The Inheritance of Energy Conditions

Heliudson Bernardo

based on 2208.09341 in collaboration with S. Brahma and M.Faruk,

and also on 2107.06900, 2108.08365 with K. Dasgupta and R. Tatar

McGill
UNIVERSITY

BIRS workshop
“Quantum Information Theory in Quantum Field Theory and Cosmology”
June 05 2023

Outline

- Comments on no-go theorems in supergravity
- Energy conditions
- Conclusion I
- Revisiting no-go theorems
- Conclusion II

Supergravity no-go theorems rely on typical assumptions [1]:

- Second derivative theory (E.H. action);
- Scalars + p-forms (no ghosts);
- $V \leq 0$;
- Time-independent compact internal space;
- Maximally symmetric external manifold;
- Finite lower-dimensional Newton constant G_d.

$$\Rightarrow \quad R_d \leq 0$$

Extended after inclusion of Dp-branes and Op-planes [2]

[1] J.M. Maldacena, C. Nunez 2001; G.W. Gibbons 2003;
[2] K. Dasgupta, R. Gwyn, E. McDonough, M. Mia, R. Tatar 2014

For the metric $ds_D^2 = \Omega^2(y)\left(dx_d^2 + \hat{g}_{mn}dy^n dy^m\right)$, Einstein's equation yields

$$R_{\mu\nu}(\eta) - \eta_{\mu\nu}\left(\hat{\nabla}^2 \log\Omega + (D-2)(\hat{\nabla}\log\Omega)^2\right) = T_{\mu\nu} - \frac{1}{D-2}\Omega^2\eta_{\mu\nu}T^L_L$$

which implies

$$\frac{1}{(D-2)\Omega^{D-2}}\nabla^2\Omega^{D-2} = R(\eta) + \Omega^2(-T^\mu_\mu + \frac{d}{D-2}T^L_L)$$

After integrating over the internal manifold, we find a non-positive Ricci scalar for the d-dimensional metric η.

- How to generalize these results? Can we drop some of these assumptions?
- From the higher-dimensional point of view, is there a way to rule out a compactification ansatz without calculating the lower-dimensional theory?

The no-go theorem assumptions are sufficient for the validity of the strong energy conditions in d dimensions, [3]

$$SEC_D \Rightarrow SEC_d$$

For a four-dimensional FLRW compactification, $w \geq -\frac{1}{3}$ and no accelerated solutions.

[3] G.W. Gibbons, 1985; J. G. Russo and P.K. Townsend 2018.

Relaxing the assumptions allows us to avoid the no-go theorem.

Time-dependent internal manifolds:

- SEC_d can be violated, but so is NEC_d [4]

- NEC_4 as a condition for controlled perturbative expansion of FLRW cosmologies in type IIB [5]

[4] J. G. Russo and P.K. Townsend 2019;
[5] H.B., S. Brahma, K. Dasgupta, M.M. Faruk, R. Tatar 2021.

Energy conditions: tools for determining the global structure of a spacetime [6]

$$\overline{R}_{MN}u^M u^N \geq 0, \quad \overline{g}_{MN}u^M u^N < 0$$ Strong (SEC)

$$\overline{R}_{MN}l^M l^N \geq 0, \quad \overline{g}_{MN}l^M l^N = 0$$ Null (NEC)

$$\overline{G}_{MN}u^M u^N \geq 0, \quad \overline{g}_{MN}u^M u^N < 0$$ Weak (WEC)

$$\overline{G}_{MN}u^M u^N \geq 0, \quad \overline{g}_{MN}u^M u^N < 0, \quad \overline{g}_{MN}\overline{G}^M{}_P\overline{G}^N{}_Q u^P u^Q \leq 0$$ Dominant (DEC)

[6] S.W. Hawking and R. Penrose, 1970.

What are the requirements for a D-dimensional energy condition to imply its lower, d-dimensional, version?

D-dimensional metric ansatz

$$
\begin{aligned}
d\overline{s}^2 &= \overline{g}_{MN} dx^M dx^N \\
&= \Omega^2(y)\tilde{g}_{\alpha\beta}(x) dx^\alpha dx^\beta + h_{mn}(x,y) dy^m dy^n
\end{aligned}
$$

- $\tilde{g}_{\alpha\beta}$ is an arbitrary *d*-dimensional metric.
- $\tilde{h}_{mn}$ is the metric of a compact space.
- Constant internal volume: $\partial_\alpha \sqrt{h} = 0$

The components of the Ricci tensor are

$$\overline{R}_{\alpha\beta} = R_{\alpha\beta}(g) - \frac{1}{4}\nabla_\alpha h^{pq}\nabla_\beta h_{pq} - \frac{1}{2}h^{pq}\nabla_\alpha\nabla_\beta h_{pq} + \frac{1}{2}g^{\sigma\rho}\nabla_m g_{\beta\sigma}\nabla^m g_{\alpha\rho} - $$

$$- \frac{1}{4}g^{\sigma\rho}\nabla_m g_{\beta\alpha}\nabla^m g_{\sigma\rho} - \frac{1}{2}h^{pq}\nabla_p\nabla_q g_{\alpha\beta}\,,$$

$$\overline{R}_{pq} = R_{pq}(h) - \frac{1}{2}g^{\mu\rho}\nabla_\mu\nabla_\rho h_{pq} + \frac{1}{4}h_{ns}\nabla^\rho h_{pq}\nabla_\rho h^{sn} + \frac{1}{2}h^{nr}\nabla_\rho h_{qr}\nabla^\rho h_{pn} - $$

$$- \frac{1}{4}\nabla_p g^{\mu\rho}\nabla_q g_{\mu\rho} - \frac{1}{2}g^{\mu\rho}\nabla_p\nabla_q g_{\mu\rho}\,,$$

$$\overline{R}_{p\beta} = h_{m[s}\nabla_{p]}\nabla_\beta h^{sm} - g_{\rho[\beta}\nabla_{\mu]}\nabla_p g^{\mu\rho} + \frac{1}{2}h_{ps}\nabla^\rho h^{ns}\nabla_n g_{\rho\beta} - \frac{1}{4}h_{ps}\nabla_\beta h^{ns} g^{\mu\rho}\nabla_n g_{\rho\mu} - $$

$$- \frac{1}{4}h_{ms}\nabla^\sigma h^{sm}\nabla_p g_{\rho\sigma}\,.$$

Strong energy condition (SEC)

$$\overline{R}_{MN}u^M u^N \geq 0, \quad \overline{g}_{MN}u^M u^N < 0$$

For the lower-dimensional theory:

$$R_{\alpha\beta}(\tilde{g})u^\alpha u^\beta + \frac{1}{4}\tilde{\nabla}_\alpha h^{pq}\tilde{\nabla}_\beta h_{pq}u^\alpha u^\beta + u^2\frac{\Omega^{-(d-2)}}{d}\nabla^2\Omega^d \geq 0$$

$$R_{\alpha\beta}(\tilde{g})u^\alpha u^\beta \geq 0, \quad \tilde{g}_{\alpha\beta}u^\alpha u^\beta < 0$$

Null energy condition (NEC)

$$\overline{R}_{MN}l^M l^N \geq 0, \quad \bar{g}_{MN}l^M l^N = 0$$

For the lower-dimensional theory:

$$R_{\alpha\beta}(\tilde{g})l^\alpha l^\beta \geq +\frac{1}{4}h^{mp}(l^\alpha \tilde{\nabla}_\alpha h_{mn})h^{nq}(l^\beta \tilde{\nabla}_\beta h_{pq}) \geq 0$$

Weak energy condition (WEC)

$$\overline{G}_{MN}u^M u^N \geq 0, \quad \bar{g}_{MN}u^M u^N < 0$$

For the lower-dimensional theory:

$$\tilde{G}_{\alpha\beta}u^\alpha u^\beta \geq T^{(h)}_{\alpha\beta}u^\alpha u^\beta + T^{(\Omega)}_{\alpha\beta}u^\alpha u^\beta,$$

where

$$T^{(h)}_{\alpha\beta} := -\frac{1}{4}\tilde{\nabla}_\alpha h^{pq}\tilde{\nabla}_\beta h_{pq} + \frac{1}{8}\tilde{g}_{\alpha\beta}\tilde{\nabla}^\sigma h^{pq}\tilde{\nabla}_\sigma h_{pq}$$

$$T^{(\Omega)}_{\alpha\beta} := \left(\frac{1}{2}\Omega^2 R(h) - \frac{2(d-1)}{d}\Omega^{2-d/2}\nabla^2\Omega^{d/2}\right)\tilde{g}_{\alpha\beta} = \Lambda(y)\tilde{g}_{\alpha\beta}$$

Dominant energy condition (DEC)

$$\overline{G}_{MN}u^M u^N \geq 0, \quad \bar{g}_{MN}u^M u^N < 0, \quad \bar{g}_{MN}\overline{G}^M{}_P \overline{G}^N{}_Q u^P u^Q \leq 0$$

For this case, we need to show that

$$U_\rho := T^{(h)}_{\rho\beta} u^\beta, \quad W_\rho := T^{(\Omega)}_{\rho\beta} u^\beta$$

are causal vectors. But,

$$T^{(\Omega)}_{\rho\alpha} T^{(\Omega)\rho}{}_\beta u^\alpha u^\beta = \Lambda^2(y) \tilde{g}_{\rho\alpha} \delta^\rho_\beta u^\alpha u^\beta = \Lambda^2(y) u^2 \leq 0$$

Conclusion I

- A constant internal volume is a sufficient condition for NEC and SEC inheritance;
- The tensor $T_{\alpha\beta} = T^{(h)}_{\alpha\beta} + T^{(\Omega)}_{\alpha\beta}$ has to satisfy WEC for we to have WEC inheritance. The simplest possibility is for $T^{(h)}$ and $T^{(\Omega)}$ to satisfy WEC independently. In this case, we get WEC for the moduli fields h_{mn} and also a condition on the curvature of the internal manifold;
- Since $T^{(\Omega)}$ is necessarily causal, the only extra condition on top of WEC for the lower-dimensional DEC to descend from the higher-dimensional one is that $T^{(h)}$ should satisfy DEC.

Are the higher-dimensional energy conditions satisfied?

$$S = \frac{1}{2\kappa_D^2}\int d^Dx\sqrt{-g}\left[R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\kappa_D^2}{2g_D^2}\sum_p e^{\beta_p\phi}|F_p|^2\right] + \sum_p \int A_p \wedge \pi_{D-p} + S_{\text{local sources}}$$

For scalar fields, we have [7]

- $V \geq 0$: the DEC is satisfied but SEC might be violated.
- $V \leq 0$: the SEC is respected but DEC might be violated.

For $p = 1$, all the energy conditions discussed are satisfied [7].

Remember: DEC implies WEC, and SEC implies NEC.

[7] H. Maeda and C. Martinez, 2020.

- For p-forms with $p > 1$,

$$T_{MN}^{(p)} = e^{\lambda\phi}\left(\frac{1}{p!}F^{M_1\cdots M_p}{}_M F_{M_1\cdots M_p N} - \frac{1}{2(p+1)!}g_{MN}F^{M_1\cdots M_{p+1}}F_{M_1\cdots M_{p+1}}\right)$$

and it can be shown that

$$\left(T_{MN}^{(p)} - \frac{1}{D-2}T^{(p)}g_{MN}\right)u^M u^N = \frac{\left(u^{\bar{0}}\right)^2}{2(p+1)!}\frac{1}{D-2}\Big[2(p+1)(D-p-2)F^{I_1\cdots I_p}{}_{\bar{0}}F_{I_1\cdots I_p\bar{0}} + \\ + 2pF^{I_1\cdots I_{p+1}}F_{I_1\cdots I_{p+1}}\Big].$$

So, the SEC is satisfied for p $< D - 2$. The same is true for DEC. [8]

[7] H. Bernardo, S. Brahma, M. Faruk, 2022.

- For p-branes, $S = -T_p \int d^{p+1}\xi e^{\alpha\phi} \sqrt{-\det(h + \mathcal{F})}, \quad \mathcal{F}_{ab} := F_{ab} + B_{ab}$

and

$$\begin{aligned} T^{MN} &= -\frac{T_p}{\sqrt{-g}} \int d^{p+1}\xi\, \delta^D(x - X(\xi))\, e^{\alpha\phi}\, \sqrt{-\mathcal{G}} \left(\mathcal{G}^{-1}\right)^{ab} \partial_a X^M \partial_b X^N \\ &= \frac{1}{\sqrt{-g}} \int d^{p+1}\xi\, \delta^D(x - X(\xi))\, \sqrt{-h}\, \mathcal{T}^{ab}\, \partial_a X^M \partial_b X^N . \end{aligned}$$

where $\mathcal{T}^{ab} = -T_p\, e^{\alpha\phi} \frac{\sqrt{-\mathcal{G}}}{\sqrt{-h}} \left(\mathcal{G}^{-1}\right)^{ab}, \quad \mathcal{G}_{ab} := h_{ab} + \mathcal{F}_{ab}$

So, the validity of the spacetime energy conditions follows from the validity of the worldvolume EC's.

- Op-planes can have negative tension. So, the would be worldvolume theory might have tachyons.

- However, they are not dynamical objects, but rather manifestation of the background orientifold structure.

- As a consistency condition, we should include in the action

$$S = -\sum_i T^i_{\mathrm{O}p} \int d^D x \int d^{p+1}\xi \, \delta^D(x - X_i(\xi))\, e^{\beta\phi} \sqrt{-h_i} + \sum_i \mu^i_{\mathrm{O}p} \int A_{p+1} \wedge *J^i_{p+1}$$

The p-brane and Op-plane configuration must satisfy the charge (tadpole) cancellation condition,

$$\sum_a \mu_p^a \int_{C_{D-p-1}} *J_{p+1}^a = 0$$

Let us write the tadpole cancellation condition explicitly for a system having $N_{\mathrm{D}p}$ Dp-branes, $N_{\mathrm{O}p_+}$ Op_+-planes, $N_{\mathrm{O}p_-}$ Op_--planes, $N_{\overline{\mathrm{D}p}}$ $\overline{\mathrm{D}p}$-(anti)branes, $N_{\overline{\mathrm{O}p}_+}$ $\overline{\mathrm{O}p}_+$-(anti)planes and $N_{\overline{\mathrm{O}p}_-}$ $\overline{\mathrm{O}p}_-$-(anti)planes. Then we have

$$\sum_a \mu_p^a = \mu_{\mathrm{D}p}\left[N_{\mathrm{D}p} - N_{\overline{\mathrm{D}p}} - 2^{p-5}\left(N_{\mathrm{O}p_+} + N_{\overline{\mathrm{O}p}_-} - N_{\overline{\mathrm{O}p}_+} - N_{\mathrm{O}p_-}\right)\right] = 0$$

where we used $\mu_{\mathrm{O}p_\pm} = \mp 2^{p-5} \mu_{\mathrm{D}p}\,.$

But since we are dealing with extremal objects, we can use the condition above to write

$$\sum_a T_p^a = T_{\mathrm{D}p}\left[N_{\mathrm{D}p} + N_{\overline{\mathrm{D}p}} - 2^{p-5}\left(N_{\mathrm{O}p_+} - N_{\overline{\mathrm{O}p}_-} + N_{\overline{\mathrm{O}p}_+} - N_{\mathrm{O}p_-}\right)\right]$$
$$= 2T_{\mathrm{D}p}\left[N_{\mathrm{D}p} - 2^{p-5}(N_{\mathrm{O}p_+} - N_{\mathrm{O}p_-})\right]$$

This is such that $\sum_a T_p^a \geq 0 \iff N_{\mathrm{D}p} \geq 2^{p-5}(N_{\mathrm{O}p_+} - N_{\mathrm{O}p_-})$

- For supersymmetric parallel configurations, the branes and orientifold planes have the same orientation and $N_{\mathrm{Dp}} = 2^{p-5} N_{\mathrm{O}p_+}$ such that the sum of tensions vanishes and all energy conditions are satisfied.

Conclusion II

- Generically, for any background metric, the field content of supergravity satisfy the SEC, NEC, WEC and DEC in for any spacetime dimension $D > 2$;
- Dp-branes also satisfy the energy conditions;
- Individually, Op-planes can violate the EC's. But the equations of motion imply physical configurations that do not do so;
- For parallel (susy) configurations, all the EC's discussed are satisfied.

Thank you for your attention!

Recently, an attempt on finding dS backgrounds in type IIB supported by all sorts of corrections was carried out by studying the metric uplift to M-theory.[35]

The type IIB metric ansatz is

$$ds^2 = \frac{1}{\Lambda \mathrm{H}^2(y)t^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \mathrm{H}^2(y)\left(F_1(t)g_{\alpha\beta}(y)dy^\alpha dy^\beta + F_2(t)g_{mn}dy^m dy^n\right)$$

where $(m, n) \in M_4$ and $(\alpha, \beta) \in M_2$. We also impose $F_1 F_2^2 = 1$

35 K. Dasgupta, M. Emelin, M.M. Faruk, R. Tatar 2019; S. Brahma, K. Dasgupta, R. Tatar 2020/2021; H.B., S. Brahma, K. Dasgupta, M.M. Faruk, R. Tatar 2021.

This background can be uplifted to M-theory with metric

$$ds^2 = g_s^{-8/3}\eta_{\mu\nu}dx^\mu dx^\nu + g_s^{-2/3}\mathrm{H}^2(y)\left(F_1(t)g_{\alpha\beta}(y)dy^\alpha dy^\beta + F_2(t)g_{mn}dy^m dy^n\right) + g_s^{4/3}|dz|^2$$

where $g_s^2 \propto \Lambda t^2 \mathrm{H}^2(y)$ and $z = x_3 + i\, x_{11}$ is the coordinate of the torus.

It turns out that we need time-dependent fluxes to support this M-theory background

However, there are no-go theorems preventing such a background to be solution to the supergravity plus fluxes and sources[36]

We need to consider curvature corrections contributions to Einstein's equations:

$$\mathbf{R}_{MN} - \frac{1}{2} g_{MN} \mathbf{R} = \mathbb{T}_{MN}^{\text{classical}} + \mathbb{T}_{MN}^{\text{corrections}}$$

What should we include in the energy-momentum tensor of the corrections? The main idea is to study all possible imageable terms!

36 J.M. Maldacena, C. Nunez 2001; G.W. Gibbons 2003; K. Dasgupta, R. Gwyn, E. McDonough, M. Mia, R. Tatar 2014

There could be an infinite number of curvature corrections. Schematically, we write a term like

$$\begin{aligned}\mathbb{Q}_{\rm T}^{(\{l_i\},n_i)} = & \ \mathbf{g}^{m_i m_i'}....\mathbf{g}^{j_k j_k'}\{\partial_m^{n_1}\}\{\partial_\alpha^{n_2}\}\{\partial_a^{n_3}\}\{\partial_0^{n_0}\}\left(\mathbf{R}_{mnpq}\right)^{l_1}\left(\mathbf{R}_{abab}\right)^{l_2}\left(\mathbf{R}_{pqab}\right)^{l_3}\left(\mathbf{R}_{\alpha ab\beta}\right)^{l_4} \\ & \times \left(\mathbf{R}_{\alpha\beta mn}\right)^{l_5}\left(\mathbf{R}_{\alpha\beta\alpha\beta}\right)^{l_6}\left(\mathbf{R}_{ijij}\right)^{l_7}\left(\mathbf{R}_{ijmn}\right)^{l_8}\left(\mathbf{R}_{iajb}\right)^{l_9}\left(\mathbf{R}_{i\alpha j\beta}\right)^{l_{10}}\left(\mathbf{R}_{0mnp}\right)^{l_{11}} \\ & \times \left(\mathbf{R}_{0m0n}\right)^{l_{12}}\left(\mathbf{R}_{0i0j}\right)^{l_{13}}\left(\mathbf{R}_{0a0b}\right)^{l_{14}}\left(\mathbf{R}_{0\alpha0\beta}\right)^{l_{15}}\left(\mathbf{R}_{0\alpha\beta m}\right)^{l_{16}}\left(\mathbf{R}_{0abm}\right)^{l_{17}}\left(\mathbf{R}_{0ijm}\right)^{l_{18}} \\ & \times \left(\mathbf{R}_{mnp\alpha}\right)^{l_{19}}\left(\mathbf{R}_{m\alpha ab}\right)^{l_{20}}\left(\mathbf{R}_{m\alpha\alpha\beta}\right)^{l_{21}}\left(\mathbf{R}_{m\alpha ij}\right)^{l_{22}}\left(\mathbf{R}_{0mn\alpha}\right)^{l_{23}}\left(\mathbf{R}_{0m0\alpha}\right)^{l_{24}}\left(\mathbf{R}_{0\alpha\beta\alpha}\right)^{l_{25}} \\ & \times \left(\mathbf{R}_{0ab\alpha}\right)^{l_{26}}\left(\mathbf{R}_{0ij\alpha}\right)^{l_{27}}\left(\mathbf{G}_{mnpq}\right)^{l_{28}}\left(\mathbf{G}_{mnp\alpha}\right)^{l_{29}}\left(\mathbf{G}_{mnpa}\right)^{l_{30}}\left(\mathbf{G}_{mn\alpha\beta}\right)^{l_{31}}\left(\mathbf{G}_{mn\alpha a}\right)^{l_{32}} \\ & \times \left(\mathbf{G}_{m\alpha\beta a}\right)^{l_{33}}\left(\mathbf{G}_{0ijm}\right)^{l_{34}}\left(\mathbf{G}_{0ij\alpha}\right)^{l_{35}}\left(\mathbf{G}_{mnab}\right)^{l_{36}}\left(\mathbf{G}_{ab\alpha\beta}\right)^{l_{37}}\left(\mathbf{G}_{m\alpha ab}\right)^{l_{38}}\end{aligned}$$

and then sum over (l_i, n_i).

$$\mathbf{S}_1 = \mathbf{M}_p^9 \int d^{11}x\sqrt{-\mathbf{g}_{11}}\Big(\mathbf{R}_{11} + \mathbf{G}_4 \wedge *\mathbf{G}_4 + \mathbf{C}_3 \wedge \mathbf{G}_4 \wedge \mathbf{G}_4 + \mathbf{M}_p^2\, \mathbf{C}_3 \wedge \mathbb{Y}_8\Big)$$

Since $\frac{g_s}{H(y)} \propto t$, we can rewrite time-dependence of all fields as g_s dependence.

The ansatz for the fluxes is then expressed as

$$\mathbf{G}_{MNPQ}(g_s, y) = \sum_k \mathcal{G}^{(k)}_{MNPQ}(y) \left(\frac{g_s}{\mathrm{H}}\right)^{2k/3}$$

We wish to solve Einstein's equation order by order in g_s. So, although we don't know the coefficients of the corrections, we can check whether our ansatz allow for a match of g_s scalings.

The energy-momentum tensor of the perturbative corrections scales as $g_s^{\theta_{kl}}$, where

$$\theta_{kl} \equiv \frac{2}{3}\sum_{i=1}^{27} l_i + \frac{1}{3}\left(\sum_{i=0}^{2} n_i - 2n_3 + l_{34} + l_{35}\right) + \frac{2}{3}(k+2)(l_{28}+l_{29}+l_{31}) + \frac{1}{3}(2k+1)(l_{30}+l_{32}+l_{33}) + \frac{2}{3}(k-1)(l_{36}+l_{37}+l_{38})$$

Note that there are relative minus signs in the n_3 term (which is the number of derivatives w.r.t to the 11th direction) and in front of l_{36}, l_{37}, l_{38}(which are powers of fluxes with the structure G_{MNab}).

If we take time-independent fluxes, $k = 0$, then there are negative definite terms in the g_s scaling:

$$\theta_{kl} = \text{positive} - \frac{2}{3}(n_3 + l_{36} + l_{37} + l_{38})$$

This means that, to a given order in g_s, there are an infinite number of higher-order terms that contributes to that order.

Hence, there is no g_s hierarchy and so no-perturbative solutions! The only possible solutions are non-perturbative ones, and they only exist if the infinite number of corrections can be resumed.

However, turning on time-dependent fluxes makes the hierarchy possible again, and there is not obstructions against a perturbative solution.