On Quantum Complexity

Mohsen Alishahiha

School of physics, Institute for Research in Fundamental Sciences (IPM)

Quantum Information Theory in Quantum Field Theory and Cosmology,
4-9 June 2023, BIRS, Canada

Based on

M. A. ,"On Quantum Complexity," PLB 2023 [arXiv:2209.14689 [hep-th]]

M. A., and Souvik Banerjee, "A universal approach to Krylov State and Operator complexities," arXiv: 2212.10583 [hep-th]

Quantum chaos is an interesting subject though it is difficult to understand. This is due to the fact that the time evolution of quantum mechanic is local and unitary and thus, in general, it is hard to study the emergence of ergodic behavior in quantum system.

Therefore it is of great interest to understand thermal behavior in quantum level in which the eigesnstate thermalization hypothesis, (ETH), plays an important role.

In the classical level the chaotic behavior may be described by the sensitivity of trajectories in the phase space to the initial conditions. Indeed, two initially near by trajectories separate exponentially fast characterized by the Lyapunov exponent.

Nonetheless, to probe the nature of quantum chaos certain quantities have been introduced. These include, for example, out-of-time-order correlators (OTOCs), Complexity ...

For chaotic systems with finite entropy S, complexity is expected to grow for exponentially large times in the entropy, long after thermal equilibrium has been reached.

Remarkably, the same growth holds for the black hole interior.

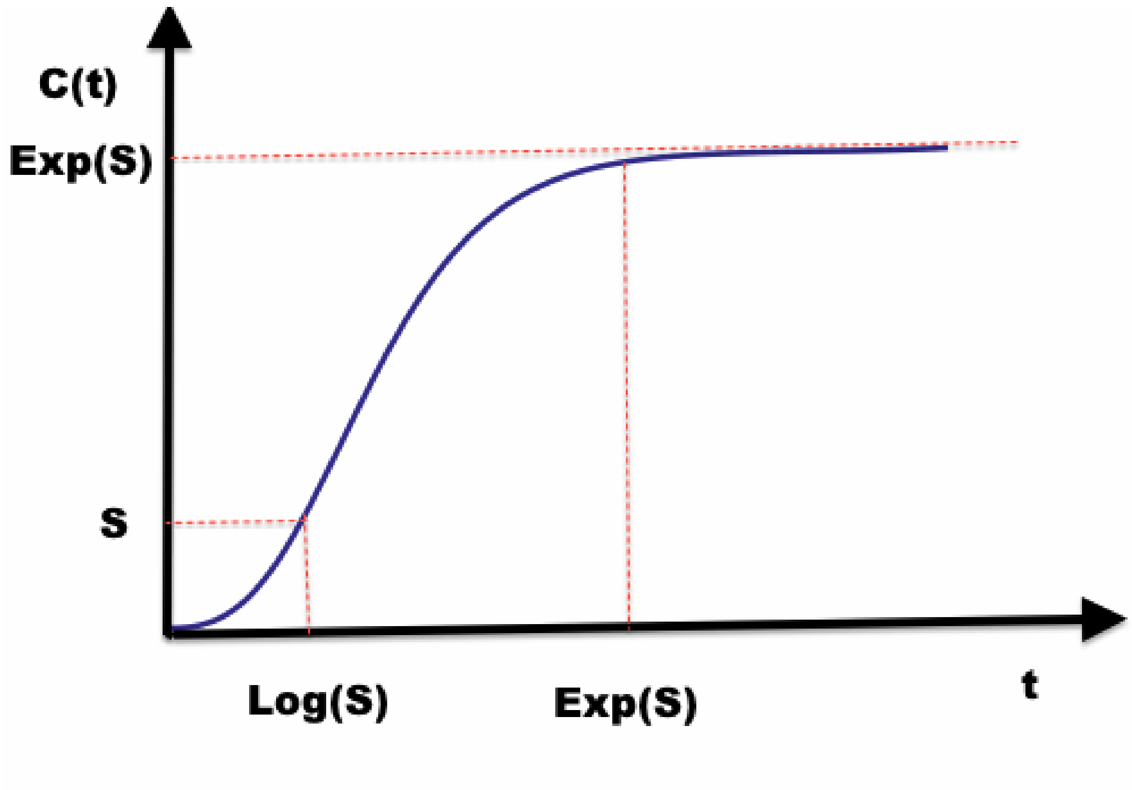
The complexity of the boundary state is proportional to the volume of a maximal codimension-one bulk surface B that extends to the AdS boundary, and asymptotes to the time slice Σ on which the boundary state is defined

$$\mathcal{C} = \frac{V(B)}{GR}\Big|_{\partial B = \Sigma}$$

This known as CV proposal. There is also another proposal known as CA.

Susskind, arXiv:1403.5695, arXiv:1402.5674

We note however, that for chaotic Hamiltonians after the aforementioned period of growth, at times $t \sim \mathcal{O}(e^S)$ we expect saturation to a plateau of size $C \sim \mathcal{O}(e^S)$.



While semi-classical contributions both in form of the CV and CA conjectures indeed provide the period of growth, the saturation to the plateau, until recently, has been illusive.

Recently, holographic complexity was calculated in JT gravity using the CV conjecture where it was shown that including higher genus geometries gives the correct late-time behavior for complexity.

The complexity is computed in terms of a non-perturbative geodesic length in JT gravity $\mathcal{C} \sim \langle \ell(t) \rangle_{QG}$

$$\langle \ell(t) \rangle = \int d\ell \, \Psi^2_{HH}(\beta + it) \, \ell$$

where Ψ^2_{HH} is Hartle-Hawking wave function.

For times $t \ll e^S$ one may evaluate the integral which at leading order takes the form

$$\langle L(t) \rangle \approx \text{const.} - C_0 e^S + C_1 \, t \, ,$$

For large t ($t \sim e^S$) the integral decays and therefore the quantum expectation value of the geodesic length becomes constant.

L.V. Iliesiu, M. Mezei and G. Sárosi, arXiv:2107.06286, M. A., S. Banerjee and J. Kames-King, arXiv:2205.01150, M. A. and S. Banerjee, arXiv:2209.02441 [hep-th].

For chaotic systems the notion of thermalization may be described by the eigenstate thermalization hypothesis (ETH) which gives an understanding of how an observable thermalizes to its thermal equilibrium value.

According to the ETH, thermalization occurs at the level of individual eigenstates of the Hamiltonian. In fact setting

$$\varepsilon = \frac{E_1 + E_2}{2}, \qquad \omega = E_1 - E_2,$$

The ETH states that the matrix elements of observables in the basis of the eigenstates of the Hamiltonian obey the following ansatz

$$\langle E_1|\mathcal{O}|E_2\rangle = \bar{\mathcal{O}}(\varepsilon)\delta_{E_1,E_2} + e^{-S} f(\varepsilon,\omega)\mathcal{R}_{E_1E_2}$$

where $\bar{\mathcal{O}}(\varepsilon)$ is the micro canonical average of the corresponding operator, S is thermodynamical entropy of the system, $f(\varepsilon,\omega)$ is a smooth function of its arguments and $\mathcal{R}$ is unit variance random function with zero mean.

J. M. Deutsch, Phys Rev A.43 (1991), M. Srednicki, cond-mat/9403051, J. Phys. A32 (1999) .

The ETH ansatz has an immediate application in understanding of the thermalization which indicates that the quantum expectation value of an observable must approach its thermal average for long enough times. We note, however, that this ansatz does not tell us how long the process of thermalization is.

Of course our main concern is not to explore the thermalization of the system. Actually the aim is to understand the late time behavior of a certain observable when the system is in the thermal equilibrium.

Indeed, within the context of the ETH we are interested in finding, if any, an observable that exhibits time growth even though the system has been already reached the thermal equilibrium.

Quantum Complexity

Our main motivation to propose a candidate for the quantum complexity comes from the holographic setup in which it is believed that the holographic complexity exhibit a linear growth at late times.

Therefore, it what follows for a given quantum system we would like to define a quantity exhibiting such a linear growth.

To proceed, let us define the quantum object $\mathcal{C}$ associated with an operator $\mathcal{O}$ as follows

$$\mathcal{C}_{\mathcal{O}}(\beta,t) \equiv \langle \mathcal{O}(t) \rangle_\beta = \langle \psi | e^{-(\frac{\beta}{2}-it)H} \mathcal{O}\, e^{-(\frac{\beta}{2}+it)H} |\psi\rangle \,.$$

Using the completeness condition of the energy eigenstates $\int dE\,|E\rangle\langle E| = 1$, one finds

$$\mathcal{C}_{\mathcal{O}}(\beta,t) = \int_0^\infty dE_1 dE_2\, e^{-\frac{\beta}{2}(E_1+E_2)} e^{it(E_1-E_2)} \rho_\psi(E_1,E_2)\, A(E_1,E_2),$$

where

$$\rho_\psi(E_1,E_2) = \langle E_1|\psi\rangle\langle\psi|E_2\rangle = \rho(E_1)\rho(E_2)\,,$$

$$A(E_1,E_2) = \langle E_1|\mathcal{O}|E_2\rangle\,,$$

with $\rho(E) = \langle E|\psi\rangle$ is the density of state.

For a typical operator the A-function follows the ETH ansatz and therefore the long time average of $\mathcal{C}_{\mathcal{O}}$ approaches that of micro canonical average of the corresponding operator.

As far as the time dependence of the corresponding quantum object is concerned, as we will see, the main role is played by the A-function.

We would like to see whether there is a condition under which the corresponding quantum quantity, $\mathcal{C}_{\mathcal{O}}$, keeps growing with time even though the whole system is reached thermal equilibrium.

To proceed, since we are interested in the late time behavior $\mathcal{C}_{\mathcal{O}}(\beta, t)$, it is useful to rewrite the corresponding expression in terms of the variables ε and ω

$$\mathcal{C}_{\mathcal{O}}(\beta, t) = \frac{1}{Z}\int_0^\infty d\varepsilon e^{-\beta\varepsilon} \int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\, \rho(\varepsilon + \frac{\omega}{2})\rho(\varepsilon - \frac{\omega}{2})\, A(\varepsilon, \omega)$$

To address this question, following the ETH idea, it is clear from this equation that the corresponding information should be encoded in the behavior of A-function.

Actually, as it is evident form the above expression, the time dependence of $\mathcal{C}_{\mathcal{O}}(\beta, t)$ should be read from the ω-integral.

Due to the simple factor of $e^{i\omega t}$ in the integrand, using the Cauchy's residue theorem with the assumption that the density of state $\rho(\varepsilon \pm \omega/2)$ is a smooth function in the limit of $\omega \to 0$, in order to get a non-trivial time dependence, the A-function must have a pole structure of order of n for $n \geq 2$.

In particular, for the case of a double pole structure where the A-function has the following limiting behavior

$$A(\varepsilon,\omega) = -\frac{a(\varepsilon)}{\omega^2} + \text{local terms}, \qquad \text{for } \omega \to 0,$$

with a positive smooth function $a(\varepsilon)$, one finds that the quantum object $\mathcal{C}_{\mathcal{O}}(\beta,t)$ exhibits a linear growth at late times

$$\mathcal{C}_{\mathcal{O}}(\beta,t) = C_0 + \int_0^\infty d\varepsilon e^{-\beta\varepsilon}\rho^2(\varepsilon)a(\varepsilon)\,(2\pi t),$$

where C_0 is a time independent constant that is a function of β.

It is worth noting that this linear growth must not be confused with that of the ramp phase in *e.g.* the spectrum form factor where the linear growth was the consequence of subleading connected part of the density-density correlation. Here we have a linear growth at leading disconnected level.

Having found a quantum object that has linear growth at late times, it is tempting to identify the corresponding quantum object, $\mathcal{C}_{\mathcal{O}}$, as the quantum complexity. To be precise, we would like to define the complexity as follows.

For a chaotic quantum system the quantum complexity is defined by $\mathcal{C}_{\mathcal{O}}$ for a particular operator $\mathcal{O}$ -to be found for a given system- so that the associated A-function exhibits a double pole structure in the limit of $E_1 \to E_2$

$$A(E_1, E_2) \approx -\frac{a(E_1, E_2)}{(E_1 - E_2)^2} + \text{local terms}$$

where $a(E_1, E_2)$ is a smooth positive function.

Of course for a given quantum system and for a given state, a priori, it is not obvious how to find $\mathcal{O}$ that results in the desired double pole structure for A-function. Moreover, in general the corresponding quantity may not be given in terms of local operators.

To further explore this observation, let us consider explicit examples in which one could identify a proper $\mathcal{O}$, that results in a linear growth for $\mathcal{C}_{\mathcal{O}}$.

Example

Let us consider a quantum system with the following Hamiltonian

$$H = \frac{P^2}{2} + 2\mu e^{-x} + 2e^{-2x}\,.$$

Then the corresponding Schrödinger equation is

$$\left(-\frac{d^2}{dx^2} + 4\mu e^{-x} + 4e^{-2x}\right)\psi(x) = 2E\psi(x)\,.$$

The eigenstate wave functions of the above equation are given in terms of the Whittaker function of the second kind with imaginary order

$$\psi_{\mu,E}(x) = e^{x/2} W_{-\mu,i\sqrt{2E}}(4e^{-x})\,.$$

Actually this Hamiltonian is used to study different aspects of two-dimensional JT gravity.

D. Harlow and D. Jafferis, arXiv:1804.01081, Z. Yang, arXiv:1809.08647, P. Saad, arXiv:1910.10311, P. Gao, D. L. Jafferis and D. K. Kolchmeyer, arXiv:2104.01184, D. Bagrets, A. Altland and A. Kamenev, arXiv:1607.00694

The orthogonality condition for the eigenstates $\psi_{\mu,E}(x)$ is

$$\int_0^\infty dx\, \psi_{\mu,E_1}(x)\, \psi_{\mu,E_2}(x) = \frac{\delta(E_1 - E_2)}{\rho_\mu(E_1)}\,,$$

where

$$\rho_\mu(E) = \left|\Gamma\left(\frac{1}{2} + \mu + i\sqrt{2E}\right)\right|^2 \frac{\sinh 2\pi\sqrt{2E}}{4\pi^2}\,,$$

where $\rho_\mu(E)$ is essentially the density of state of the system.

Following our proposal, the quantum complexity is given by $\mathcal{C}$ whose A-function, using the coordinate system, is

$$A(E_1, E_2) = \int_0^\infty dx\, dx'\, \psi_{\mu,E_1}(x)\psi_{\mu,E_2}(x')\, f(x, x')\,.$$

where $f(x, x') = \langle x|\mathcal{O}|x'\rangle$.

We will consider $f(x,x') = \delta(x-x')\,x$ by which the associated A-function reads

$$A(E_1,E_2) = \int_0^\infty dx\;\psi_{\mu,E_1}(x)\psi_{\mu,E_2}(x)x\,.$$

Actually, since the function f may be interpreted as matrix elements in the coordinate basis, the above choice corresponds to the matrix elements of position operator that is obviously diagonal leading to a delta function. On the other hand since the wave function satisfies the Schrödinger equation, essentially in this case what we are evaluating is the average of position operator.

M. A., S. Banerjee and J. Kames-King, arXiv:2205.01150.

By making use of the explicit expression for the wave function in terms of the Whittaker function, it is then straightforward to study the pole structure of the A-function. Indeed, using the variables (ε, ω) and in the limit of $E_1 \to E_2$ one has

$$A(\varepsilon, \omega) = -\frac{\sqrt{2\varepsilon}}{\pi \rho_\mu(\varepsilon)} \frac{1}{\omega^2} + \text{local terms.}$$

Therefore one finds the late time behavior as follows

$$\mathcal{C}(\beta, t) = C_0 + \int_0^\infty d\varepsilon \, e^{-\beta\varepsilon} \rho_\mu(\varepsilon) \sqrt{2\varepsilon}\, (2t)$$

that is the linear growth, as expected.

If one recalls that the Hamiltonian we considered describes two dimensional JT-gravity it is possible to identify what exactly the quantity $\mathcal{C}$ is. Indeed in this case it can be interpreted as the quantum expectation value of the geodesic length (wormhole) connecting two boundaries of a two sided 2d black hole. This means that the function $f(x, x')$ is just the (regularized) geodesic length.

M. A., S. Banerjee and J. Kames-King, arXiv:2205.01150.

Double pole and Krylov complexity

Let us see if there is a systematic way to observe the double pole behavior in the Krylov complexity.

To proceed let us consider a quantum system describing by a time independent Hamiltonian H whose eigenstates and eigenvalues are denoted by $|E_a\rangle$ and E_a, respectively. Here $a = 1, 2, \cdots \mathcal{D}$ with $\mathcal{D}$ being the dimension of the associated Hilbert space $\mathcal{H}$.

In this system the time evolution of a state is given by

$$|\psi(t)\rangle = e^{iHt}|\psi(0)\rangle \, .$$

Then the density matrix associated with this state at any time is

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| = e^{iHt}\,\rho(0)\,e^{-iHt}$$

where $\rho(0) = |\psi(0)\rangle\langle\psi(0)|$.

D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi and E. Altman, arXiv:1812.08657, J. L. F. Barbón, E. Rabinovici, R. Shir and R. Sinha, 1907.05393, S. K. Jian, B. Swingle and Z. Y. Xian,, 2008.12274,E. Rabinovici, A. Sánchez-Garrido, R. Shir and J. Sonner, 2009.01862 [hep-th]]. V. Balasubramanian, P. Caputa, J. M. Magan and Q. Wu, arXiv:2202.06957.

Using the Hamiltonian one can construct an orthonormal and ordered basis associated with any state of the Hilbert space. Denoting the corresponding state by $|\psi(0)\rangle$ the orthonormal, ordered basis $\{|n\rangle, n = 0, 1, 2, \cdots, \mathcal{D}_\psi - 1\}$ may be constructed using the Gram-Schmidt process.

The first element of the basis is indeed the original given state of the Hilbert state $|0\rangle = |\psi(0)\rangle$ (which is assumed to be normalized). Then the other elements are constructed recursively as follows

$$|\widehat{n+1}\rangle = (H - a_n)|n\rangle - b_n|n-1\rangle$$

where $|n\rangle = b_n^{-1}|\widehat{n}\rangle$ and

$$a_n = \langle n|H|n\rangle, \qquad b_n = \sqrt{\langle \widehat{n}|\widehat{n}\rangle}$$

The procedure stops where ever b_n vanishes which occurs for $n = \mathcal{D}_\psi$ which is the dimension of subspace $\mathcal{H}_\psi$ expanded by the basis $\{|n\rangle\}$.

This procedure produces an orthogonal basis together with coefficients a_n and b_n known as Lanczos coefficients.

Since the basis of the subspace $\mathcal{H}_\psi$ is an ordered basis one may label any element of the subspace by a number which amounts to define a label operator as follows

$$\ell = \sum_{n=0}^{\mathcal{D}_\psi - 1} c_n |n\rangle\langle n| \,,$$

for arbitrary function c_n which is the label associated with the state $|n\rangle$. Note $c_n > c_{n'}$ for $n > n'$. Since the basis $\{|n\rangle\}$ is already ordered a natural choice for the coefficient c_n is $c_n = n$.

Then the Krylov complexity is defined as

$$\mathcal{C}(t) = \mathrm{Tr}(\ell\rho(t)) = \sum_{n=0}^{\mathcal{D}_\psi - 1} n\, |\psi_n(t)|^2 \,,$$

where

$$|\psi(t)\rangle = \sum_{n=0}^{\mathcal{D}_\psi - 1} \psi_n(t)|n\rangle,$$

A comment

It has an advantage to express the Krylov complexity in terms of the trace of density matrix.

$$\mathcal{C}(t) = \mathrm{Tr}(\ell\rho(t)),$$

One can extend it to Krylov subregion complexity and also define Krylov mutual complexity.

M. A., and S. Banerjee, arXiv: 2212.10583 [hep-th]

To further proceed exploring the complexity let us assume that the Hamiltonian of the system has continues spectrum.

Using the energy eigenstates, the Krylov complexity may be recast into the following form

$$\mathcal{C}(t) = \int dE_a \, dE_b \, e^{i(E_a - E_b)t} \rho_0(E_a, E_b) \, A(E_a, E_b) \, ,$$

where

$$\rho_0(E_a, E_b) = \langle E_a | \rho(0) | E_b \rangle, \quad A(E_a, E_b) = \langle E_a | \ell | E_b \rangle \, ,$$

which is the same expression proposed for complexity in which the A-function is given by the matrix elements of label operator in energy basis.

At leading order in the dimension of the Hilbert space the density matrix $\rho_0(E_a, E_b)$ is

$$\rho_0(E_a, E_b) = \rho(E_a)\rho(E_b).$$

The behavior of Krylov complexity is encoded in the behavior of Lanczos coefficients. It was conjectured that for a chaotic system, one has large n linear growth; $b_n \approx \alpha n$ for large n.

Actually the large n linear growth is a typical behavior for Lanczos coefficients and has noting to do with the chaotic nature of the system.

Under certain condition it may also exhibit saturation phase on which the Lanczos coefficients saturate to a constant.

The saturation of Lanczos coefficients results in a linear growth for complexity at late times.

A. Dymarsky and M. Smolkin, [arXiv:2104.09514 [hep-th]], B. Bhattacharjee, X. Cao, P. Nandy and T. Pathak,[arXiv:2203.03534 [quant-ph]], A. Avdoshkin, A. Dymarsky and M. Smolkin, [arXiv:2212.14429 [hep-th]], H. A. Camargo, V. Jahnke, K. Y. Kim and M. Nishida, [arXiv:2212.14702 [hep-th]].

This behavior at late times imposes a condition on the function $A(E_a, E_b)$ to have a double pole structure

$$A(E_a, E_b) = -\frac{a(E)}{\omega^2} + \text{local terms}, \qquad \text{for } \omega \to 0\,,$$

where $\omega = E_a - E_b$ and $2E = E_a + E_b$.

Actually using the expression for the matrix elements of the label operator in the continuum limit, it is straightforward to see that for $x(y) = y$, it exhibits a double point behavior.

(Continuum limit: $x = \epsilon n$, $b(x) = 2\epsilon b_n$, $dy = \frac{dx}{b(x)}$.)

Indeed in continuum limit one has

$$\langle E_1|\ell|E_2\rangle = \frac{1}{\epsilon^2}\int_0^\infty dy\, x(y)\, e^{-i\omega y}$$

To conclude we note that the double pole structure is a consequence of saturation of Lanczos coefficients.

M. A., and S. Banerjee, arXiv: 2212.10583 [hep-th]

So far we have shown that if for a given operator -to be found for given system- its matrix elements in energy basis exhibit a double pole structure at late times, one may define a quantum object exhibiting the late time linear growth which could be interpreted as quantum complexity.

Generally, in order to get a non-trivial time dependence at late times, the A-function should have poles of order of n with $n \geq 2$.

For general $n > 2$ one generally gets power low growth at late times, though for $n = 2$ one has a linear growth.

Since having a linear growth at late times might be a signature of the complexity that is expected to be the fastest growth, one may propose a hypothesis:

The double pole structure is the highest pole structure the A-function could have.

Complexity Saturation

An other interesting feature of complexity is that it saturates at the very late times given by the exponential of the entropy of the system. It is then natural to see how the saturation could occur in this context.

To address this question we note that the density matrix $\rho(E_1, E_2)$ appearing in the expression of the quantum object $\mathcal{A}_{\mathcal{O}}$ has the following general form

$$\rho(E_1, E_2) = \rho(E_1)\rho(E_2) + \rho_c(E_1, E_2)$$

where ρ_c represents the connected term meaning that it cannot be written in a factorized form of $g_1(E_1)g_2(E_2)$.

The connected terms could have either perturbative or non-perturbative origins which may have generally non-trivial pole structure that could result in the saturation phase at very late times.

This is a well known structure which has been seen in the literature for the spectrum form factor of chaotic models such as JT-gravity in which the connected part of $\rho(E_1, E_2)$ results in the ramp phase.

Of course for the spectrum form factor there is no an A-function and the whole time dependence is controlled by the density-density correlator.

On the other hand, for the holographic complexity of JT-gravity where there is an A-function, the connected part of $\rho(E_1, E_2)$, which has non-trivial pole structure at late times is, indeed, responsible for the saturation phase.

We note, however, that in the present case, where we are dealing with a general formalism which is not directly related to the holography picture, it is not clear how the full expression of the connected term could be computed.

For a chaotic system, as far as the late time behavior is concerned, one would expect that the main contribution comes from the short range correlation which is given by the universal sine-kernel term

$$\rho_c(\varepsilon,\omega) \approx -\frac{\sin^2(D\omega\rho(\varepsilon))}{(D\omega)^2}, \qquad \text{for } \omega \ll 1\,.$$

Here D is the dimension of Hilbert space which is given by the exponential of the entropy of the system.

The double point structure of the A-function leads to linear growth at the leading disconnected part of the density-density correlation, while there is the saturation phase which can be described by subleading connected term given in the universal sine-kernel term multiplied by the double pole structure of the A-function.

One can see that the saturation occurs at $t \sim D$.

M. L. Mehta, " Random Matrices." (Elsevier, San Diego, 2004) Third edition

Using the general expression for $\rho(E_1.E_2)$ one finds

$$\mathcal{C} = \text{Constant} - \int_0^\infty d\epsilon e^{-\beta\epsilon}\rho^2(\epsilon)a(\epsilon)\int_{-\infty}^\infty d\omega \frac{e^{-it\omega}}{\omega^2}\left(1 - \frac{\sin^2(\rho(\epsilon)D\omega)}{(\rho(\epsilon)D\omega)^2}\right).$$

From this expression one observes that at late times when $\omega \sim \frac{1}{t} \to 0$ and for $\rho\,\omega \gg 1$ essentially the first term in the bracket on the r.h.s of the above equation dominates leading to a linear growth, while for $\rho\,\omega \ll 1$ which is the case at $t \sim D$, the second term starts dominating that essentially cause the whole integral to approach zero leading to a constant complexity which is the saturation phase.

L.V. Iliesiu, M. Mezei and G. Sárosi, arXiv:2107.06286, M. A., S. Banerjee and J. Kames-King, arXiv:2205.01150, M. A. and S. Banerjee, arXiv:2209.02441 [hep-th].

Summary

Quantum complexity may be defined as a quantum object associated with an operator whose matrix elements in energy basis exhibit a double pole structure at late times.

$$\mathcal{C}(\beta,t)=\int dE_1 dE_2 e^{-i\frac{E_1+E_2}{2}\beta}e^{i(E_1-E_2)t}\rho(E_1,E_2)\,A(E_1,E_2)$$

The double pole structure is the highest pole structure the A-function could have.

For JT-gravity it reduces to the quantum length connecting two boundaries.

It also reduces to Krylov complexity if one compute the quantum object for label operator of Keylov basis.

There is a universal form for the saturation phase which occurs due to the connected part of $\rho-\rho$ correlation.

Thank you