, ordered configurations $\nu \leq \eta$:



Sccond class particle: Results

Balazs-Seppäläinen: position $Q(T)$ of second class particle started at 0 in a stationary (Bernoulli b) environment for $k < 3$

$$E[|Q(T) - x_0(T)|^k] \asymp T^{\frac{2}{3}k},$$

where

$$x_0(T) = (L - R)(2b - 1)T.$$

We show:

$$P\left[|Q(T) - x_0| > u\right] \leq C\mathrm{e}^{-cu^3 T^{-2}}.$$

Ingredients

Two technical inputs come from some of my favorite papers of Timo's:

1. Microscopic concavity coupling (Bálasz-Seppäläinen)
2. Exponential formula (Elnur-Jianjigian-Seppäläinen)

This is combined with

3. Degeneration from stationary stochastic six vertex model (Borodin-Corwin-Gorin, Aggarwal)

Stochastic Six Vertex Model

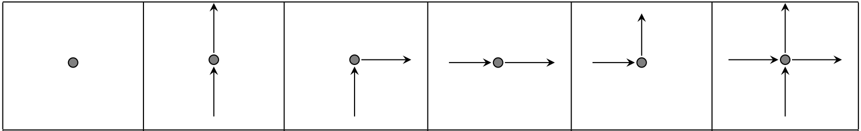
Configurations of arrows entering/exiting vertices of domain in $\mathbb{Z}^2$.

At each vertex:

$$\text{Number of incoming arrows} = \text{Number outgoing arrows}$$

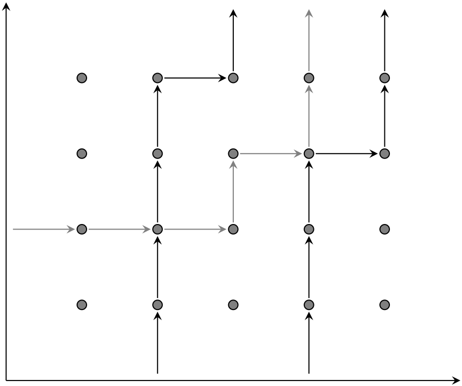
Stochastic Six Vertex Model

Six possible configurations, with weights:



1	δ_1	$1-\delta_1$	δ_2	$1-\delta_2$	1

Up-right paths



Weight of configuration:

$$1^{\#\text{type } 1}\delta_1^{\#\text{ type } 2}(1-\delta_1)^{\#\text{type } 3}\delta_2^{\#\text{type } 4}(1-\delta_2)^{\#\text{type } 5}1^{\#\text{type } 6}$$

Stationary Model

There is a stationary version of the model, considered by Aggarwal.

Choose arrow configurations along the boundaries $(x, 0)$ and $(0, y)$ to be Bernoulli with parameters b_1 and b_2 such that

$$\frac{b_1}{1 - b_1} = \kappa \frac{b_2}{1 - b_2}.$$

Then the probabilities of incoming and outgoing arrows along down-right paths are *invariant* for S6V.

Degneration to ASEP

Consider S6V with

$$\delta_1 = \epsilon L, \delta_2 = \epsilon R$$

with initial data being (b_1, b_2) such that

$$\frac{b_1}{1-b_1} = \frac{1-\delta_1}{1-\delta_2}\frac{b_2}{1-b_2}.$$

$p_i(t)$: particles in S6V at height t

$X_i(t)$ particles in ASEP Bernoulli b_2 data

$q_i(t) = p_i(t) - t$.

Degneration to ASEP

For finite $S \subseteq \mathbb{Z}^n$:

$$\lim_{\epsilon \to 0} \mathbb{P}\left[q_{i_1}(\lfloor \epsilon^{-1} t_1 \rfloor), \ldots, q_{i_n}(\lfloor \epsilon^{-1} t_n \rfloor) \in S\right]$$
$$= \mathbb{P}\left[X_{i_1}(t_1), \ldots, X_{i_n}(t_n) \in S\right].$$

As a consequence,

$$\lim_{\epsilon \to 0} P\left[H^{(b_1,b_2)}(x + \lfloor \epsilon^{-1} t \rfloor, \lfloor \epsilon^{-1} t \rfloor) > r\right] = P\left[J_t(x) \geq r\right].$$

Observed in Borodin-Corwin-Gorin, proved by Aggarwal, used extensively in Aggarwal and Borodin-Aggarwal.

Height function

For a S6V configuration in the rectangle

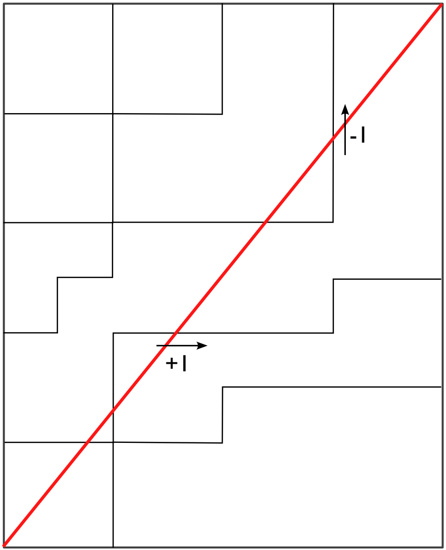
$$\{x \leq X, y \leq Y\}$$

the **height function** is defined by

$$H(X,Y) = \text{net flux of paths across line from origin to } (X,Y).$$

$+1$ if the line is traversed left-right, -1 if right-left.

Height function



Asymptotics of the height function

Borodin-Corwin-Gorin identify scaling limit:

$$\lim_{\epsilon \to 0} \epsilon H(\epsilon^{-1} x, \epsilon^{-1} y) = \mathcal{H}(x, y),$$

for explicit$\mathcal{H}$ and Riemann boundary conditions (fixed densities along the axes). Also show Tracy-Widom limit after centering and rescaling by $\epsilon^{-1/3}$

Hydrodynamic limit for general boundary conditions by Aggarwal.

Tail estimates

Subject to a **characteristic direction** condition, for

$$(y(1-\kappa))^{1/3} \leq u \leq cy(1-\kappa),$$

we have

$$\mathbb{E}\left[H^{(b_1,b_2)}(x,y) - \mathbb{E}[H^{(b_1,b_2)}(x,y)] > u\right] \geq c\mathrm{e}^{-Cu^{3/2}(y(1-\kappa))^{-1/2}},$$

and

$$\mathbb{E}\left[H^{(b_1,b_2)}(x,y) - \mathbb{E}[H^{(b_1,b_2)}(x,y)] > u\right] \leq C\mathrm{e}^{-cu^{3/2}(y(1-\kappa))^{-1/2}}.$$

Ideas in the proofs.

Coupling method (a.k.a. *Seppäläinen machine*)

General strategy, originating in Balázs-Cator-Seppäläinen.

Appears in Timo's works on O'Connell-Yor, ASEP, zero-range process, log-Gamma polymer, LPP, etc.

Based on simultaneously estimating two quantities:

Height function (passage time, log partition function, current...)

$$H(\theta, \eta)$$

Derivative (time spent on the boundary, first jump, second class particle...)

$$Q = \partial_\theta H.$$

The coupling method heuristic

Stationarity:

$$H = B(\theta) + R(\theta, \eta),$$

B: increments along $[(1,1),(N,1)]$

R: increments along $[(N,1),(N,M)]$.

Rearrange:

$$\overline{R} = \overline{H} - \overline{B}$$

$$\mathrm{Var}(R) = \mathrm{Var}(H) + \mathrm{Var}(B) - 2E[B\overline{H}].$$

The coupling method heuristic

$$
\begin{aligned}
E[B\overline{H}] &= \frac{\mathrm{d}}{\mathrm{d}\delta} E[e^{\delta \bar{B}}\overline{H}]\Big|_{\delta=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\delta}(E[\overline{H}(B_{\theta,\delta})])|_{\delta=0} \\
&:= E[Q]
\end{aligned}
$$

Assume $\mathrm{Var}(R) = \mathrm{Var}(B)$ (characteristic direction):

$$\mathrm{Var}(H) = 2E[Q].$$

Exact version of KPZ relation

$$2\chi = \xi.$$

Convexity

$$Q = \frac{\mathrm{d}}{\mathrm{d}\delta} H(B_{\theta,\delta}, \eta) =: \partial_\theta H.$$

In his papers, Timo (**master of couplings**), bounds $P(Q > u)$ in terms of H by ingenious couplings.

Generally: if $\theta \mapsto H(\theta, \eta)$ is convex, then for $\lambda > 0$:

$$Q \leq \frac{H(\lambda, \theta) - H(\theta, \theta)}{\lambda - \theta}.$$

Upper bound for χ

For example, for known integrable models, can typically show:

$$Q \lesssim \frac{1}{\lambda - \theta}\big(|\overline{H}(\theta,\theta)| + |\overline{H}(\lambda,\lambda)| + |E[H(\theta,\theta)] - E[H(\lambda,\lambda)]|\big).$$

We get

$$\mathrm{Var}(H) := V \leq \frac{C}{\lambda - \theta}(V^{1/2} + N^{1/2}(\lambda - \theta) + N(\lambda - \theta)^2).$$

Optimize:

$$\lambda - \theta \sim N^{-1/3} \Rightarrow \chi = \frac{1}{3}.$$

A remarkable formula

Want to run the previous argument on an exponential scale.

Elnur, Jianjigian and Seppäläinen (EJS): general methdology to get concentration in stationary models (in their case, exponential LPP) using a formula due to Rains (2001), with a simple proof.

EJS formula

EJS's observation: for "any" model with product invariant measure on a quadrant, in the characteristic direction,

$$\mathbb{E}\Big[\exp\left((\theta-\eta)(H(\theta,\eta)-\mathbb{E}[H(\theta,\theta)])\right)\Big]=\exp\left(c(\eta)N(\theta-\eta)^{3}\right).$$

Cubic behavior $\to N^{1/3}$ fluctuations.

Consequence

For $\eta < \theta$:

$$H(\theta, \theta) = H(\theta, \nu) + \int_\eta^\theta Q(\theta, u)\mathrm{d}u$$
$$\leq H(\theta, \eta) + (\theta - \eta)Q(\theta, \theta)$$

Subtract $E[H(\theta, \theta)]$, multiply by $\theta - \eta$ and exponentiate. Done provided we can control

$$E[e^{\epsilon^2 Q(\theta,\theta)}].$$

This argument gives moderate deviations on $N^{1/3}$ scale with $u^{3/2}$ exponent for *all* known integrable polymer models, and works for some non-integrable interacting diffusion models (Landon-S., 2022)

Back to S6V

The height function in the stationary six vertex model has the form:

$$\begin{aligned} H(x,y) &= \text{horizontal arrows through right side} \\ &\quad - \text{vertical arrows through bottom} \\ &:= R - B. \end{aligned}$$

EJS for S6V

Choose parameters:

$$e^{\varepsilon}\frac{a_1}{1-a_1} = \frac{1-\delta_1}{1-\delta_2}\frac{a_2}{1-a_2}$$

Then:

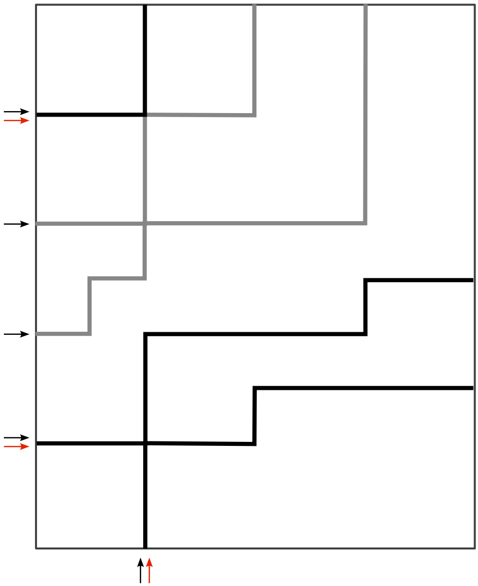
$$\boxed{\mathbb{E}\left[\exp\left(\varepsilon H^{(a_1,a_2)}(x,y)\right)\right] = (e^{\varepsilon}a_1 + (1-a_1))^y(e^{-\varepsilon}a_2 + (1-a_2))^x.}$$

Couplings and 2nd class paths

The role of Q is played by second class paths. As in ASEP, we can couple different boundary conditions for S6V, and look at the discrepancies between them (“grey paths"). Their distribution

Balázs and Seppäläinen's famous second class particle arguments extend to that setting.

Second class paths



Exit point estimate

Let $a_1 < b_1$ and $a_2 < b_2$. Start a second class path at the bottom left corner.

$$\begin{aligned}&\mathbb{P}[\text{second class path exits through north}]\\ &\lesssim e^{-ck} + e^{k}\mathbb{E}\left[e^{\epsilon H^{(a_1,a_2)}(x,y)}\right]^{1/2}\mathbb{E}\left[e^{-\epsilon H^{(b_1,a_2)}(x,y)}\right]^{1/2}.\end{aligned}$$

Future directions

- Lower tail bounds: for some models, geometric argument is available.
- Non product form invariant measures: half-space models.

Thanks a lot for listening & Happy Birthday to Timo!