Temperature dependence of Lanczos coefficients and integrability

Nick Angelinos

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Outline

Introduction

Krylov space and iteration algorithm

Lanczos coefficients as dynamical variables

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The exponent of Krylov complexity bounds the exponent of OTOC

$$\lambda_{OTOC} \leq \lambda_K.$$

Parker, Cao, Avdoshkin, Scaffidi, Altman 2019

Krylov space

Lanczos coefficients b_n, obtained from the iteration algorithm, encode chaotic behavior. Linear growth of $b_n \sim n$ is an indicator of chaos.

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We will show that $b_n(\beta)$ satisfy a completely integrable (non-linear) system of equations.

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Consider a Hamiltonian H and an operator A

$$A(t) = e^{-iHt} A e^{iHt} = \sum_{n=0}^{\infty} \frac{(-it)^n \mathcal{L}^n}{n!} A,$$

where we defined the "super-operator" $\mathcal{L}$ (Liouvillian)

$$\mathcal{L} A \equiv [H, A].$$

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Under time-evolution, $A(t)$ remains inside the Krylov space

$$\mathbb{K} = \text{span}\{A, \mathcal{L}A, \mathcal{L}^2 A, \mathcal{L}^3 A, \dots\}.$$

Inner product

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We require that ρ_1, ρ_2 commute with H and that the inner product is semi-positive definite and non-degenerate.

Lanczos algorithm

We obtain an orthonormal basis for $\mathbb{K}$ using the Lanczos algorithm.

Lanczos algorithm

Let $O_0 = A$.
For $n = 0, 1, 2, \ldots$:

$$a_n = \frac{\langle O_n | \mathcal{L} | O_n \rangle}{\langle O_n | O_n \rangle}, \quad b_{n-1}^2 = \frac{\langle O_n | O_n \rangle}{\langle O_{n-1} | O_{n-1} \rangle},$$

$$O_{n+1} = \mathcal{L} O_n - a_n O_n - b_{n-1}^2 O_{n-1},$$

$$A_n = O_n / \sqrt{\langle O_n | O_n \rangle}.$$

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The set of operators $\{A_0, A_1, A_2, \ldots\}$ is called the Krylov basis.
The sequences a_n, b_n are called Lanczos coefficients.

Representation of Liouvillian

The representation of $\mathcal{L}$ in Krylov space written in Krylov basis is, by construction, tridiagonal

$$\mathcal{L}A_n = \sum L_{nm}A_m,$$

$$L = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & \cdots \\ 0 & 0 & b_2 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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Krylov basis, Lanczos coefficients and representation L of Liouvillian depend on choice of inner product.

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The Lanczos coefficients acquire time-dependence: $a_n(\tau), b_n(\tau)$.
Their evolution is governed by a system of completely integrable non-linear equations.
Dymarsky, Gorsky 2019

Toda chain

Toda chain equations in Lax form

$$\frac{d}{d\tau}L = [B, L], \quad B = L_+ - L_-.$$

Completely integrable, with the following independent integrals of motion

$$H_k = tr(L^k).$$

Explicitly, the equations read

$$\frac{d}{d\tau}b_n = b_n(a_{n+1} - a_n),$$

$$\frac{d}{d\tau}a_n = 2(b_n^2 - b_{n-1}^2).$$

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Simplification: Assume that $A \in im(\mathcal{L})$, and let $\dim(\mathbb{K}) = 2N$.

Representation of $\{H, \cdot\}$

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The matrix J satisfies the Lax equation

$$\frac{d}{d\beta} J = [B, J], \quad B = J_+ - J_-.$$

This looks similar to Toda, however J is not tridiagonal.

Even-odd decoupling

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Since $\langle A_{2n+1}(\beta)|A_{2m}(\beta')\rangle = 0$, we can write

$$J = J_{even} \oplus J_{odd}.$$

Better, but we still have $O(N^2)$ parameters.

Relation between L, J

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The identity

$$[H, \{H, \cdot\}] = \{H, [H, \cdot]\}$$

can be written as

$$[\mathcal{L}, \mathcal{J}] = 0 \implies [L, J] = 0.$$

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The independent integrals of motion are

$$\mathcal{I}_k = tr(J_{even}^k), \quad k = 1, 2, \dots, N$$

$$\mathcal{M}_k = tr(L^{2k}), \quad k = 1, 2, \dots, N.$$

We have $4N$-dimensional phase-space and $2N$ integrals of motion, so this is a fully integrable system.

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In this basis, we have 2 decoupled Toda chains J_{even}, J_{odd}.

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- Potential as a powerful numerical or analytical tool.
- Temperature dependence of Lanczos coefficients can be solved as an initial value problem.
- Given a 2pf at $\beta = 0$, we can calculate the 2pf at finite β.
- Study scaling of b_n with n as β is varied.