Spread Complexity and Topological Transitions in the Kitaev Chain

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based on [2208.05520] with P Caputa, N Gupta, J Murugan, S Shajidul Haque, S Liu

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Talk Layout

Complexity

- Central question: How hard is it to synthesize a desired target state with the gates at your disposal?
- Need, $|\phi_r\rangle$, $|\phi_t\rangle$, $\{U_1, U_2, \cdots, U_n\}$, $g(U_1, U_2, \cdots, U_n)$

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- E.g. $U_1 U_2 U_1 U_3 (U_1)^3 U_2 |\phi_r\rangle = U_3 U_1 U_2 U_1 U_3 (U_1)^3 U_2 U_3 |\phi_r\rangle$, "complexity $= 8$"

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- Discrete notion of complexity closely related to quantum computational setups
- We will, however, be interested in a continuous notion of complexity

Nielsen Complexity

- Accessible gates are taken to be from some symmetry group

  [Nielsen, quant-ph/0502070]

- E.g. $SU(2)$: Gates $U = e^{i(s_1 J_1 + s_2 J_2 + s_3 J_3)}$
- Target states: $|\phi_t(s_1, s_2, ..., s_n)\rangle = U(s_1, \cdots, s_n)|\phi_r\rangle$

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- Target states: $|\phi_t(s_1, s_2, ..., s_n)\rangle = U(s_1, \cdots, s_n)|\phi_r\rangle$
- We have a manifold of target states on which one can define a metric
- Complexity = shortest distance connecting points
- Can introduce a circuit parameter $s_i = s_i(\sigma)$

Nielsen Complexity

- Two examples of metrics
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- Group symmetries are encoded as metric isometries
- $\mathcal{F}_1 : \quad F_i = \partial_i \left(\langle\phi_t(s'_1, s'_2, \cdots, s'_n)|\phi_t(s_1, s_2, ..., s_n)\rangle\right)|_{s'=s}$
- FS metric: $g_{ij} = \partial_i \partial'_j \log\left(\langle\phi_t(s'_1, s'_2, \cdots, s'_n)|\phi_t(s_1, s_2, ..., s_n)\rangle\right)\Big|_{s'=s}$

Nielsen Complexity

- The overlap $\langle\phi_r|U^\dagger(s')U(s)|\phi_r\rangle$ is thus a key quantity
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- The overlap $\langle\phi_r|U^\dagger(s')U(s)|\phi_r\rangle$ is thus a key quantity
- The states $U(s)|\phi_r\rangle$ are generalized coherent states [Perelomov, 1972]
- Stability subgroup $H \subset G$ such that $U_h|\phi_r\rangle = e^{i\phi_h}|\phi_r\rangle$
- Manifold of states $\Leftrightarrow$ group elements of G/H

Spread Complexity

- A notion of complexity without the need to specify gates
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- Given a Hamiltonian and reference state one first builds the basis $|O_n) = H^n|\phi_r\rangle$
- From a Gram-Schmidt process one then obtains the Krylov basis $|K_n\rangle$
- The K-complexity of a state (or spread complexity) is then given by $C_K = \sum_n n\langle\phi_t|K_n\rangle\langle K_n|\phi_t\rangle \equiv \langle\phi_t|\hat{K}|\phi_t\rangle$
- The Krylov basis provides an **ordered** basis for the Hilbert space of target states

Spread Complexity

- Given some basis for the Hilbert space of target space in increasing complexity $|B_n\rangle$
- We can define complexity as $C = \sum_n c_n \langle \phi_t | B_n \rangle \langle B_n | \phi_t \rangle$
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- With c_n strictly increasing
- The choice $|B_n\rangle = |K_n\rangle$ minimises the complexity of the time-evolved reference state

[Balasubramanian, Caputa, Magan, Wu, arXiv:2202.06957]

Some Comments

- Complexity is an ambiguous quantity - can likely be a proxy for many physical quantities
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- Spread complexity is dependent on the choice of reference state - this may be unsatisfactory
- Could average over different choices
- Are there features that can be expected to be robust?
- Topological phase transitions appear to be such a feature

[Caputa, Liu, arXiv:2205.05688], [Caputa, Gupta, Murugan, Haque, Liu, HJRvZ, arXiv:2208.06311]

Low rank algebras

- Fully analytic results can be obtained for $su(1,1)$, $su(2)$, Heisenberg-Weyl [Caputa, Magan, Patramanis, arXiv:2109.03824]
- $L_+ = L_-^\dagger \quad ; \quad [[L_-, L_+], L_\pm] = \pm 2 f L_\pm$
- Highest weight state $L_-|w\rangle = 0$, $[L_-, L_+]|w\rangle = w_0|w\rangle$
- An arbitrary group element action may be written as $e^{i(a_+ L_+ + a_+^* L_- + a_0 [L_-, L_+])}|w\rangle = N e^{z L_+}|w\rangle$
- The manifold of target states is a two-dimensional manifold $\Leftrightarrow$ elements of $G/([L_-, L_+])$

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- The manifold of target states is a two-dimensional manifold $\Leftrightarrow$ elements of $G/([L_-, L_+])$
- Krylov basis $|K_n\rangle = \frac{(L_+)^n|w\rangle}{\sqrt{\langle w|(L_-)^n (L_+)^n|w\rangle}}$
- Spread complexity $C = z\partial_z \log\langle w|e^{\bar{z}L_-} e^{zL_+}|w\rangle$

Low rank algebras

- Can do a little better than this
- If the Krylov basis is known for H, $|\phi_r\rangle$ then the Krylov basis for $UHU^\dagger$, $U|\phi_r\rangle$ is given by $|K_n\rangle \to U|K_n\rangle$
- This is particularly useful for the low-rank algebras, since the Krylov basis is rather insensitive to the choice of H
- Spread complexity $C = z'\partial_{z'} \log\langle w|e^{\bar{z}' L_-} e^{z' L_+}|w\rangle$

Tensor Products

- Suppose we have a Hamiltonian $H = \sum_i H_i$ with $[H_i, H_j] = 0$
- Krylov basis, by definition, is the ordered orthonormal basis obtained from $|O_n) = H^n|\phi_{r,1}, \phi_{r,2} \cdots\rangle$

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- For **many** spin $\frac{1}{2}$ $SU(2)$ tensor products they are equal

Kitaev Chain

- A model of Dirac fermions on an L-site lattice [Kitaev, 2001]
- $H = \sum_{j=1}^{L}\left[-\frac{J}{2}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu(c_j^\dagger c_j - \frac{1}{2}) + \frac{1}{2}(\Delta c_j^\dagger c_{j+1}^\dagger + \Delta^* c_{j+1} c_j)\right]$
- Hopping amplitude J, chemical potential μ and superconducting pairing strength Δ
- c_j's can be redefined to always produce a real Δ
- Topological phase transition occurs at $|J| = |\mu|$, gapless for $|\mu| < |J|$

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Kitaev Chain

- $c_j = \frac{1}{\sqrt{L}} \sum_n e^{ik_n j} a_{k_n}$

Kitaev Chain

- $c_j = \frac{1}{\sqrt{L}} \sum_n e^{ik_n j} a_{k_n}$
- $H = -\sum_{k_n>0} \left[2(\mu + J\cos(k_n)) J_0^{(k_n)} - i\Delta \sin(k_n) \left(J_+^{(k_n)} - J_-^{(k_n)} \right) \right]$
- $J_0^{(k_n)} = \frac{1}{2}(a_{k_n}^\dagger a_{k_n} - a_{-k_n} a_{-k_n}^\dagger) \quad J_+^{(k_n)} = a_{k_n}^\dagger a_{-k_n}^\dagger \quad J_-^{(k_n)} = a_{-k_n} a_{k_n}$
- Spin-$\frac{1}{2}$ representation of $su(2)$
- $\left[J_0^{(k_n)}, J_\pm^{(k_n)} \right] = \pm J_\pm^{(k_n)} \qquad \left[J_+^{(k_n)}, J_-^{(k_n)} \right] = 2J_0^{(k_n)}$

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Kitaev Chain

- Eigenstates can be written as $SU(2)$ coherent states - Krylov complexity for simple groups such as $SU(2)$ is well understood
- To determine the Krylov basis we need to specify a reference state (i.e. the zero complexity state)
- Natural choices include the lowest energy state when $\Delta \to 0$ or $J, \mu \to 0$ as well as the fermion vacuum
- In principle the Krylov basis needs to be recomputed for all these choices. Here they are related by a unitary transformation

Reference States

- Our circuits will connect these different choices of reference state ($s = 0$) to the Kitaev chain ground state ($s = 1$)
- Reference state 1:
$|\Omega_k(s=0)\rangle = e^{-i\frac{\pi}{2}\theta(\mu+J\cos(k))\left(J_+^{(k)}+J_-^{(k)}\right)}|\tfrac{1}{2},-\tfrac{1}{2}\rangle_k$
- Reference state 2:
$|\Omega_k(s=0)\rangle = e^{-i\frac{\pi}{4}\left(J_+^{(k)}+J_-^{(k)}\right)}|\tfrac{1}{2},-\tfrac{1}{2}\rangle_k$
- Reference state 3:
$|\Omega_k(s=0)\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle_k$

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Target State

- Ground state $|\Omega_k(s=1)\rangle = \prod_k \sin|\phi_k| e^{-i\cot\phi_k J_+^{(k)}} |\frac{1}{2}, -\frac{1}{2}\rangle_k$
- $\phi_k = \frac{1}{2}\tan^{-1}\frac{\Delta\sin k}{\mu + J\cos k}$

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- Can readily cast the above in the form
  $|\Omega_k(s=1)\rangle = U(s)|\frac{1}{2}, -\frac{1}{2}\rangle_k = e^{z(s)J_+^{(k)}}|\frac{1}{2}, -\frac{1}{2}\rangle_k$
- $C_k(s) = z\partial_z \log \ {}_k\langle\frac{1}{2}, -\frac{1}{2}| e^{\bar{z}(s)J_-^{(k)}} e^{z(s)J_+^{(k)}} |\frac{1}{2}, -\frac{1}{2}\rangle_k$
- $C(J,\mu,\Delta) = \frac{1}{L}\sum_{n>0} C_{k_n} \to \frac{1}{\pi}\int_0^\pi dk C_k$

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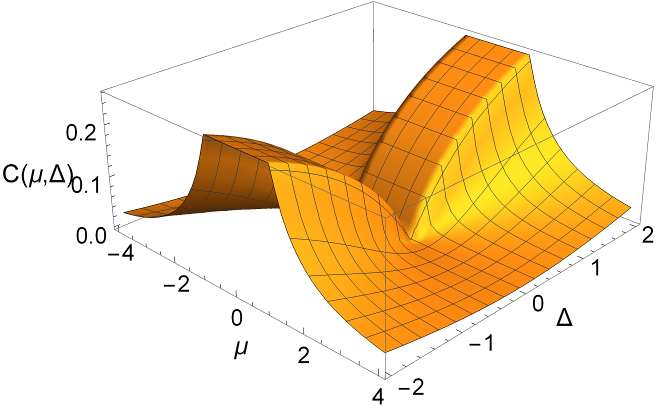
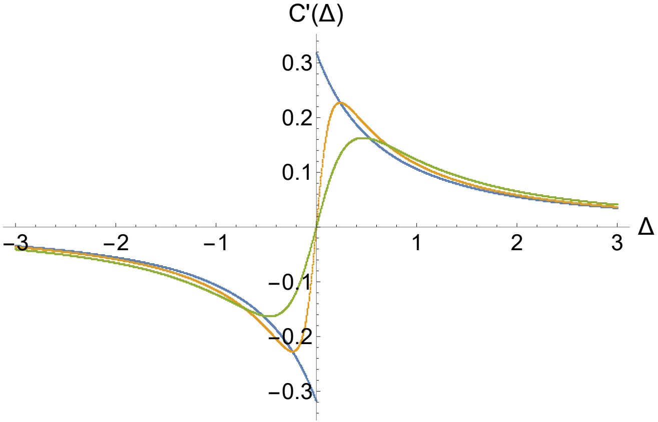
- $C(J, \mu, \Delta) = \frac{1}{L}\sum_{n>0} C_{k_n} \to \frac{1}{\pi}\int_0^\pi dk C_k$
- Will set $J = 1$

Circuit 1



- Complexity takes a Δ-dependent constant value in the topological phase

Circuit 1



- $\mu = 1.1, 1.02, 0.98$. A discontinuity develops when $|\mu| < 1$

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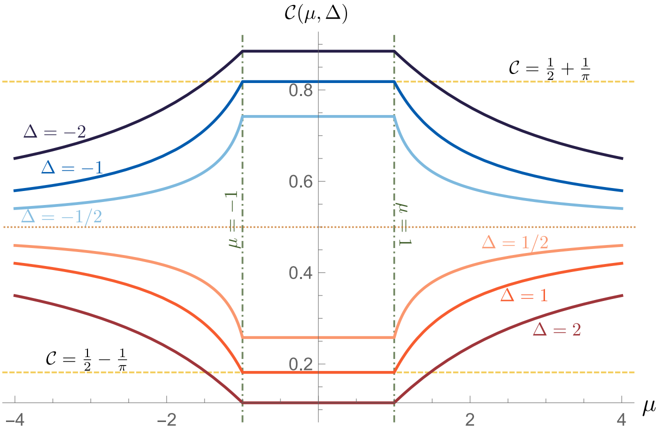
Circuit 2



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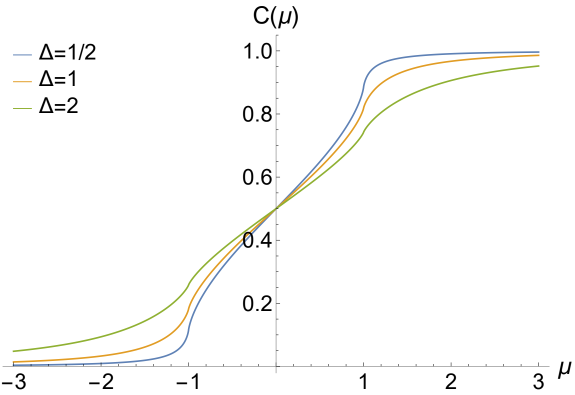
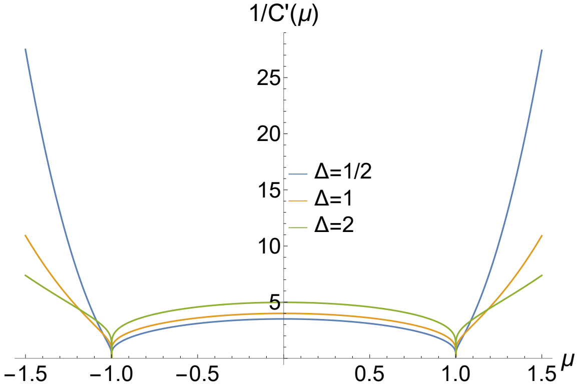
Circuit 3



- Complexity asymptotes between 0 and 1, the expected values

Circuit 3



- Derivative diverges as the topological phase transition is crossed

Outlook

- Spread Complexity is sensitive to the topological phase transition in the Kitaev chain see also [Caputa, Liu, arXiv:2205.05688]
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- This appears to be a rather robust feature
- Which choices of reference state exhibit the plateau feature? Presumably related to symmetries...
- What are the effects of twisted boundary conditions? Gauging the model?
- In general, what features of quantum many-body systems can be probed with spread complexity

Thank you for your attention!

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