From the KPZ equation to the directed landscape

Xuan Wu

University of Chicago

Random growth models and KPZ universality

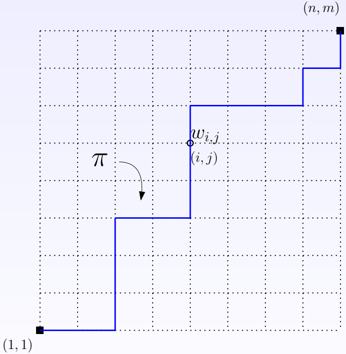
Outline

- Directed polymer and KPZ equation
- Main result
- Ideas of proof

Directed polymer and KPZ equation

Discrete directed polymers

$$w(\pi) = \sum_{(i,j)\in\pi} w_{i,j}$$



Partition function: $\mathcal{Z} = \sum_{\pi} \exp(w(\pi))$

Free energy: $\mathcal{H} = \log \mathcal{Z}$

Discrete directed polymers

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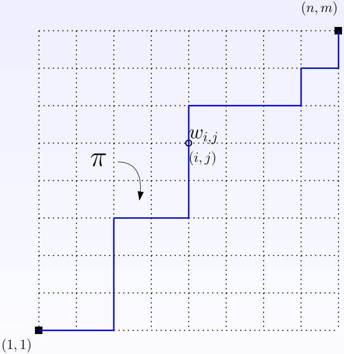


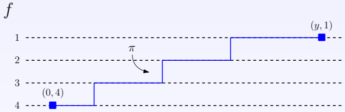
Partition function: $\mathcal{Z} = \sum_{\pi} \exp(w(\pi))$

Free energy: $\mathcal{H} = \log \mathcal{Z}$

Integrable weight: $-w_{ij} \sim$ log-gamma [Seppäläinen 12]

Semi-discrete directed polymers

$$f(\pi) = \sum_i (f_i(t_i) - f_i(t_{i+1})$$



$$\mathcal{Z} = \int \exp(f(\pi))\, d\pi$$

$$\mathcal{H} = \log \mathcal{Z}$$

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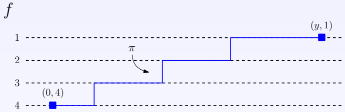

$$\mathcal{Z} = \int \exp(f(\pi))\, d\pi$$

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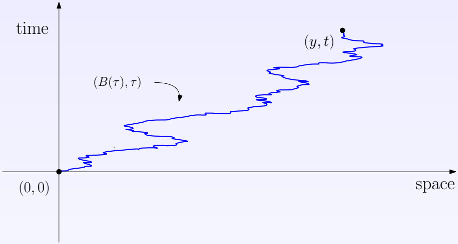
Integrable weight: $f \sim$ i.i.d. Brownian motion [O'Connell-Yor 02]

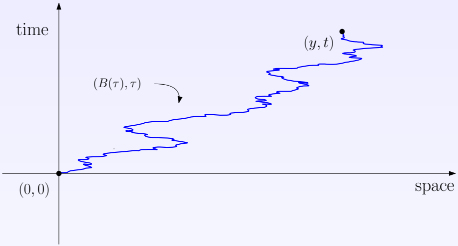
Continumm directed polymers



$$\mathcal{Z}(t,y)$$
$$= \mathbb{E}\left[\exp\left(\int_0^t \xi\left(\tau, B(\tau)\right) d\tau\right)\right]$$

$$\mathcal{H}(t,y) = \log \mathcal{Z}(t,y)$$

Continumm directed polymers



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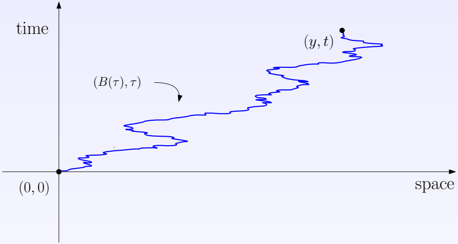
Theorem (Alberts-Khanin-Quastel 14)

Discrete directed polymers $\longrightarrow$ *Continumm directed polymers*

Continumm directed polymers



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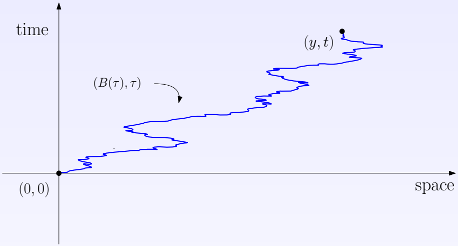
Theorem (Nica 21)

OY semi-discrete directed polymers $\longrightarrow$ *Continumm directed polymers*

Continumm directed polymers

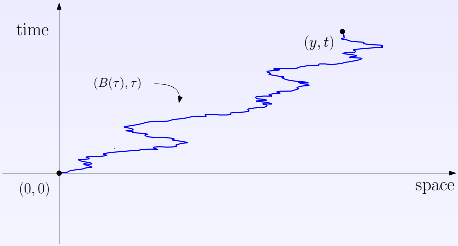


$$\mathcal{Z}(t,y) = \mathbb{E}\left[\exp\left(\int_0^t \xi\left(\tau, B(\tau)\right) d\tau\right)\right]$$

$$\mathcal{H}(t,y) = \log \mathcal{Z}(t,y)$$

$$\begin{aligned} \partial_t \mathcal{Z}(t,y) =& \frac{1}{2}\partial_{yy}\mathcal{Z}(t,y) + \xi \cdot \mathcal{Z}(t,y) \\ \mathcal{Z}(0,y) =& \delta(y) \end{aligned}$$

Continumm directed polymers



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$$\partial_t \mathcal{H}(t,y) = \frac{1}{2}\partial_{yy}\mathcal{H}(t,y) + \frac{1}{2}(\partial_y \mathcal{H}(t,y))^2 + \xi$$
$$\mathcal{H}(0,y) = \log \delta(y)$$

Kardar-Parisi-Zhang equation

$$\begin{aligned} \partial_t \mathcal{H}(t,y) = &\quad \tfrac{1}{2}\partial_{yy}\mathcal{H}(t,y) + \tfrac{1}{2}\left(\partial_y \mathcal{H}(t,y)\right)^2 + \xi \\ \mathcal{H}(0,y) = &\quad f(y) \end{aligned}$$

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Theorem (Balázs-Quastel-Seppäläinen 11)

Take $f(y) = \exp(B(y))$. *Then when* t *goes to infinity,*

$$\mathrm{Var}(\mathcal{H}(t,y)) \sim t^{2/3}.$$

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Theorem (Amir-Corwin-Quastel 11)

Take $f(y) = \log \delta(y)$. *Then* $\mathcal{H}(t,0) \xrightarrow{1:2:3}$ *Tracy-Widom GUE.*

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Take $f(y) = \log \delta(y)$. Then $\mathcal{H}(t,0) \xrightarrow{1:2:3}$ Tracy-Widom GUE.

Theorem (Quastel-Sarkar 23 and Virág 20)

For a wide class of inital data $f(y)$, $\mathcal{H}(t,y) \xrightarrow{1:2:3}$ KPZ fixed point.

Kardar-Parisi-Zhang equation

$$\begin{aligned} \partial_t \mathcal{H}_i(t,y) = & \quad \tfrac{1}{2}\partial_{yy}\mathcal{H}_i(t,y) + \tfrac{1}{2}\left(\partial_y \mathcal{H}_i(t,y)\right)^2 + \xi \\ \mathcal{H}_i(s_i,y) = & \quad f_i(y) \end{aligned}$$

$$i = 1, 2, \ldots, m$$

Kardar-Parisi-Zhang equation

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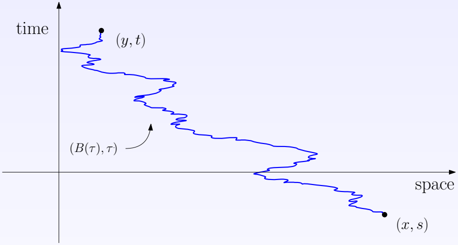
$$i = 1, 2, \ldots, m$$

$$\mathcal{H}_i(t,y) = \log \int \exp\left(\mathcal{H}(t, y \mid s_i, x) + f_i(x)\right)\, dx$$

Narrow wedge solutions



$$\mathcal{Z}(t, y \,|\, s, x) \\ = \mathbb{E}\left[\exp\left(\int_s^t \xi\left(\tau, B(\tau)\right) d\tau\right)\right]$$

$$\mathcal{H}(t, y \,|\, s, x) = \log \mathcal{Z}(t, y \,|\, s, x)$$

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Theorem (Alberts-Janjigian-Rassoul-Agha-Seppäläinen 22)
$\mathcal{H}(t,y \,|\, s,x)$ *form a random continuous function.*

Main result: convergence of the KPZ equation

Theorem 1 (W. 23)

$\mathcal{H}(t, y \,|\, s, x) \xrightarrow{1:2:3} \mathcal{L}(t, y \,|\, s, x).$

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Implication: **all initial value problems converge simultaneously!**

Directed landscape

$\mathcal{L}(t, y \mid s, x)$: the directed landscape

Directed landscape

$\mathcal{L}(t, y \mid s, x)$: the directed landscape

- constructed by Dauvergne-Ortmann-Virág 23

Directed landscape

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- $\mathcal{L}(1, 0 \,|\, 0, 0)$ is the Tracy-Widom GUE

Directed landscape

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- constructed by Dauvergne-Ortmann-Virág 23
- $\mathcal{L}(1, 0 \,|\, 0, 0)$ is the Tracy-Widom GUE
- $\mathcal{L}(1, y \,|\, 0, 0) + y^2$ is the Airy_2 process

Main result: convergence of the KPZ equation

Theorem 1 (W. 23)
$\mathcal{H}(t, y \,|\, s, x) \xrightarrow{1:2:3} \mathcal{L}(t, y \,|\, s, x)$.

- To $\mathcal{L}(1, 0 \,|\, 0, 0)$. [Amir-Corwin-Quastel 11]
- To $\mathcal{L}(t, y \,|\, 0, 0)$. [Quastel-Sarkar 23] and [Virág 20]

Today:

- To $\mathcal{L}(t, y \,|\, s, x)$ [W. 23]

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Today:

- To $\mathcal{L}(t, y \,|\, s, x)$ [W. 23]

Airy sheet: $\mathcal{S}(x, y) := \mathcal{L}(1, y \,|\, 0, x)$

KPZ sheet: $\mathcal{H}^T(x, y) := \mathcal{H}(T, y \,|\, 0, x)$

Main result: convergence of the KPZ equation

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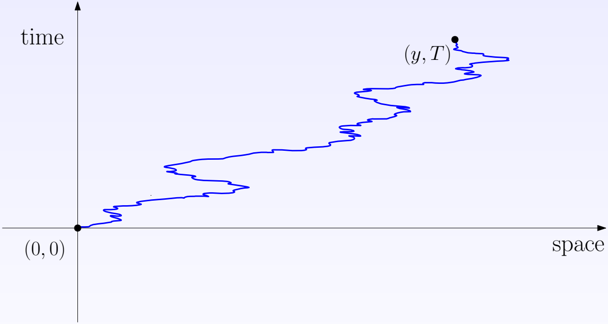
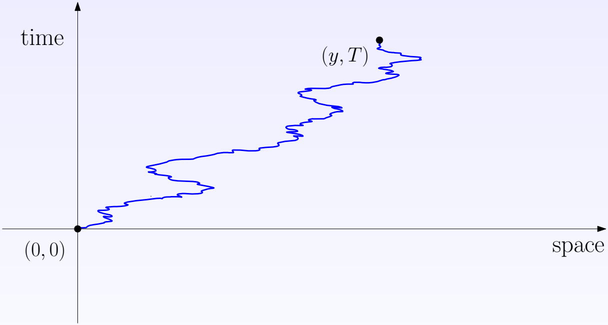
Theorem 2 (W. 23)

KPZ sheet $\xrightarrow{1:2:3}$ *Airy sheet.*

RSK correspondence of white noise



RSK correspondence of white noise



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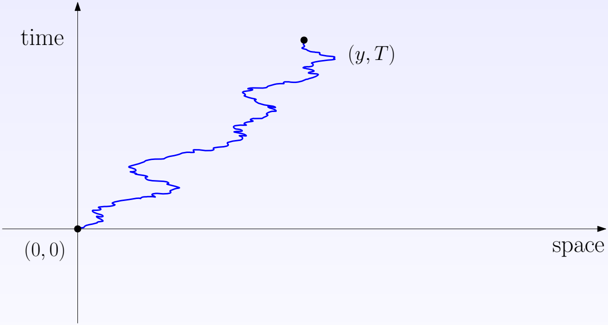

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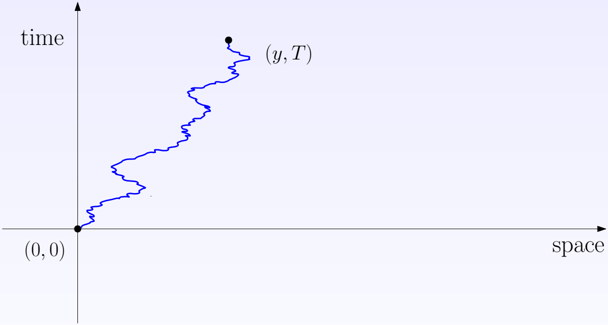

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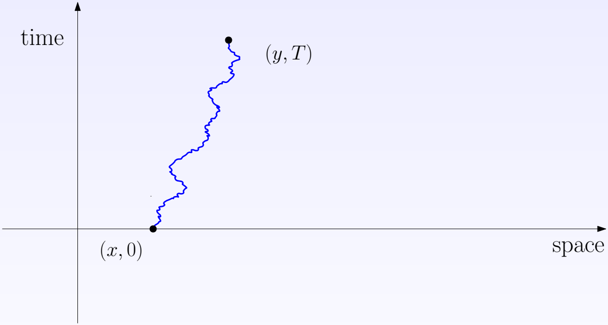

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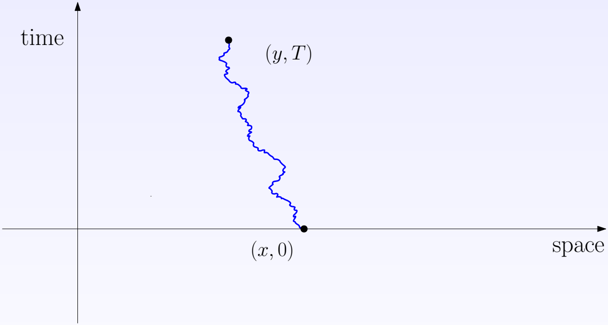

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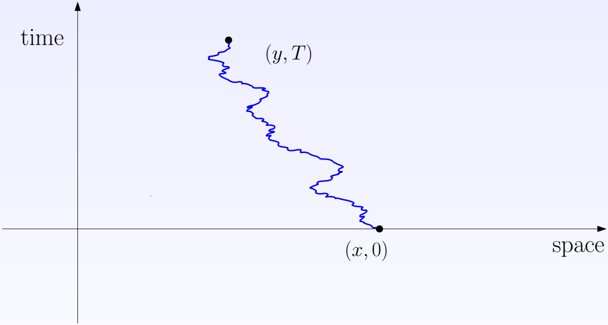

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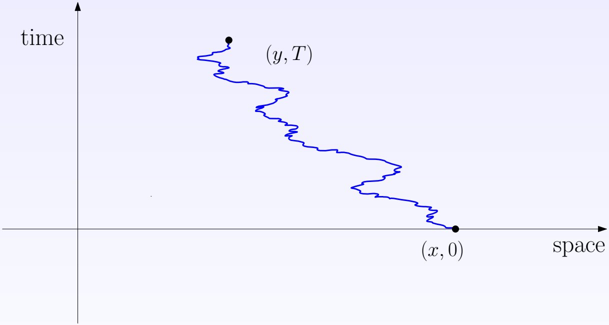

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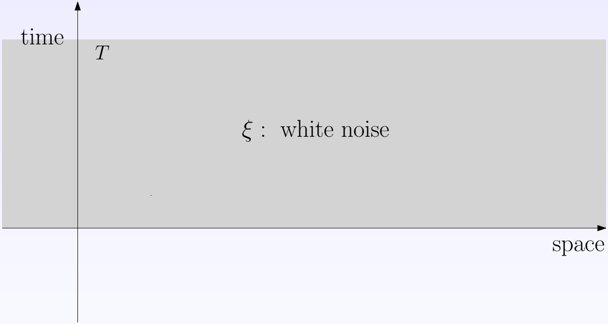

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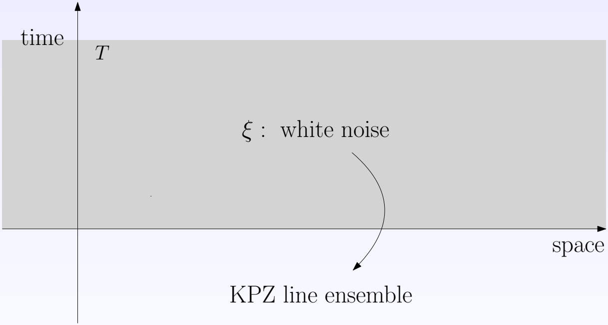
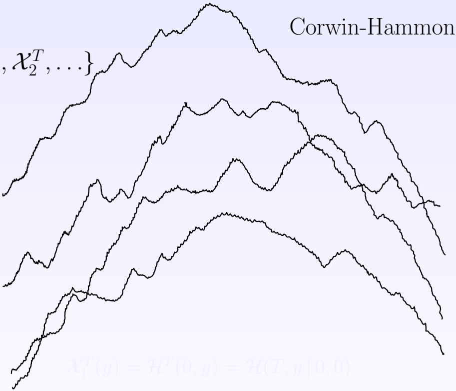


KPZ line ensemble

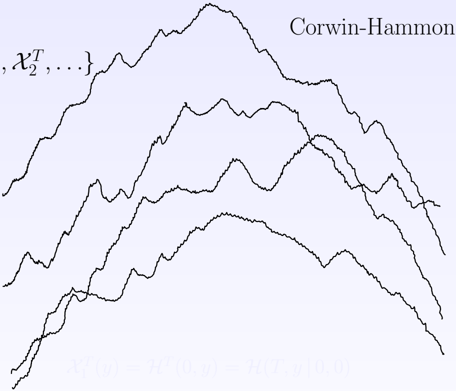


$$\mathcal{X}^T = \{\mathcal{X}_1^T, \mathcal{X}_2^T, \dots\}$$

Corwin-Hammond 16

$$\mathcal{X}_1^T(y) = \mathcal{H}^T(0,y) = \mathcal{H}(T,y \,|\, 0,0)$$

KPZ line ensemble



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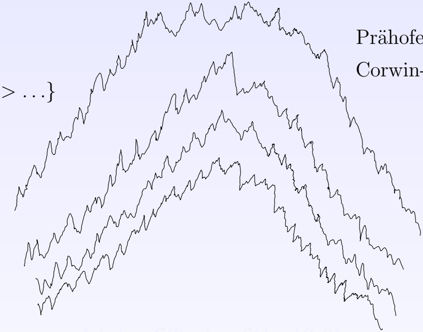
Theorem (QS 23, V 20, Dimitrov-Matetski 18, W. 22)

KPZ line ensemble $\xrightarrow{1:2:3}$ *Airy line ensemble.*

Airy line ensemble

$\mathcal{A} = \{\mathcal{A}_1 > \mathcal{A}_2 > \dots\}$



Prähofer-Spohn 02

Corwin-Hammond 14

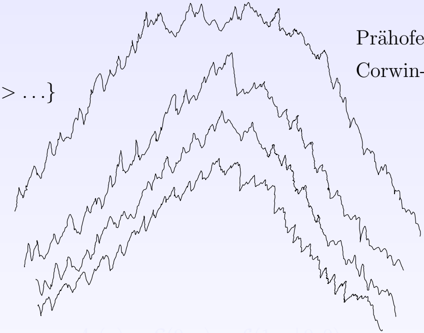
$$\mathcal{A}_1(y) = \mathcal{S}(0, y) = \mathcal{L}(1, y \,|\, 0, 0)$$

Dauvergne-Ortmann-Virág 23

$$\lim_{k\to\infty} \mathcal{A}[(z_k, k) \to (y_2, 1)] - \mathcal{A}[(z_k, k) \to (y_1, 1)] = \mathcal{S}(x, y_2) - \mathcal{S}(x, y_1)$$

$$z_k = -\sqrt{k/(2x)}$$

Airy line ensemble



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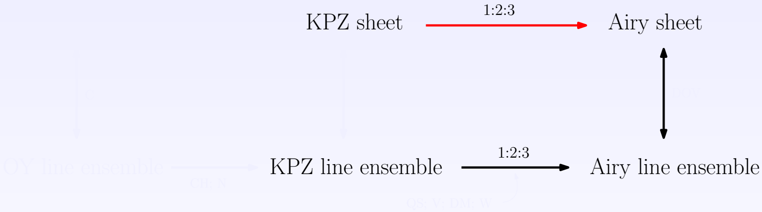
Big picture



KPZ sheet

1:2:3

Airy sheet

OY line ensemble

CH; N

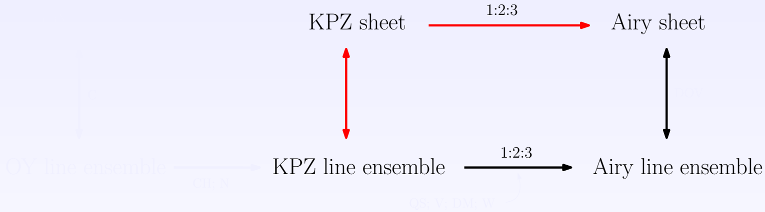
KPZ line ensemble

1:2:3

QS; V; DM; W

Airy line ensemble

Big picture


KPZ sheet

1:2:3

Airy sheet

C

OY line ensemble

CH; N

KPZ line ensemble

1:2:3

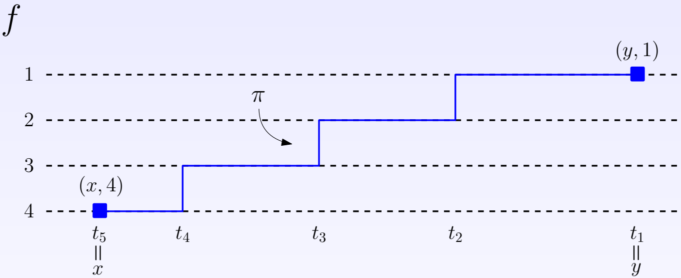
Airy line ensemble

QS; V; DM; W

Big picture

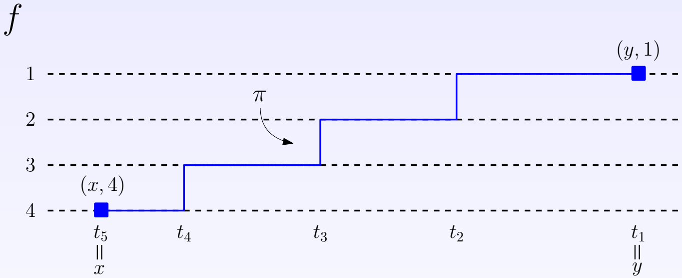
$$\begin{tikzcd}
\text{OY sheet} \arrow[r] \arrow[d, leftrightarrow] & \text{KPZ sheet} \arrow[r, red, "1:2:3"] \arrow[d, red, leftrightarrow] & \text{Airy sheet} \arrow[d, leftrightarrow] \\
\text{OY line ensemble} \arrow[r, "\text{CH; N}"'] & \text{KPZ line ensemble} \arrow[r, "1:2:3", "\text{QS; V; DM; W}"'] & \text{Airy line ensemble}
\end{tikzcd}$$

Semi-discrete directed polymers



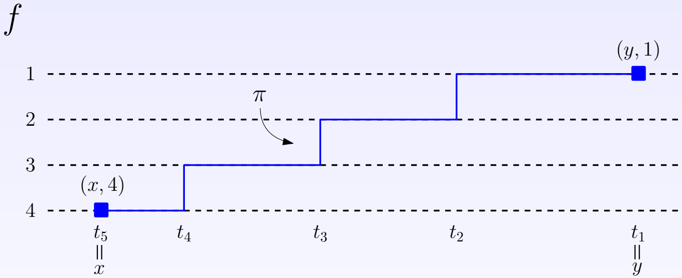
$$f(\pi) = \big(f_4(t_4) - f_4(t_5)\big) + \big(f_3(t_3) - f_3(t_4)\big)$$
$$+ \big(f_2(t_2) - f_2(t_3)\big) + \big(f_1(t_1) - f_1(t_2)\big)$$

Semi-discrete directed polymers



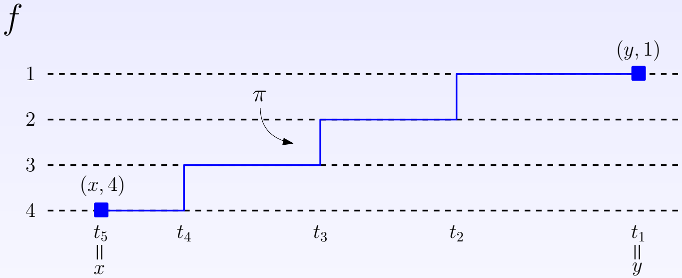
$$f[(x,4) \longrightarrow (y,1)] = \log \int \exp(f(\pi))\, d\pi$$

Semi-discrete directed polymers



$$f[(x,4) \xrightarrow{\beta} (y,1)] = \beta^{-1} \log \int \exp(\beta f(\pi))\, d\pi$$

Semi-discrete directed polymers



$$f[(x,4) \xrightarrow{\infty} (y,1)] = \max f(\pi)$$

Big picture

$$\begin{tikzcd}
\text{OY sheet} \arrow[r] \arrow[d, leftrightarrow] & \text{KPZ sheet} \arrow[r, "1:2:3"] \arrow[d, leftrightarrow] & \text{Airy sheet} \arrow[d, leftrightarrow] \\
\text{OY line ensemble} \arrow[r, "\text{CH; N}"'] & \text{KPZ line ensemble} \arrow[r, "1:2:3", "\text{QS; V; DM; W}"'] & \text{Airy line ensemble}
\end{tikzcd}$$

Theorem (Nica 21)

O'Connell-Yor sheet ⟶ KPZ sheet.

Big picture

$$\begin{tikzcd}
\text{OY sheet} \arrow[r] \arrow[d, leftrightarrow] & \text{KPZ sheet} \arrow[r, "1:2:3"] \arrow[d, leftrightarrow] & \text{Airy sheet} \arrow[d, leftrightarrow] \\
\text{OY line ensemble} \arrow[r, "\text{CH; N}"'] & \text{KPZ line ensemble} \arrow[r, "1:2:3", "\text{QS; V; DM; W}"'] & \text{Airy line ensemble}
\end{tikzcd}$$

OY line ensemble $=$ gRSK transform of Brownian motions [O'Connell 12]

Big picture

$$\begin{tikzcd}
\text{OY sheet} \arrow[r] \arrow[d, leftrightarrow] & \text{KPZ sheet} \arrow[r, "1{:}2{:}3"] \arrow[d, leftrightarrow] & \text{Airy sheet} \arrow[d, leftrightarrow] \\
\text{OY line ensemble} \arrow[r, "\text{CH; N}"'] & \text{KPZ line ensemble} \arrow[r, "1{:}2{:}3"] \arrow[r, "\text{QS; V; DM; W}"'] & \text{Airy line ensemble}
\end{tikzcd}$$

OY line ensemble = gRSK transform of Brownian motions [O'Connell 12]

Theorem (Corwin-Hammond 16, Nica 21)

OY line ensembles $\longrightarrow$ KPZ line ensemble.

Big picture

$$\begin{tikzcd}
\text{OY sheet} \arrow[r] \arrow[d, leftrightarrow, blue] & \text{KPZ sheet} \arrow[r, red, "1:2:3"] \arrow[d, leftrightarrow, red] & \text{Airy sheet} \arrow[d, leftrightarrow] \\
\text{OY line ensemble} \arrow[r] & \text{KPZ line ensemble} \arrow[r, "1:2:3"] & \text{Airy line ensemble}
\end{tikzcd}$$

Theorem (Corwin 21)

$f = \{f_1, f_2, \ldots, f_n\}$

$$f[(x,n) \to (y,1)] = \mathcal{W}f[(x,n) \to (y,1)]$$

Big picture

$$\begin{tikzcd}
\text{OY sheet} \arrow[r] \arrow[d, leftrightarrow] & \text{KPZ sheet} \arrow[r, "1:2:3"] \arrow[d, leftrightarrow] & \text{Airy sheet} \arrow[d, leftrightarrow] \\
\text{OY line ensemble} \arrow[r, "\text{CH; N}"'] & \text{KPZ line ensemble} \arrow[r, "1:2:3", "\text{QS; V; DM; W}"'] & \text{Airy line ensemble}
\end{tikzcd}$$

Ideas of proof

Main difficulty

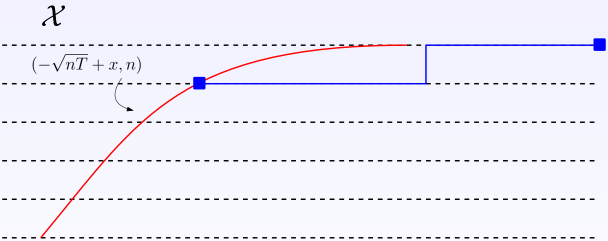
$$\mathcal{H}^{T,n}(x,y) = \mathcal{X}^{T,n}[(-\sqrt{nT}+x,n) \to (y,1)] + C(T,n)$$

Main difficulty

$$\mathcal{H} \quad (x, y) = \mathcal{X} \quad [(-\sqrt{nT} + x, n) \to (y, 1)]$$

Main difficulty

$$\mathcal{H}(x,y) = \mathcal{X}[(-\sqrt{nT}+x,n) \to (y,1)]$$



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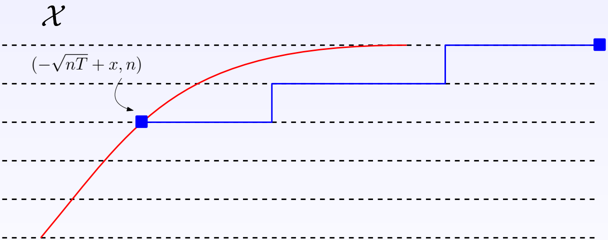

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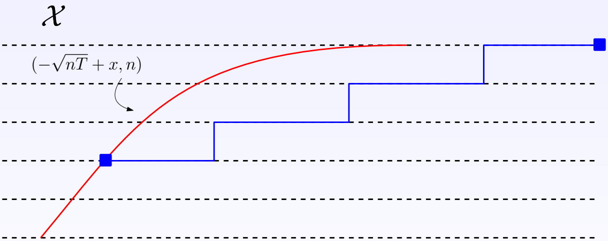

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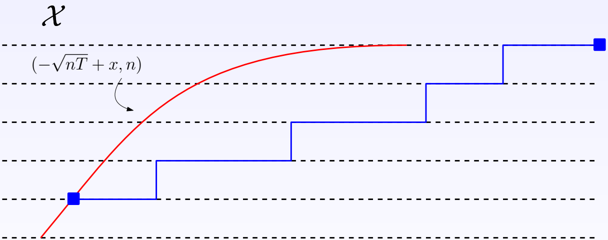

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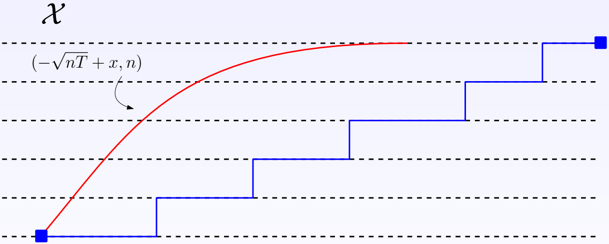
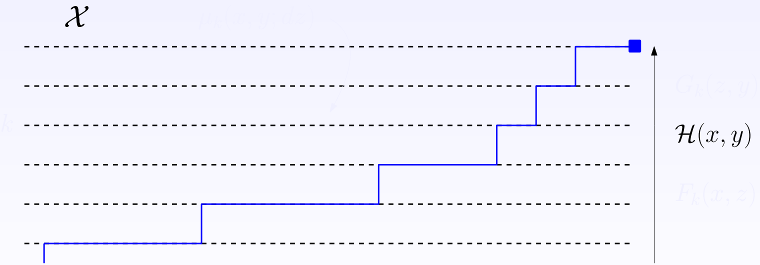


Truncation

$$\mathcal{H}(x, y) = \mathcal{X}[(-\sqrt{nT} + x, n) \to (y, 1)]$$



Truncation

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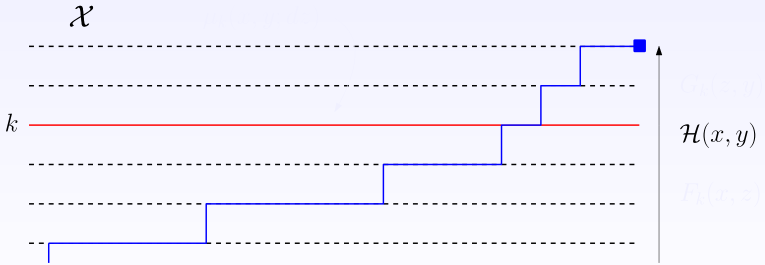
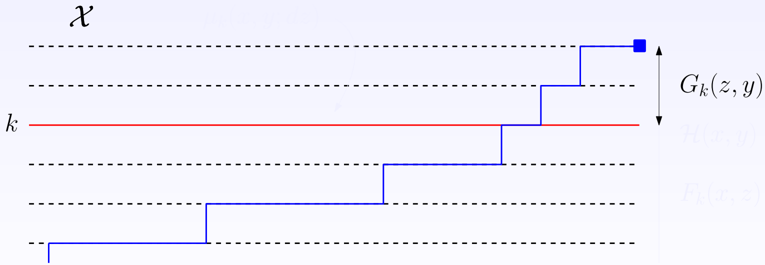


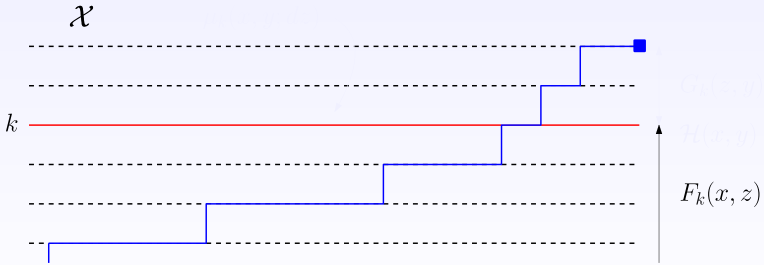
Truncation

$$G_k(z,y) = \mathcal{X}[(z,k) \to (y,1)]$$



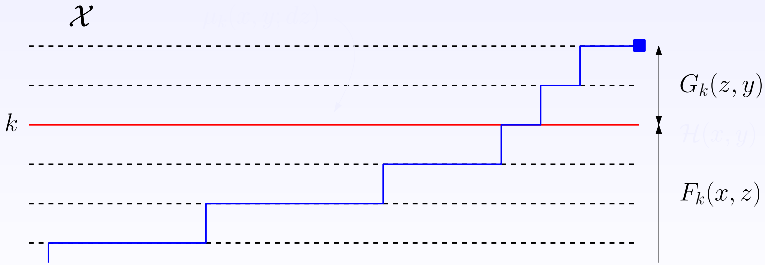
Truncation

$$F_k(x, z) = \mathcal{X}[(-\sqrt{nT} + x, n) \to (z, k + 1)]$$



Truncation

$$\exp(\mathcal{H}(x,y)) = \int \exp(F_k(x,z) + G_k(z,y))dz$$



Busemann functions

$$\lim_{k\to\infty} \mathcal{A}[(z_k, k) \xrightarrow{\infty} (y_2, 1)] - \mathcal{A}[(z_k, k) \xrightarrow{\infty} (y_1, 1)] = \mathcal{S}(x, y_2) - \mathcal{S}(x, y_1)$$

$$G_k(z_k, y_2) - G_k(z_k, y_1) \qquad \mathcal{H}(x, y_2) - \mathcal{H}(x, y_1)$$

Compare $G_k(z_k, y_2) - G(z_k, y_1)$ and $\mathcal{H}(x, y_2) - \mathcal{H}(x, y_1)$

Busemann functions

$$\lim_{k\to\infty} \mathcal{A}[(z_k, k) \xrightarrow{\infty} (y_2, 1)] - \mathcal{A}[(z_k, k) \xrightarrow{\infty} (y_1, 1)] = \mathcal{S}(x, y_2) - \mathcal{S}(x, y_1).$$

$$\begin{array}{ccc} \wr & & \wr \\ G_k(z_k, y_2) - G_k(z_k, y_1) & & \mathcal{H}(x, y_2) - \mathcal{H}(x, y_1) \end{array}$$

Compare $G_k(z_k, y_2) - G_k(z_k, y_1)$ and $\mathcal{H}(x, y_2) - \mathcal{H}(x, y_1)$

Busemann functions

$$\begin{array}{cccc} \lim\limits_{k\to\infty} & \underset{\displaystyle G_k(z_k, y_2) - G_k(z_k, y_1)}{\underset{\wr}{\mathcal{A}[(z_k, k) \xrightarrow{\infty} (y_2, 1)] - \mathcal{A}[(z_k, k) \xrightarrow{\infty} (y_1, 1)]}} & = & \underset{\displaystyle \mathcal{H}(x, y_2) - \mathcal{H}(x, y_1)}{\underset{\wr}{\mathcal{S}(x, y_2) - \mathcal{S}(x, y_1)}}. \end{array}$$

Compare $G_k(z_k, y_2) - G_k(z_k, y_1)$ and $\mathcal{H}(x, y_2) - \mathcal{H}(x, y_1)$

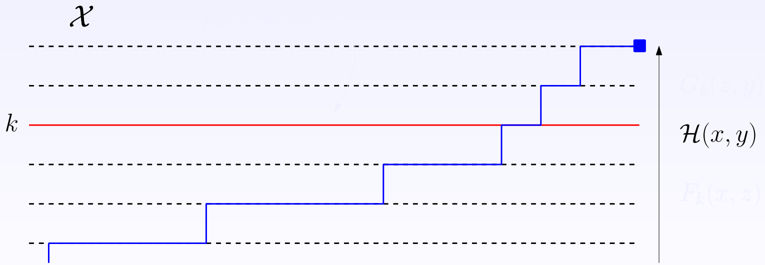
Busemann functions

$$\lim_{k\to\infty} \mathcal{A}[(z_k, k) \xrightarrow{\infty} (y_2, 1)] \underset{\substack{\wr \\ G_k(z_k, y_2) - G_k(z_k, y_1)}}{-} \mathcal{A}[(z_k, k) \xrightarrow{\infty} (y_1, 1)] = \mathcal{S}(x, y_2) \underset{\substack{\wr \\ \mathcal{H}(x, y_2) - \mathcal{H}(x, y_1)}}{-} \mathcal{S}(x, y_1).$$

$$\boxed{\text{Compare } G_k(z_k, y_2) - G_k(z_k, y_1) \text{ and } \mathcal{H}(x, y_2) - \mathcal{H}(x, y_1)}$$

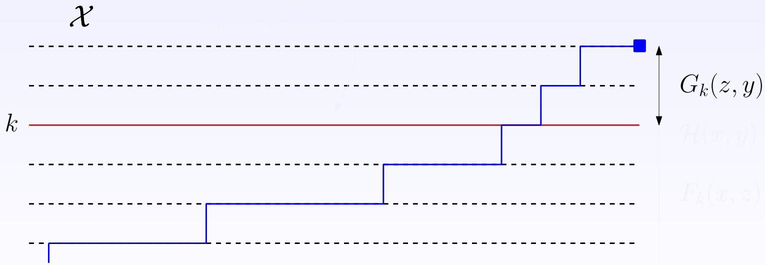
Truncation

$$\mathcal{H}(x, y) = \mathcal{X}[(-\sqrt{nT} + x, n) \to (y, 1)]$$



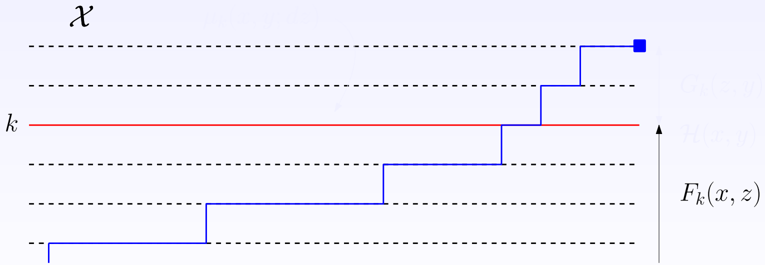
Truncation

$$G_k(z,y) = \mathcal{X}[(z,k) \to (y,1)]$$



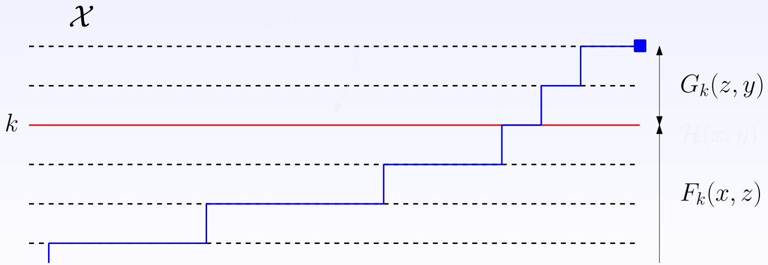
Truncation

$$F_k(x, z) = \mathcal{X}[(-\sqrt{nT} + x, n) \to (z, k+1)]$$



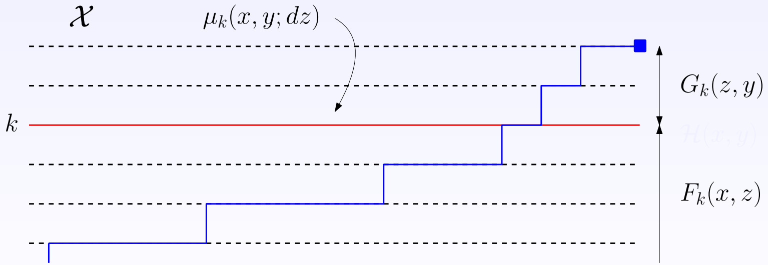
Truncation

$$\exp(\mathcal{H}(x,y)) = \int \exp(F_k(x,z) + G_k(z,y))dz$$



Truncation

$$\mu_k(x, y; dz) = \exp(-\mathcal{H}(x, y) + F_k(x, z) + G_k(z, y))dz$$



Ideas

- Control the difference between $\mathcal{H}(x, y_2) - \mathcal{H}(x, y_1)$ and $G_k(z, y_2) - G_k(z, y_1)$ by μ_k.
- Determine μ_k by F_k.
- Compute the limit of F_k.

Busemann functions

Lemma

For $y_2 \geq y_1$,

$$-\log \int_{-\infty}^{z} \mu_k(x, y_2; dz') \geq$$

$$\mathcal{H}(x, y_2) - \mathcal{H}(x, y_1) - G_k(z, y_2) + G_k(z, y_1)$$

$$\geq \log \int_{z}^{\infty} \mu_k(x, y_1; dz')$$

Busemann functions

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$$\geq \log \int_{z}^{\infty} \mu_k(x, y_1; dz')$$

Busemann functions

Proof.

$$\begin{aligned}
&\exp(\mathcal{H}(x, y_2) - \mathcal{H}(x, y_1)) \\
&= \int \exp(G_k(z', y_2) - G_k(z', y_1))\mu_k(x, y_1; dz') \\
&\geq \int_z^\infty \exp(G_k(z', y_2) - G_k(z', y_1))\mu_k(x, y_1; dz') \\
&\geq \exp(G_k(z, y_2) - G_k(z, y_1)) \cdot \int_z^\infty \mu_k(x, y_1; dz').
\end{aligned}$$

□

Busemann functions

Proof.

$$
\begin{aligned}
&\exp(\mathcal{H}(x, y_2) - \mathcal{H}(x, y_1)) \\
=&\int \exp(G_k(z', y_2) - G_k(z', y_1))\mu_k(x, y_1; dz') \\
\geq&\int_z^\infty \exp(G_k(z', y_2) - G_k(z', y_1))\mu_k(x, y_1; dz') \\
\geq&\exp(G_k(z, y_2) - G_k(z, y_1)) \cdot \int_z^\infty \mu_k(x, y_1; dz')
\end{aligned}
$$

$\square$

Busemann functions

Proof.

$$
\begin{aligned}
&\exp(\mathcal{H}(x, y_2) - \mathcal{H}(x, y_1)) \\
=&\int \exp(G_k(z', y_2) - G_k(z', y_1))\mu_k(x, y_1; dz') \\
\geq&\int_{z}^{\infty} \exp(G_k(z', y_2) - G_k(z', y_1))\mu_k(x, y_1; dz') \\
\geq&\exp(G_k(z, y_2) - G_k(z, y_1)) \cdot \int^{\infty} \mu_k(x, y_1; dz')
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$\square$

Busemann functions

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□

Busemann functions

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$$\mathcal{H}(x, y_2) - \mathcal{H}(x, y_1) - G_k(z, y_2) + G_k(z, y_1)$$

$$\geq \log \int_{z}^{\infty} \mu_k(x, y_1; dz')$$

Busemann functions

Lemma

For $y_2 \geq y_1$,

$$-\log \int_{-\infty}^{z} \mu_k(x, y_2; dz') \geq \mathcal{H}(x, y_2) - \mathcal{H}(x, y_1) - G_k(z, y_2) + G_k(z, y_1) \geq \log \int_{z}^{\infty} \mu_k(x, y_1; dz')$$

$$\boxed{\text{Need to understand } \mu_k}$$

Determine μ_k

$\mu_k(x, y; dz)$ is largely determined by $F_k(x, z)$ because $F_k(x, z)$ is very sensitive in x.

Determine μ_k

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Lemma

Suppose $\mathcal{H}(x, y)$ *does not depend on* x*.*

Determine μ_k

$\mu_k(x, y; dz)$ is largely determined by $F_k(x, z)$ because $F_k(x, z)$ is very sensitive in x.

Lemma

Suppose $\mathcal{H}(x, y)$ *does not depend on* x*. Then* $\mu_k(x, y; dz)$ *is a delta measure at* z_k*.* z_k *solves* $(\partial F_k/\partial x)\,(x, z_k) = 0$*.*

Determine μ_k

$$\mathcal{H}(x, y) = \log \int \exp(F_k(x, z) + G_k(z, y)) dz$$

Determine μ_k

$$\mathcal{H}(x, y) = \log \int \exp(F_k(x, z) + G_k(z, y))dz$$

Proof.

$$0 = \int \frac{\partial F_k}{\partial x}(x, z)\mu_k(x, y; dz) \tag{1}$$

Determine μ_k

$$\mathcal{H}(x,y) = \log \int \exp(F_k(x,z) + G_k(z,y))dz$$

Proof.

$$0 = \int \frac{\partial F_k}{\partial x}(x,z)\mu_k(x,y;dz) \tag{1}$$

$$\begin{aligned} 0 = &\int \frac{\partial F_k}{\partial x}(x,z)\frac{\partial G_k}{\partial y}(z,y)\mu_k(x,y;dz) \\ &- \int \frac{\partial F_k}{\partial x}(x,z)\mu_k(x,y;dz)\int \frac{\partial G_k}{\partial y}(z,y)\mu_k(x,y;dz) \geq 0 \end{aligned} \tag{2}$$

$\square$

Determine μ_k

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Lemma

Suppose $\mathcal{H}(x, y)$ does not depend on x. Then $\mu_k(x, y; dz)$ is a delta measure at z_k. z_k solves $(\partial F_k/\partial x)(x, z_k) = 0$.

Determine μ_k

$\mu_k(x, y; dz)$ is largely determined by $F_k(x, z)$ because $F_k(x, z)$ is very sensitive in x.

Lemma

Suppose $\mathcal{H}(x, y)$ *does not depend on* x*. Then* $\mu_k(x, y; dz)$ *is a delta measure at* z_k*.* z_k *solves* $(\partial F_k/\partial x)(x, z_k) = 0$*.*

$$\boxed{\text{Need to understand } F_k}$$

Limit of F_k

$$F_k^{T,n}(x,z) = \mathcal{X}^{T,n}[(-\sqrt{nT}+x,n) \to (z,k+1)] \quad \leftarrow \; n-k+1 \text{ levels}$$

Lemma

$$\lim_{n\to\infty} F_k^{T,n}(x,z) \overset{d}{=} \mathcal{X}^T[(0,k+1)\to(x,1)] + T^{-1}zx \quad \leftarrow \; k \text{ levels}$$

Key lemma $f = \{f_1, f_2, \ldots, f_n\}$

[illegible]

$n-k-1$ levels $\quad$ k levels

Limit of F_k

$$F_k^{T,n}(x,z) = \mathcal{X}^{T,n}[(-\sqrt{nT}+x,n) \to (z,k+1)] \leftarrow \; n-k-1 \text{ levels}$$

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Key $f = \{f_1, f_2, \ldots, f_n\}$

[illegible]

$n-k-1$ levels [illegible] levels

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$$\lim_{n\to\infty} F_k^{T,n}(x,z) \overset{d}{=} \mathcal{X}^T[(0,k+1) \to (x,1)] + T^{-1}zx \leftarrow \ k \text{ levels}$$

Key $\quad f = \{f_1, f_2, \dots, f_n\}$

$$(\mathcal{W}f)[(x,n) \to (z,k+1)] = (\mathcal{W}R_z f)[(z-x,k+1) \to (z,1)] + \mathcal{W}f_{k+1}(z)$$

Limit of F_k

$$F_k^{T,n}(x,z) = \mathcal{X}^{T,n}[(-\sqrt{nT}+x,n) \to (z,k+1)] \leftarrow \ n-k+1 \text{ levels}$$

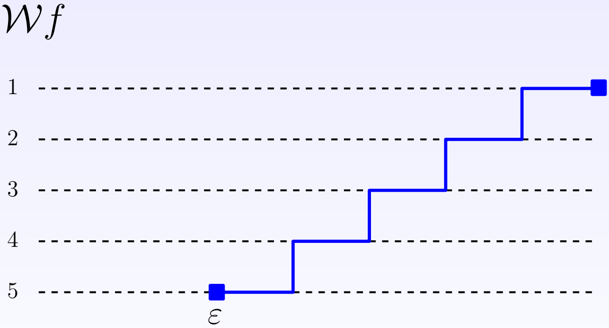
Lemma

$$\lim_{n\to\infty} F_k^{T,n}(x,z) \overset{d}{=} \mathcal{X}^{T}[(0,k+1) \to (x,1)] + T^{-1}zx \leftarrow \ k \text{ levels}$$

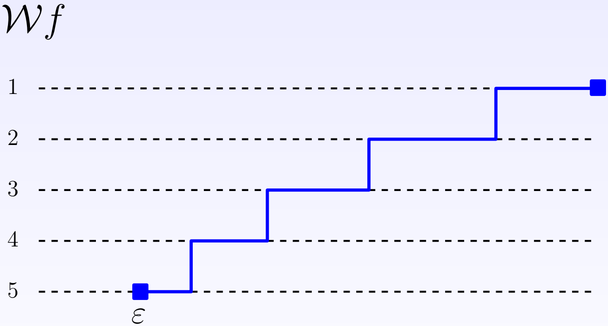
Key $f = \{f_1, f_2, \ldots, f_n\}$

$$(\mathcal{W}f)[\underbrace{(x,n) \to (z,k+1)}_{n-k-1 \text{ levels}}] = (\mathcal{W}R_z f)[\underbrace{(z-x,k+1) \to (z,1)}_{k \text{ levels}}] + \mathcal{W}f_{k+1}(z)$$

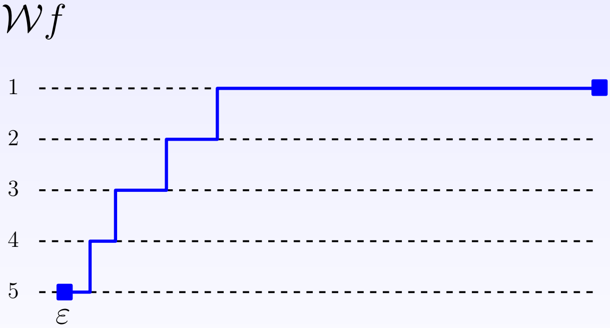
Concentration



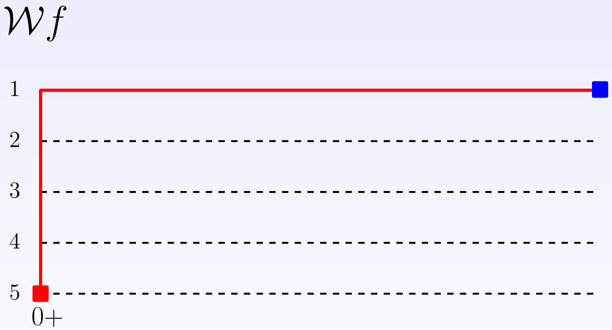
Concentration



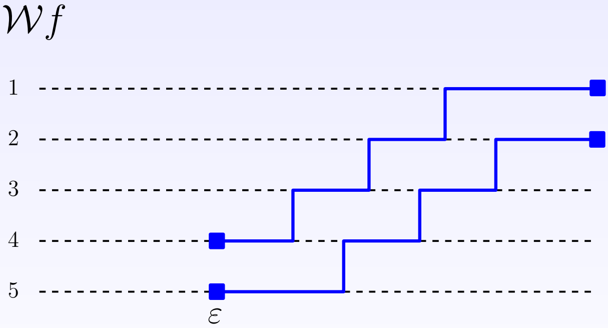
Concentration



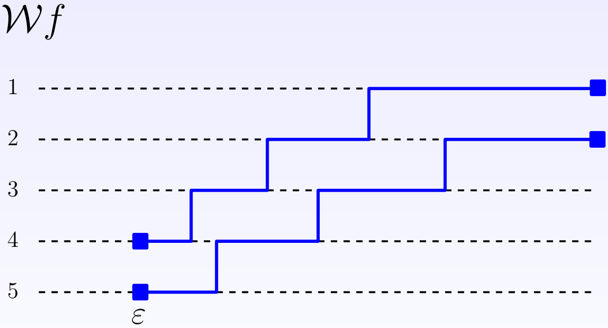
Concentration



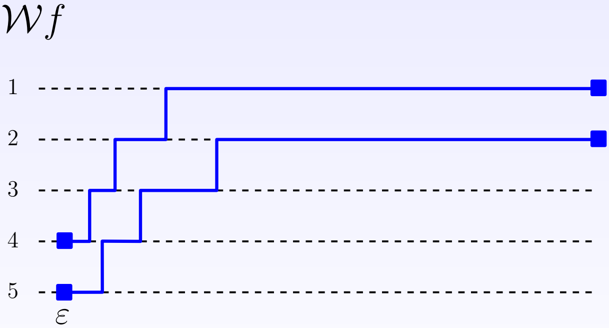
Concentration



Concentration



Concentration



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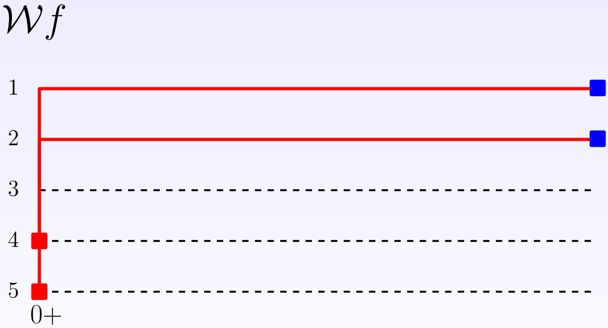

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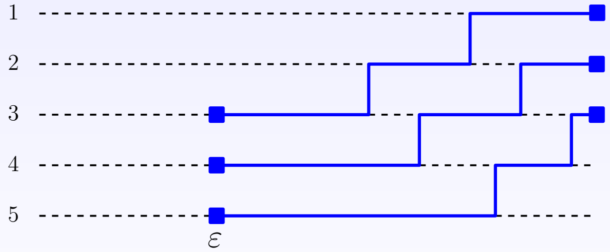

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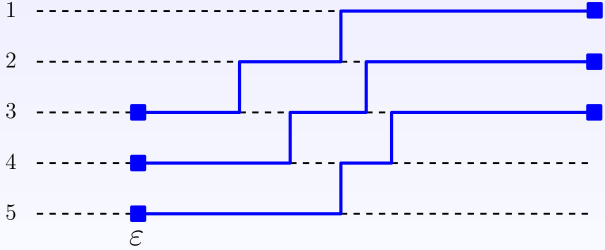

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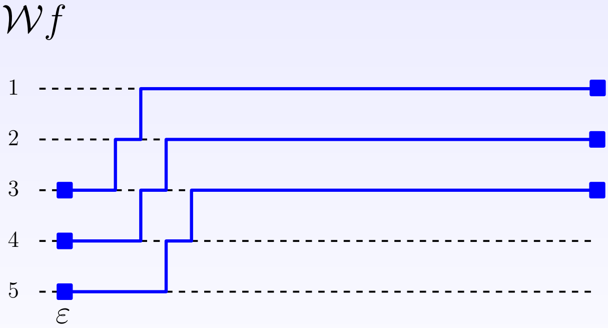

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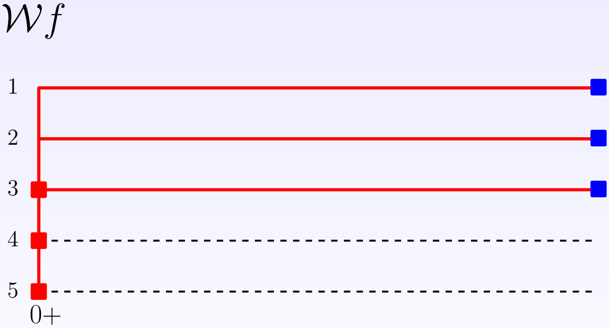

Happy birthday Timo!