State-Changing Modular Berry Phases

Claire Zukowski

University of Minnesota, Duluth

Quantum Information in QFT and Cosmology, Banff, Canada
June 8, 2023

J. de Boer, R. Espindola, J. van der Heijden, B. Najian, D. Patramanis and C.Z. [2111.05345]

J. de Boer, B. Czech, R. Espindola, J. van der Heijden, B. Najian and C.Z. [2305.16384]

]

Outline

- Holography and bulk reconstruction
- Modular Berry transport
- State-changing Berry phases for 2d CFT
- State-changing Berry phases in higher dimensions
- Future directions and summary

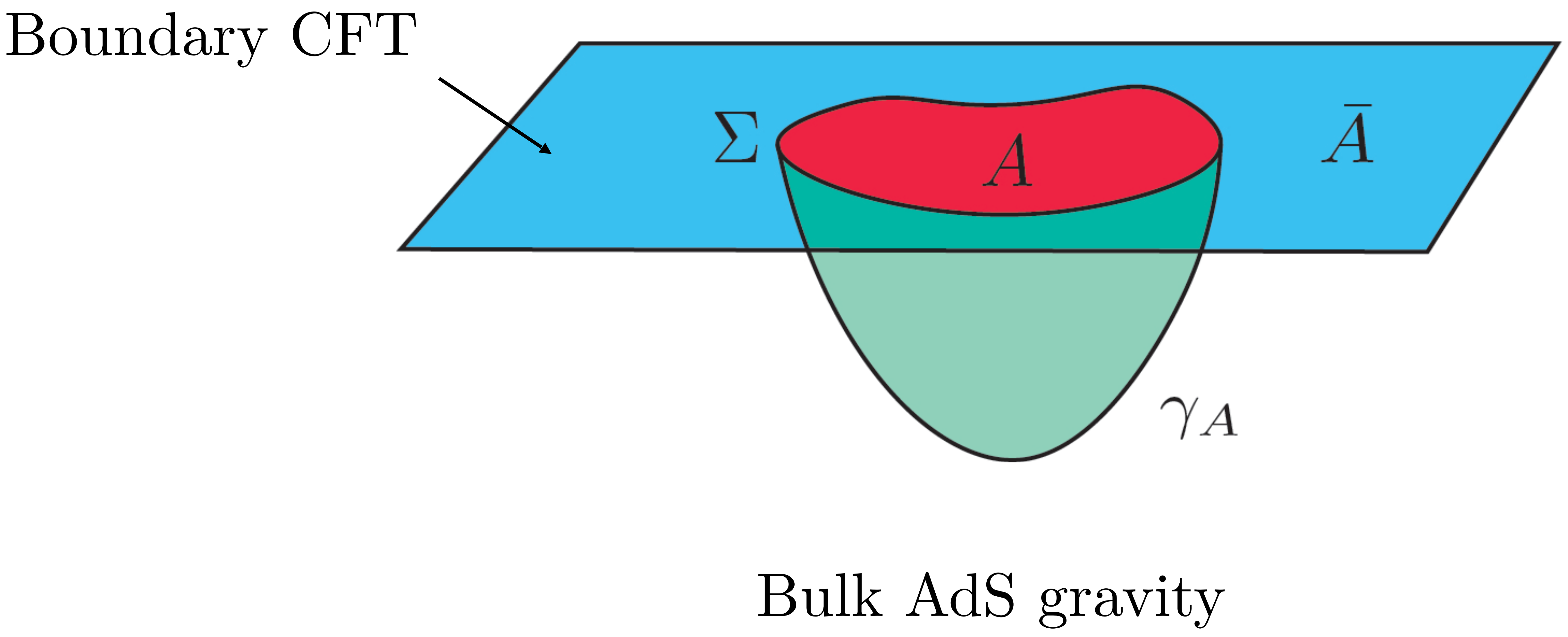
Introduction

Holography and Quantum Information

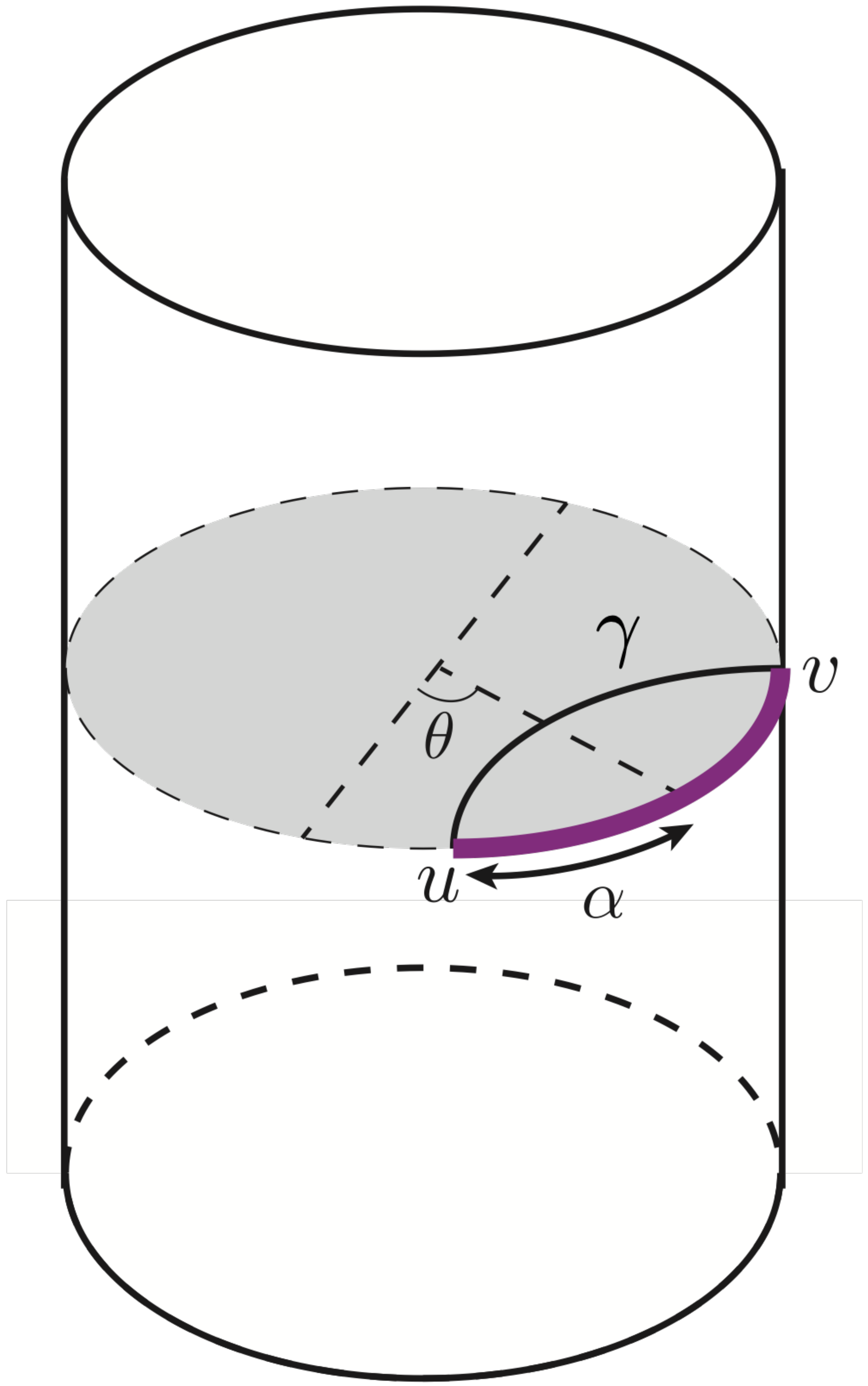
Bekenstein-Hawking formula (1973): $S = \dfrac{A}{4G_N}$

Ryu-Takayanagi formula (2006):

$$S_{EE} = \min_{\Sigma \sim \partial\gamma_A} \frac{\text{Area}(\gamma_A)}{4G_N}$$



RT surface in AdS_3 = spacelike geodesic



$$\frac{\text{length}(\gamma)}{4G} = S_{\text{ent}}(u, v)$$

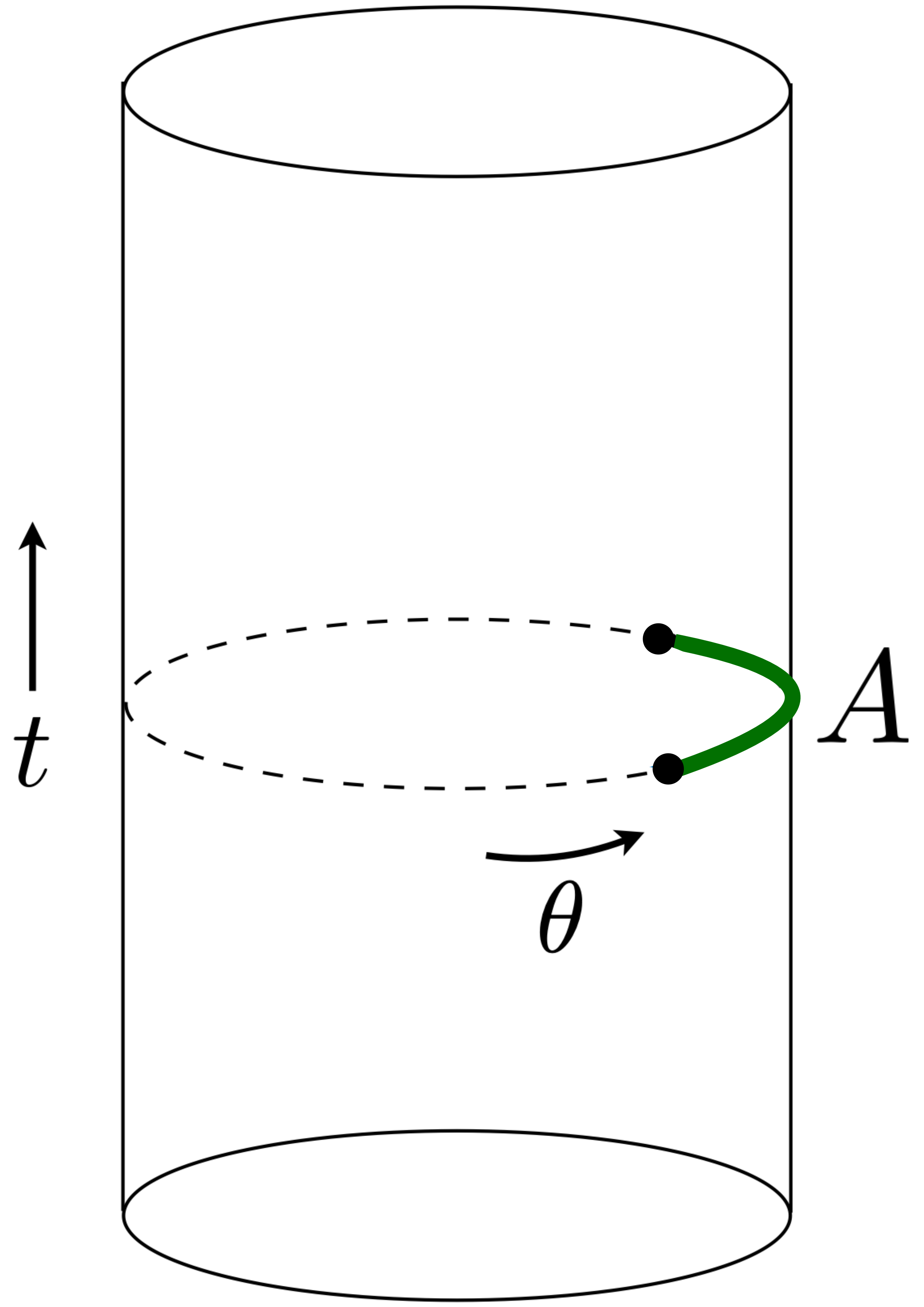
Objectives

Goal: Move beyond entanglement entropy to find additional CFT quantities connected to quantum information that can probe bulk geometry

It turns out we can be guided by symplectic geometry and group theory in identifying some appropriate CFT quantities. These will in turn also have connections to quantum information: transport of the modular Hamiltonian, complexity, etc.

Modular Berry phases

Modular Berry transport



Input: Interval A, reduced density matrix ρ_A, algebra of operators $\mathcal{O}$ on A

Modular Hamiltonian: $H_{\text{mod}} = -\log \rho_A$

Modular zero modes: Operators that commute with the modular Hamiltonian: $[Q_i, H_{\text{mod}}] = 0$

$$V = e^{-i\sum_i s_i Q_i} \qquad \mathcal{O} \to V^\dagger \mathcal{O} V$$

Take the algebra to itself and leave expectation values of algebra elements unchanged

Modular Berry transport

Consider a family of $H_{\text{mod}}(\lambda)$ depending on the parameter λ

Diagonalize modular Hamiltonian:

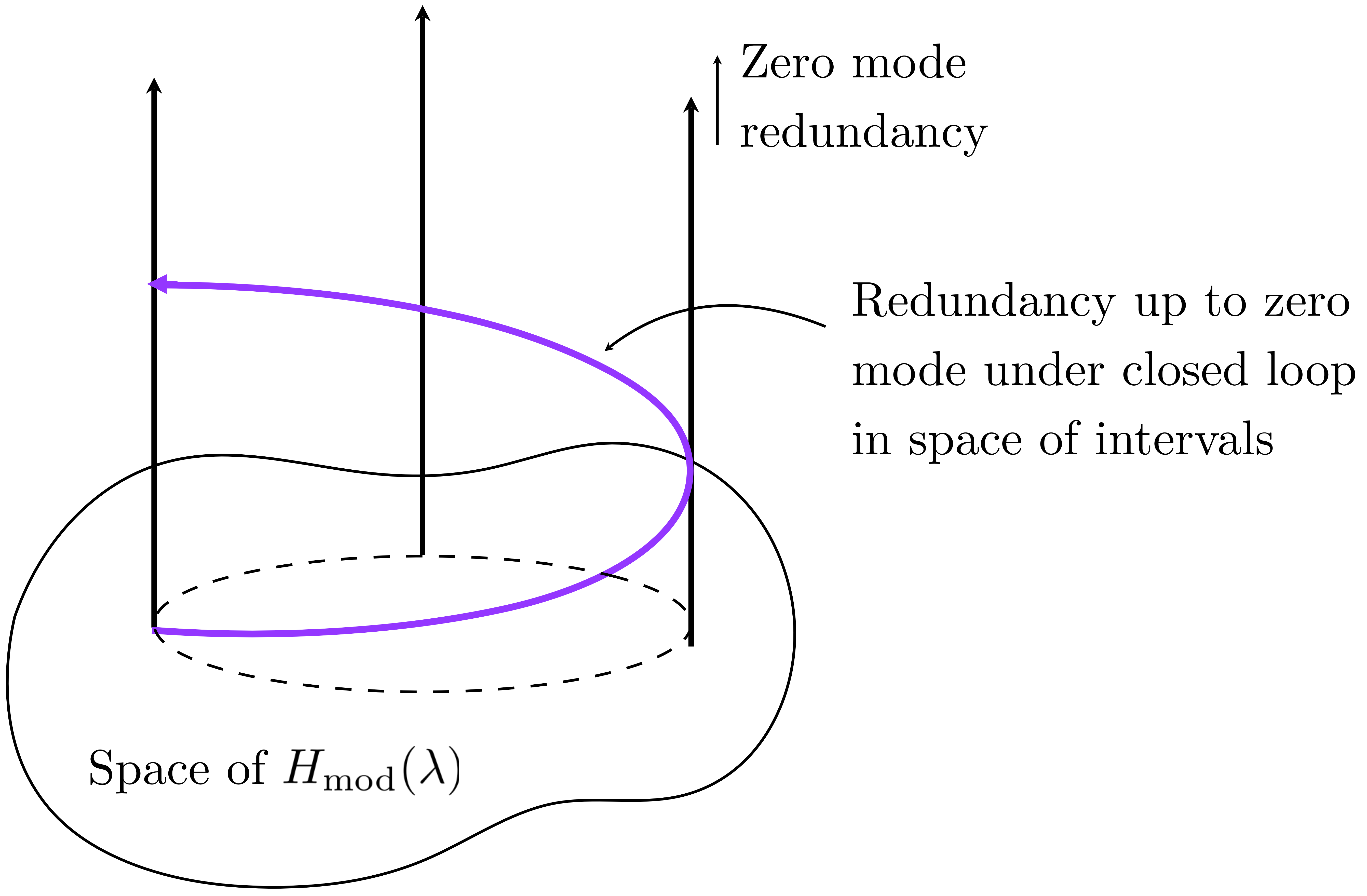
$$H_{\text{mod}} = U^\dagger \Delta U$$

Parallel transport equation:

$$\dot{H}_{\text{mod}} = [\dot{U}^\dagger U, H_{\text{mod}}] + U^\dagger \dot{\Delta} U$$

Redundancy by a modular zero mode:

$$H_{\text{mod}} = V^\dagger U^\dagger \Delta U V$$



Fiber bundle

Modular Berry transport

Fix redundancy: $H_{\text{mod}} = \tilde{U}^\dagger \Delta \tilde{U} \qquad \tilde{U} = UV$

$$P_0[\partial_\lambda \tilde{U}^\dagger \tilde{U}] = 0$$

$$\Rightarrow \quad (\partial_\lambda - \Gamma)\, V = 0 \qquad \Gamma = P_0[\dot{U}^\dagger U]$$

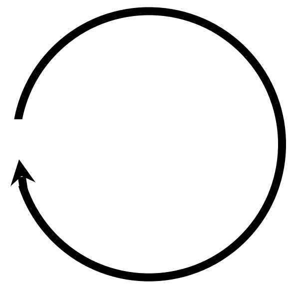
$\Gamma =$ Berry connection

$\Gamma \to V^\dagger \Gamma V - V^\dagger \dot{V}$ under $U \to UV$

Berry curvature = holonomy around a small loop

$$\exp\left(\int_{\lambda_i}^{\lambda_i+\delta\lambda} \tilde{U}^\dagger \delta_\lambda \tilde{U}\right)$$

$\lambda \to \lambda + \delta\lambda$



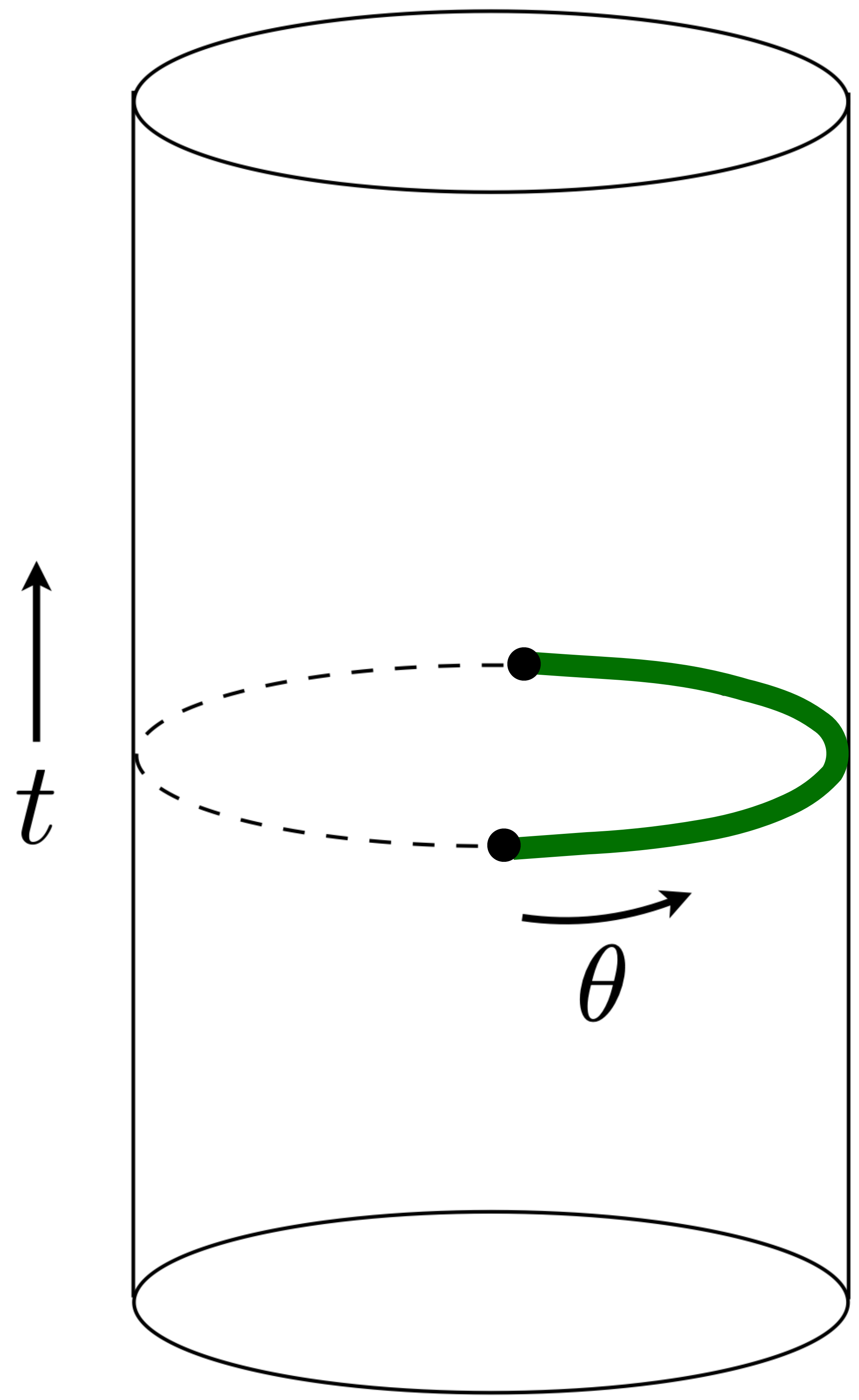
Berry phase: $$\theta(\gamma) = \int_{B|\partial B=\gamma} F$$

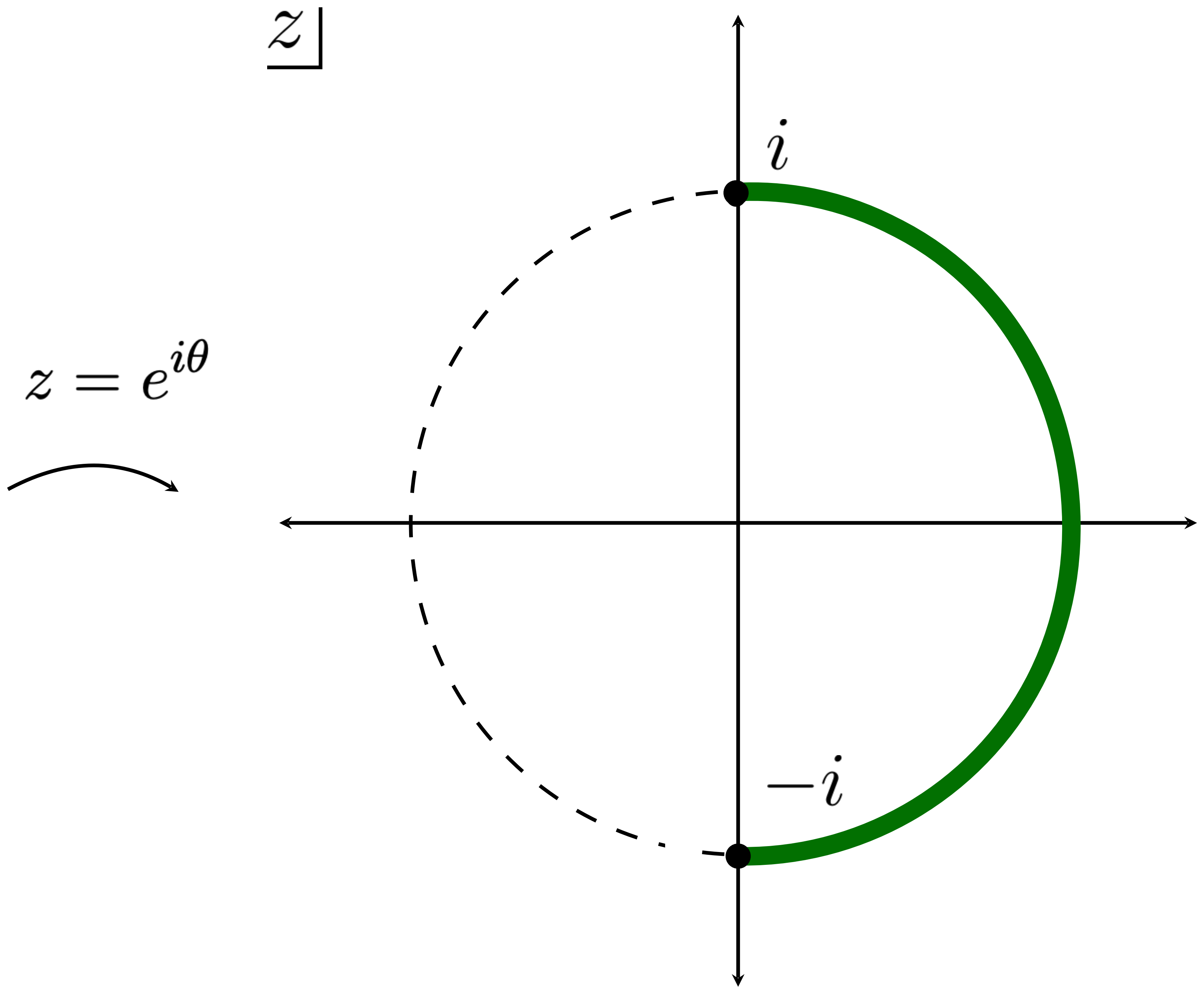
State-changing parallel transport in 2d CFT

de Boer, Espindola, Najian, Patramanis, van der Heijden, CZ (2021)

State-changing parallel transport

One option is to change the state with a fixed interval





Half interval modular Hamiltonian in the vacuum:

$$H_{\text{mod}} = \frac{1}{2i} \oint (1 + z^2) T(z)\, dz$$

$$H_{\text{mod}} = \pi(L_{-1} + L_1)$$

Act with diffeomorphism:

$$X_\xi = \frac{1}{2\pi i} \oint \xi(z) T(z)\, dz$$

$$\delta_\xi H_{\text{mod}} = [X_\xi, H_{\text{mod}}]$$

$$F = P_0([X_{\xi_1}, X_{\xi_2}])$$

de Boer, Espindola, Najian, Patramanis,
van der Heijden, CZ (2021)

State-changing parallel transport

More precisely, we work with diffeomorphisms that diagonalize the adjoint action but are non-differentiable at interval endpoints:

$$[H_{\text{mod}}, X_\lambda] = \lambda X_\lambda$$

Solution: $\xi_\lambda(z) = \pi(1+z^2)\left(\frac{1-iz}{z-i}\right)^{-i\lambda/2\pi}$

$\xi_\lambda(z) \to 0 \quad \text{as} \quad z \to \pm i$

$\lambda = 0 \qquad \xi_\lambda \to$ modular Hamiltonian

Zero mode projection: $$P_0(X_\xi) = \frac{1}{\pi}\int_{-i}^{i} \frac{\xi(z)}{(1+z^2)^2}\,dz$$

de Boer, Espindola, Najian, Patramanis,
van der Heijden, CZ (2021)

Virasoro-like coadjoint orbits

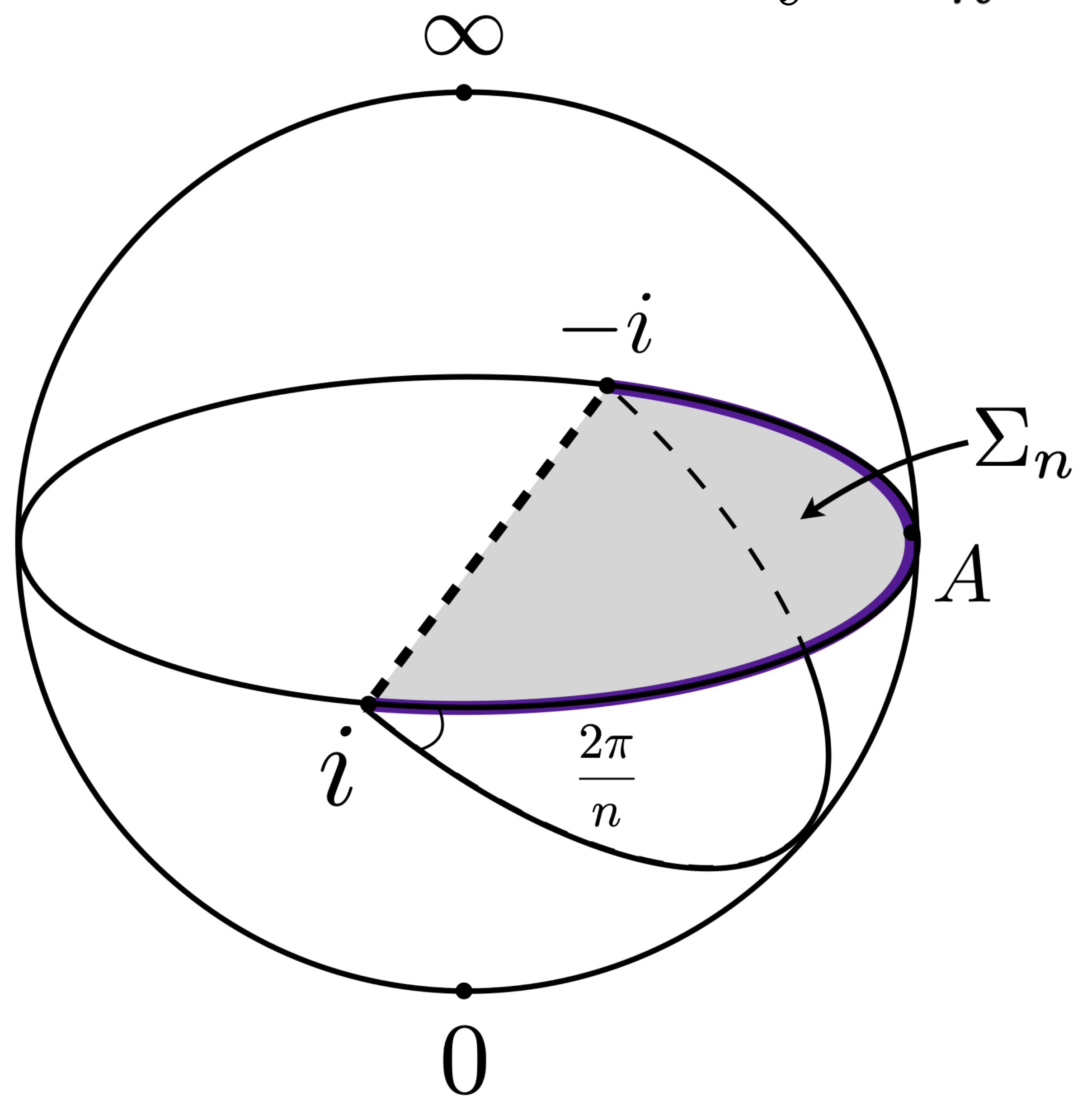
Coadjoint orbits are special symplectic geometries that are dictated by symmetry.
They admit a symplectic form known as the Kirillov-Kostant (KK) symplectic form.

We found a Virasoro-like orbit such that: KK symplectic form = Berry curvature

Lewkowycz and Maldacena (2013)
Dong (2014, 2016)
de Boer, Espindola, Najian, Patramanis, van der Heijden, CZ (2021)

Euclidean cosmic brane geometry

On the boundary $\mathcal{B}_n$: Insertion of twist fields at the interval endpoints



In the bulk:

Creates a brane with tension $\mathcal{T}_n = \frac{n-1}{4nG}$ ending on z_1, z_2

$n \to 1$ settles onto RT surface

$$z' = f(z) = \left(\frac{z-z_1}{z-z_2}\right)^{\frac{1}{n}} + \mathcal{O}(w^2)$$

$$ds^2 = \frac{dw^2}{w^2} + \frac{1}{w^2}\left(dz - w^2\frac{6}{c}\bar{T}(\bar{z})d\bar{z}\right)\left(d\bar{z} - w^2\frac{6}{c}T(z)dz\right) \longrightarrow ds^2 = \frac{dw'^2 + dz'd\bar{z}'}{w'^2}$$

$$T(z) = \frac{c}{12}\{f(z), z\} = \frac{c}{6}\left(\frac{1}{n^2} - 1\right)\frac{1}{(1+z^2)^2}$$

← Part of integrand for projection operator

de Boer, Espindola, Najian, Patramanis,
van der Heijden, CZ (2021)

Euclidean cosmic brane geometry

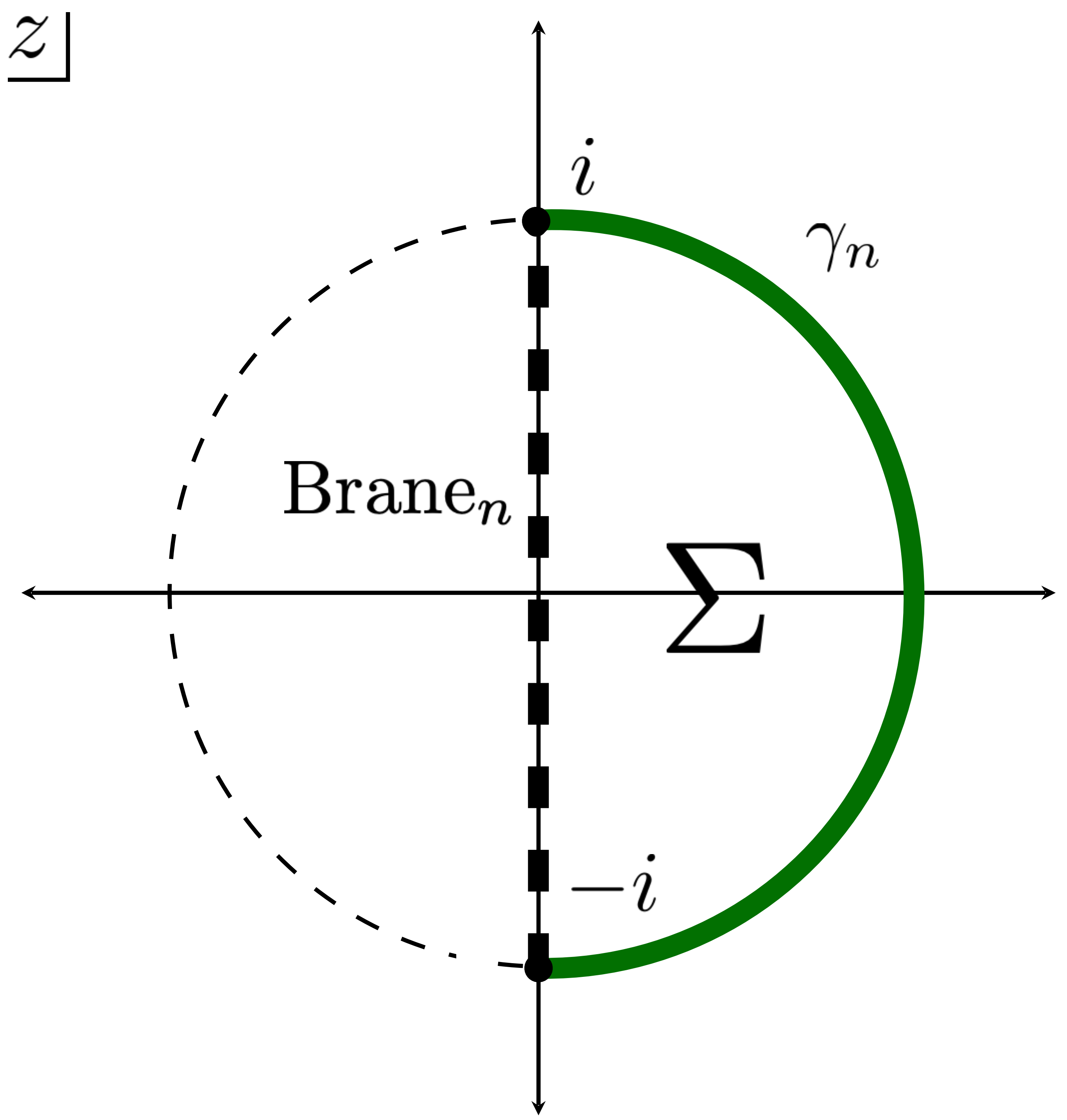
Chern-Simons symplectic form on entanglement wedge: $\omega = \frac{k}{4\pi}\int_\Sigma \mathrm{tr}(\delta_1 A \wedge \delta_2 A)$

$$A = \frac{1}{2w}\begin{pmatrix} dw & -2\,dz \\ w^2 \frac{12}{c} T(z)\,dz & -dw \end{pmatrix}, \qquad \bar{A} = -\frac{1}{2w}\begin{pmatrix} dw & w^2 \frac{12}{c}\bar{T}(\bar{z})\,d\bar{z} \\ -2\,d\bar{z} & -dw \end{pmatrix}$$

Dirichlet boundary conditions: $\delta A = 0$ on Brane_n

$$\omega_n = \frac{c}{12\pi}\left(1 - \frac{1}{n^2}\right)\int_{\gamma_n} \frac{[\xi_1, \xi_2]}{(z^2+1)^2}\,dz$$

$$\omega \equiv \lim_{n\to 1} \frac{\partial}{\partial n}\frac{\omega_n}{k} \qquad \Longrightarrow \qquad \omega = P_0([X_{\xi_1}, X_{\xi_2}])$$



Bulk symplectic form = Berry curvature

State-changing parallel transport in higher dimensions

de Boer, Czech, Espindola, Najian, van der Heijden, CZ (2023)

What changes in higher dimensions?

Imagine changing the state by a coordinate transformation:

$$x^\mu \mapsto x'^\mu = x^\mu + \xi^\mu(x)$$

Unlike in 2d, we are not implementing parallel transport with symmetry generators (~Virasoro).

In the general case, we do not expect a coadjoint orbit interpretation. But we can still find a match to the bulk.

A useful tool for this is coherent states and the Euclidean path integral.

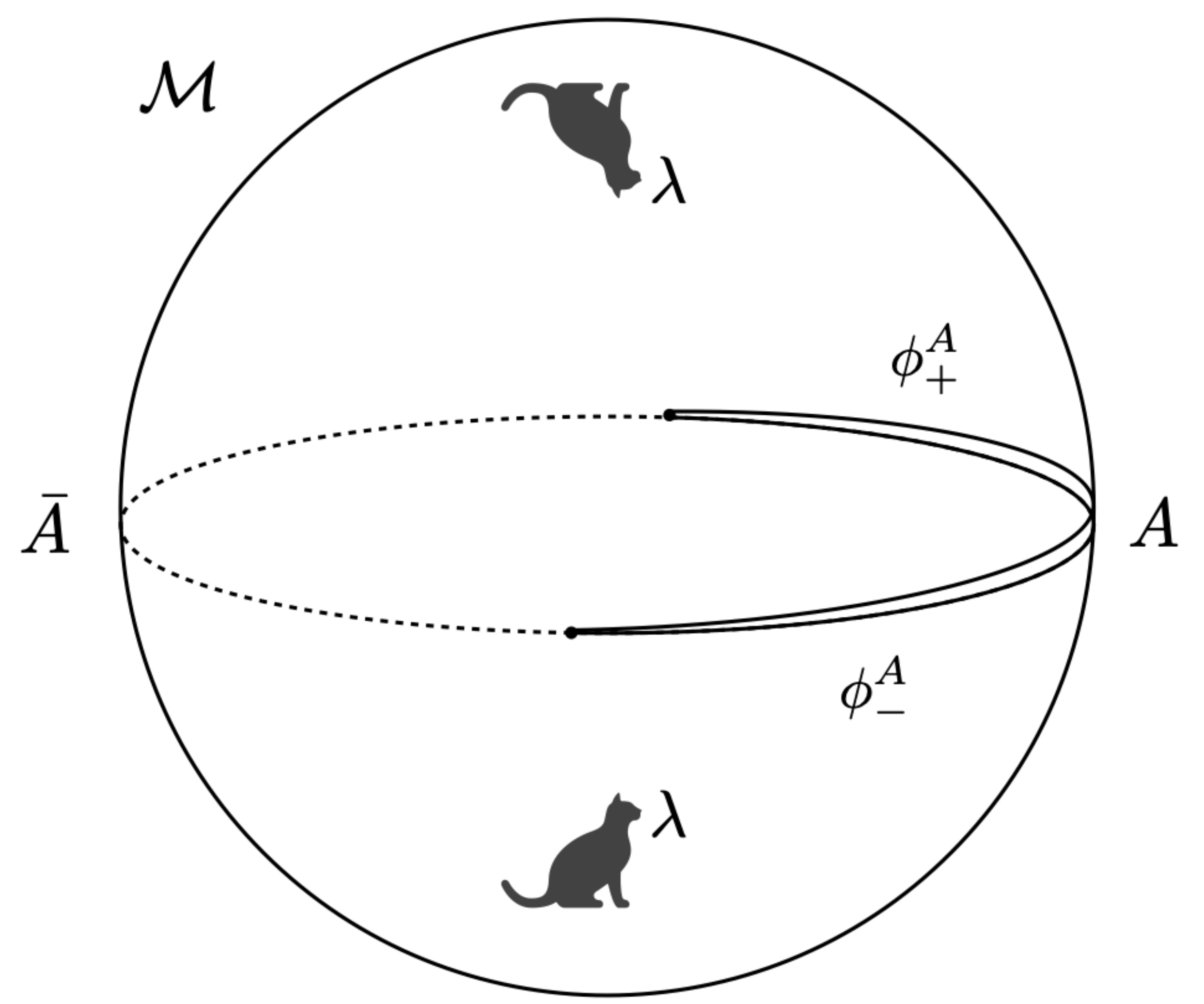
de Boer, Czech, Espindola, Najian, van der Heijden, CZ (2023)

Euclidean path integral

Prepare a coherent state $|\Psi\rangle$ with source $\lambda(x)$ using the Euclidean path integral.

Matrix elements of the density matrix:

$$\langle\phi_+^A|\,\rho\,|\phi_-^A\rangle = \frac{1}{Z}\int_{\phi(0^-)=\phi_-^A}^{\phi(0^+)=\phi_+^A}[\mathcal{D}\phi]e^{-S[\phi]}$$



Perturb the state:

$$\langle\phi_+^A|\,\rho'\,|\phi_-^A\rangle = \frac{1}{(Z+\delta Z)}\int_{\phi(0^-)=\phi_-^A}^{\phi(0^+)=\phi_+^A}[\mathcal{D}\phi]e^{-S[\phi]-\int d^dx\delta\lambda(x)\mathcal{O}(x)}$$

$$\Rightarrow \quad \delta\rho \equiv \rho' - \rho \qquad \Rightarrow \quad \delta H_{\text{mod}}$$

Faulkner, Lewkowycz (2017)

Modular frequency basis

Introduce a modular frequency basis: $H_{\text{mod}}|\omega\rangle = \omega|\omega\rangle$

Matrix elements for perturbed modular Hamiltonian:

$$\langle\omega|\delta H_{\text{mod}}|\omega'\rangle = \int d^dx\, \delta\lambda(x)\langle\omega|\mathcal{O}(x)|\omega'\rangle \frac{\omega-\omega'}{e^{\omega-\omega'}-1}$$

Fourier decompose operators:

$$\mathcal{O}_\omega = \int_{-\infty}^{\infty} ds\, e^{-is\omega}\mathcal{O}_s \qquad \mathcal{O}_s = e^{iH_{\text{mod}}s}\mathcal{O}e^{-iH_{\text{mod}}s}$$

$$[H_{\text{mod}}, \mathcal{O}_\omega] = \omega\mathcal{O}_\omega$$

de Boer, Czech, Espindola, Najian,
van der Heijden, CZ (2023)

Parallel transport in modular frequency basis

Parallel transport equation:

$$\langle\omega|(\delta H_{\rm mod} - P_0(\delta H_{\rm mod}))|\omega'\rangle = \langle\omega|[X, H_{\rm mod}]|\omega'\rangle$$

Generator of parallel transport:

$$\langle\omega|\, X\, |\omega'\rangle = -\int d^dx \delta\lambda(x) n(\omega-\omega')\, \langle\omega|\, \mathcal{O}(x)\, |\omega'\rangle \qquad n(\omega) \equiv \frac{1}{e^\omega - 1}$$

Zero mode projection:

$$P_0(\mathcal{O}) \equiv \int d\omega\, \langle\omega|\mathcal{O}|\omega\rangle|\omega\rangle\langle\omega|$$

$$P_0(H_{\rm mod}) = H_{\rm mod}$$

$$P_0([H_{\rm mod}, X]) = 0$$

de Boer, Czech, Espindola, Najian, van der Heijden, CZ (2023)

Berry curvature and information metric

We can now compute the Berry curvature:

$$F = P_0([X_1, X_2]) \qquad\qquad F_\Psi \equiv \langle \Psi | F | \Psi \rangle$$

Result in the modular frequency basis:

$$F_\Psi = \int d^d x \int d^d x' \delta_1 \lambda(x) \delta_2 \lambda(x') \int d\omega \, n(\omega) \, \langle \mathcal{O}(x) \mathcal{O}_\omega(x') \rangle$$

$$n(\omega) \equiv \frac{1}{e^\omega - 1}$$

Jafferis, Lewkowycz, Maldacena, Suh (2015)
Faulkner, Lewkowycz (2017)
May, Hijano (2018)

Bulk modular flow

Decompose bulk fields:

$$\Phi_\omega = \int_{-\infty}^{\infty} ds\, e^{-i\omega s} \rho_{\text{bulk}}^{-is} \Phi \rho_{\text{bulk}}^{is}$$

$$\Phi_\omega(X) = \int_\Sigma d^{d+1}Y \left[\alpha(X,Y)\Phi(Y) + \beta(X,Y)\Pi(Y)\right]$$

Modular extrapolate dictionary $\lim_{z\to 0} z^{-\Delta_+}\Phi_\omega(x,z) = \mathcal{O}_\omega(x)$

$\Rightarrow$ Move F_Ψ in from the boundary to the bulk

de Boer, Czech, Espindola, Najian, van der Heijden, CZ (2023)

Bulk symplectic form

$$F_\Psi = \int d^d x \int d^d x' \, \delta_1\lambda(x)\delta_2\lambda(x') \int d\omega \, n(\omega) \, \langle \mathcal{O}(x)\mathcal{O}_\omega(x')\rangle$$

$\Big\downarrow$ Modular extrapolate dictionary

$$F_\Psi = \int_\Sigma d^{d+1}Y \int d\omega \; i \; [\delta_2\pi_{-\omega}(Y)\delta_1\phi(Y) - \delta_2\phi_{-\omega}(Y)\delta_1\pi(Y)]$$

$\delta\phi, \delta\pi =$ exp. value in $\delta\rho_{\text{bulk}}$

Bulk symplectic form: $\Omega(\delta_1\phi, \delta_2\phi) = \int_\Sigma d^{d+1}Y \, [\delta_1\phi(Y)\delta_2\pi(Y) - \delta_1\pi(Y)\delta_2\phi(Y)]$

$$\Rightarrow \quad \boxed{F_\Psi = i\,\Omega(\delta_1\phi, \delta_2\phi)}$$

Conclusions

Summary

- We considered a new quantum information theoretic probe of bulk geometry: the parallel transport of modular Hamiltonians under a change of state.
- In a varied setup and in both 2d and higher dimensions, this computes a bulk symplectic form for the entanglement wedge.
- When the state-change is implemented by symmetry generators, this has connections to the geometry of coadjoint orbits.

Summary: The Triangle

Boundary/QI ⟷ **Bulk**

- Shape modular transport/kinematic space
- State modular transport

- Bulk lengths
- Entanglement wedge symplectic form

Auxiliary coadjoint orbit geometry

- $\frac{SO(d,2)}{SO(d-1,1)\times SO(1,1)}$ orbit
- New Virasoro-like orbit

de Boer, Chagnet, Chapman, CZ (2021)

Brief advertisement: Complexity

Boundary/QI ⟷ **Bulk**

Boundary/QI

- Shape modular transport/kinematic space
- State modular transport
- CFT Circuit complexity

Bulk

- Bulk lengths
- Entanglement wedge symplectic form
- Distances between timelike geodesics

Auxiliary coadjoint orbit geometry

- $\frac{SO(d,2)}{SO(d-1,1)\times SO(1,1)}$ orbit
- New Virasoro-like orbit
- $\frac{SO(d,2)}{SO(d)\times SO(2)}$ orbit

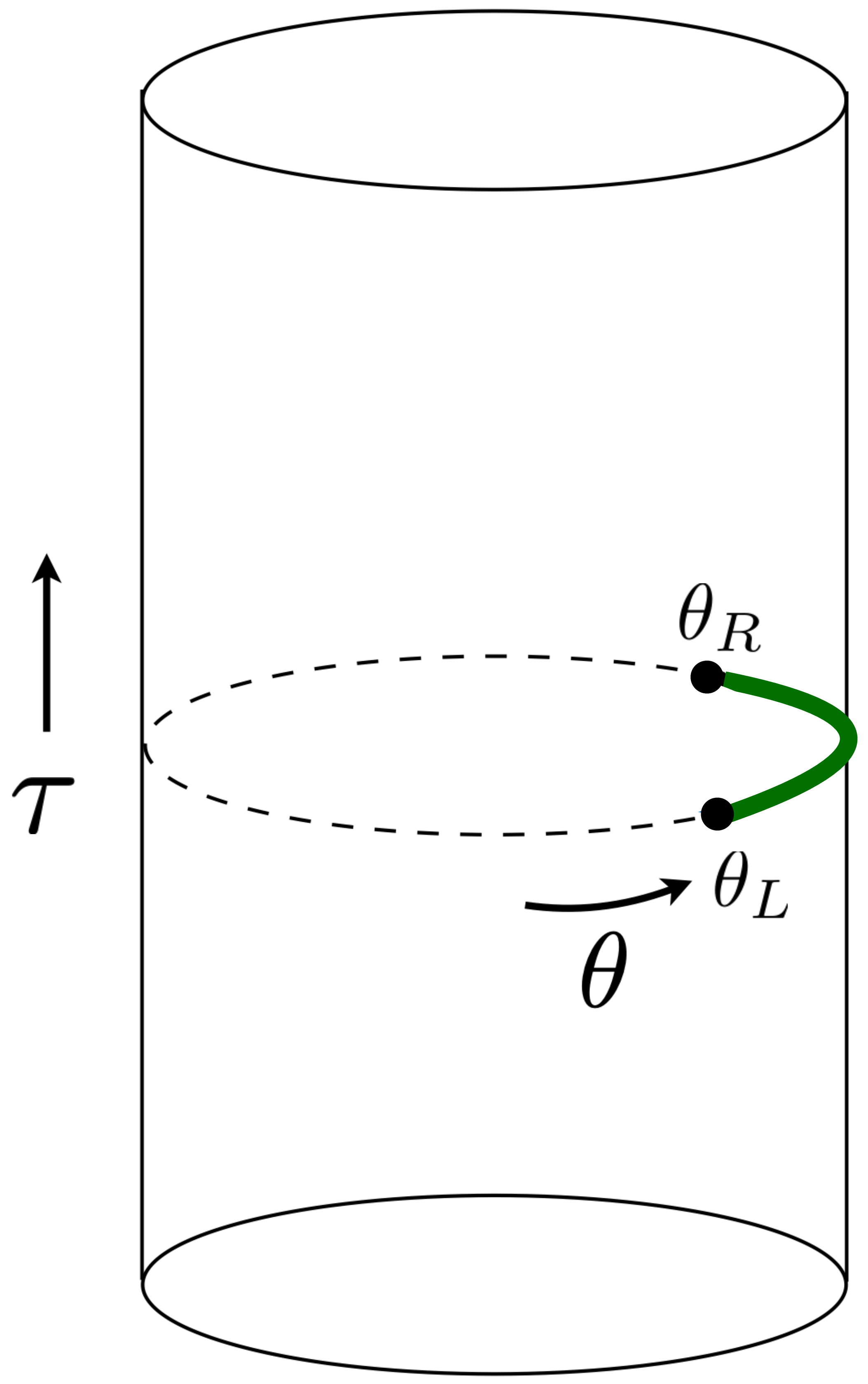
Extra Slides

Czech, Lamprou, McCandlish, Sully (2018)

Example: Kinematic space

Kinematic space = space of intervals (= bulk geodesics)

Consider parallel transport under a change of interval location



$$H_{\text{mod}} = s_1 L_1 + s_0 L_0 + s_{-1} L_{-1} \qquad s_{0,\pm 1}(\theta_L, \theta_R)$$

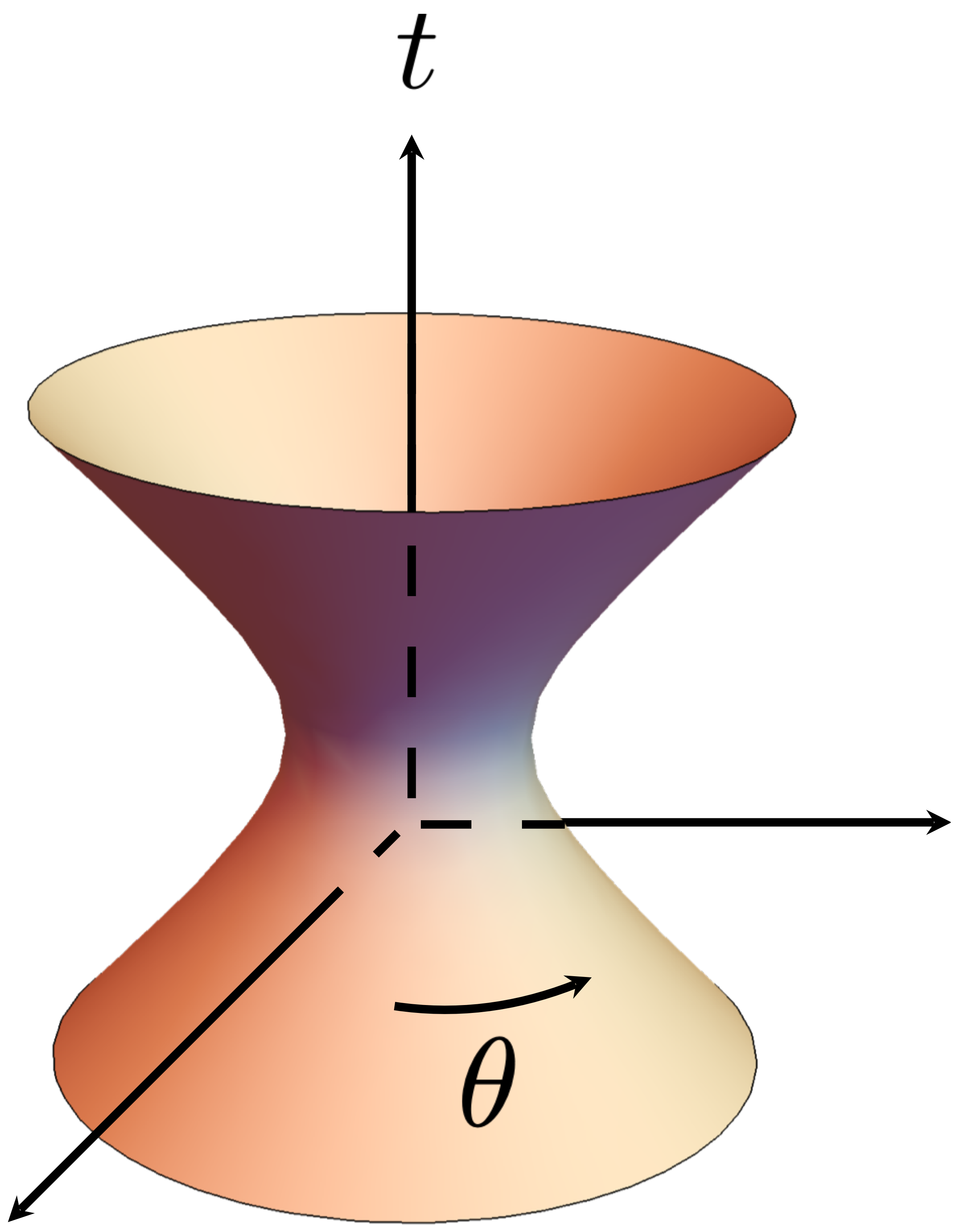
$$\delta_{\theta_L} H_{\text{mod}} = [S_{\delta\theta_L}, H_{\text{mod}}]$$

$$F = [S_{\delta\theta_L}, S_{\delta\theta_R}] = -\frac{i}{4\pi} \frac{H_{\text{mod}}}{\sin^2\left(\frac{\theta_R - \theta_L}{2}\right)}$$

$$\theta(\gamma) = \int_{B|\partial B = \gamma} \frac{1}{\sin^2 t} dt \wedge d\theta \qquad t = \frac{1}{2}(\theta_R - \theta_L)$$

Penna, CZ (2018)

Example: Kinematic space



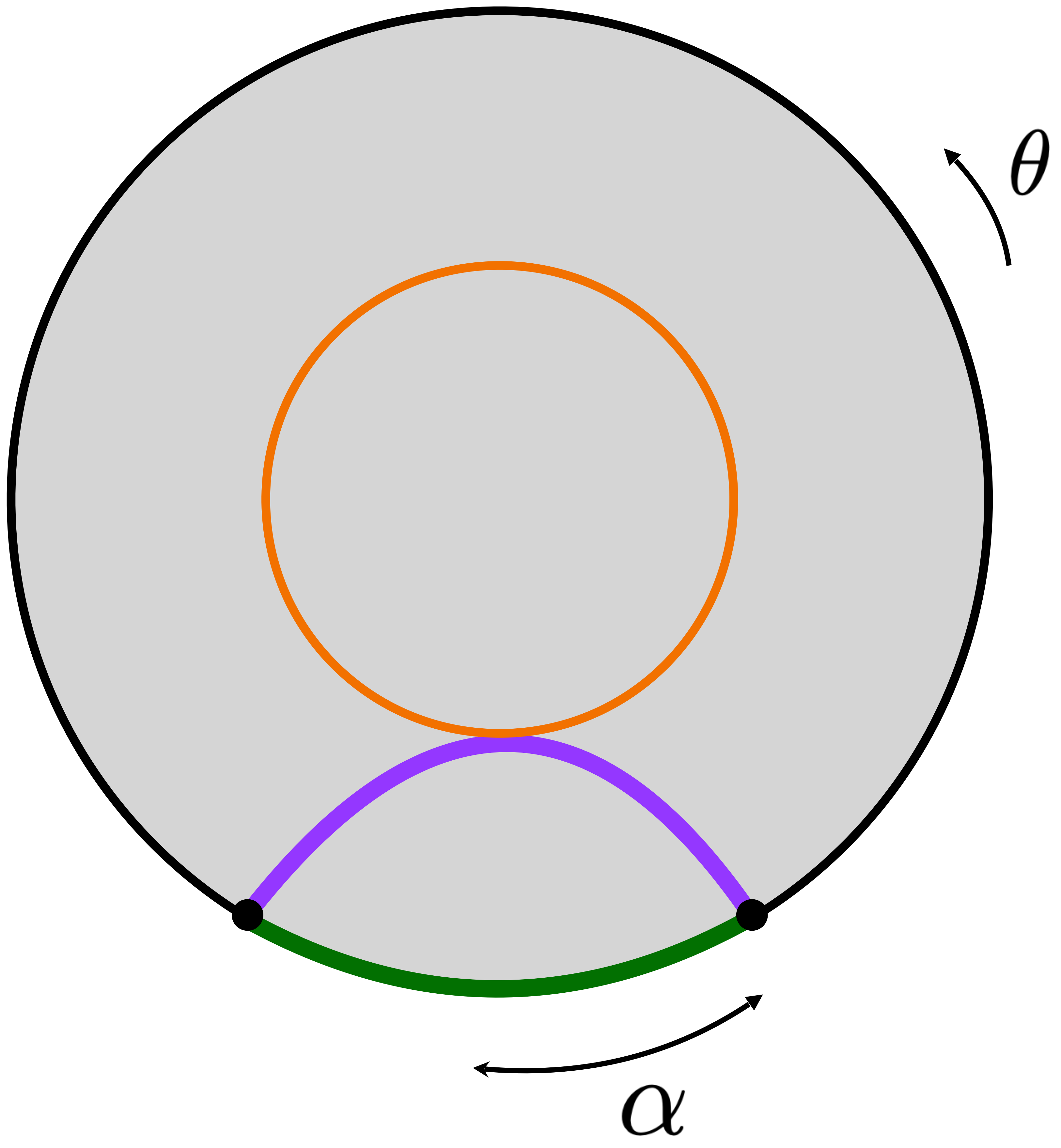
$$\omega = \frac{1}{\sin^2 t} dt \wedge d\theta \quad = \text{volume form on } \mathrm{dS}_2$$

$$\mathrm{dS}_2 = \text{space of spacelike geodesics in } \mathrm{AdS}_3$$

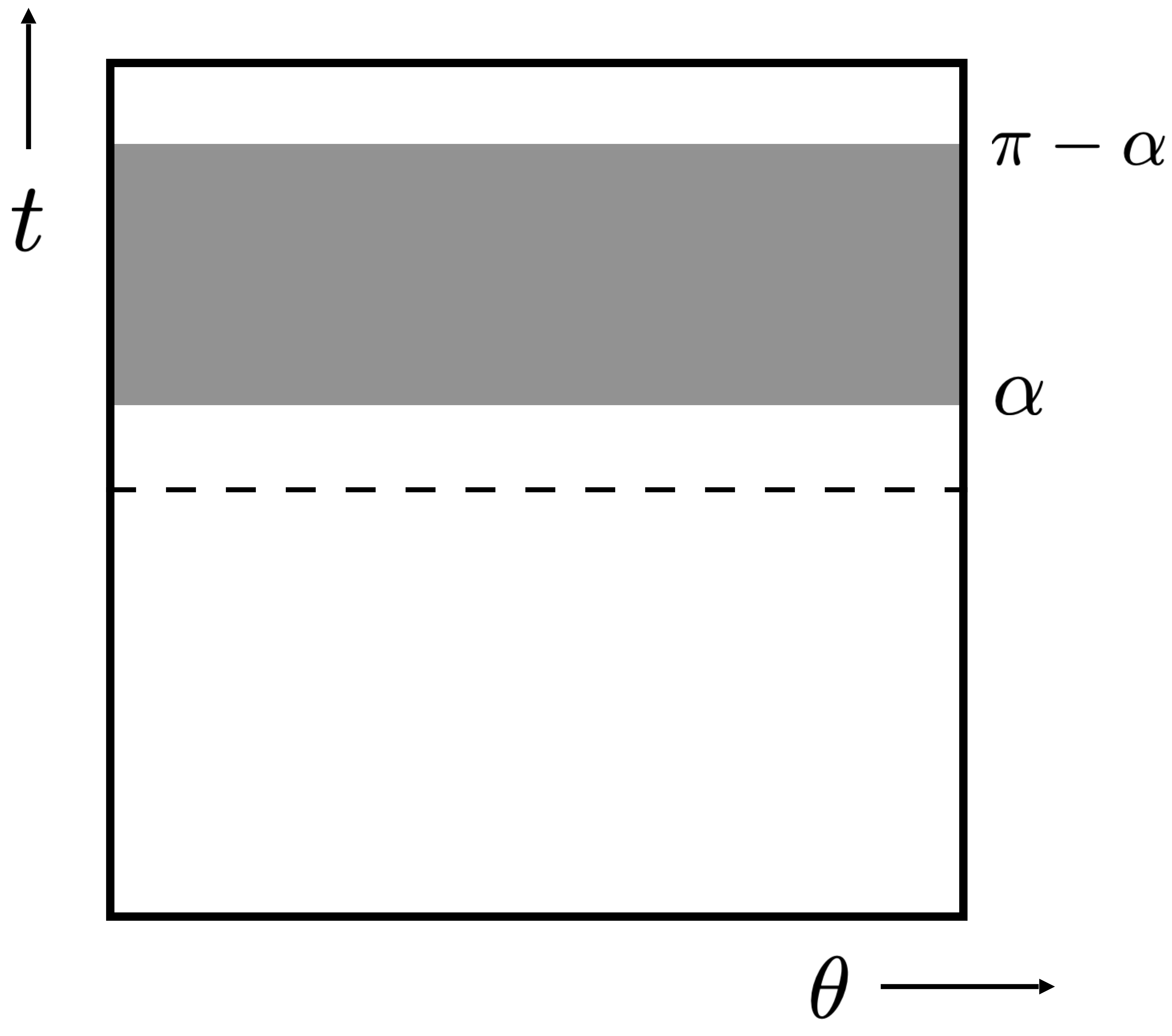
$$= \text{coadjoint orbit of } SO(2,1)$$

KK symplectic form = Berry curvature

Example: Kinematic space



AdS_3 time slice



dS_2 kinematic space