Quantum Information Theory in Quantum Field Theory and Cosmology @ BIRS, June 8, 2023

Krylov complexity and chaos in quantum mechanics

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based on [2305.16669] with

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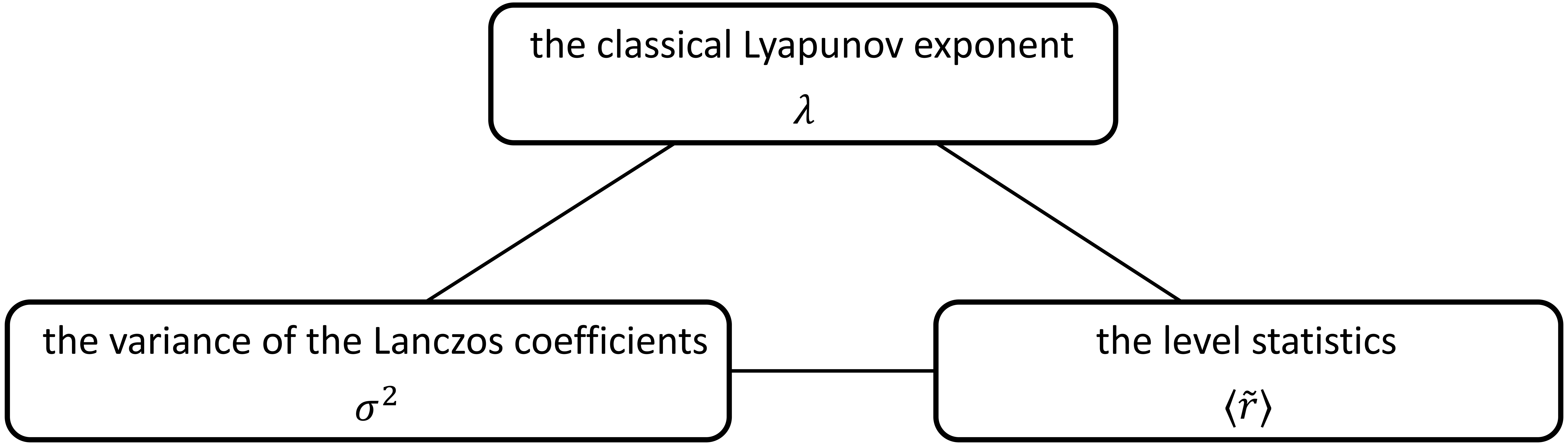
How should we characterize quantum chaos?
Can complexity measure quantum chaos?

Summary

- In the stadium billiard system, we find a significant correlation between…



- The variance of the Lanczos coefficients can be a measure of quantum chaos.
- Similar results were confirmed for the Sinai billiard.

Outline

1. Review on classical/quantum chaos (3 slides)
2. Review on Krylov complexity (6 slides)
3. Krylov complexity and chaos in QM (9 slides)
4. Summary

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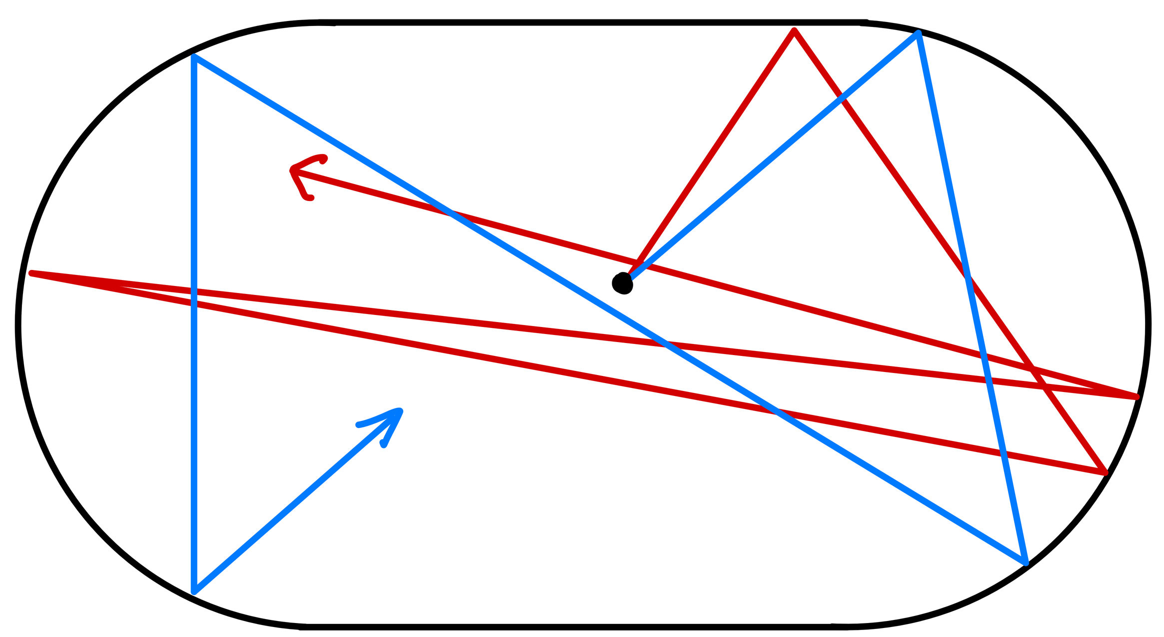
Classical Chaos: Lyapunov exponent

Unpredictable complex motion in deterministic nonlinear dynamical systems

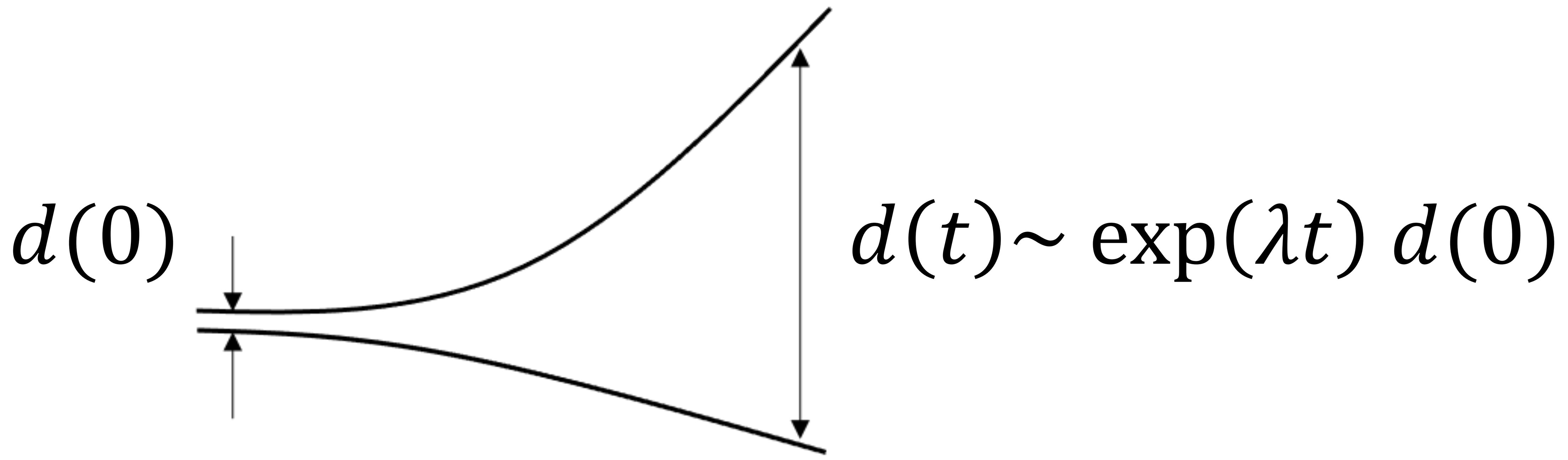
e.g.) stadium billiard, Sinai billiard, double pendulum, …

- Sensitive dependence on initial conditions



- Exponential divergence of trajectories in the phase space:

  The Lyapunov exponent λ measures chaoticity.



Quantum chaos: Spectral statistics

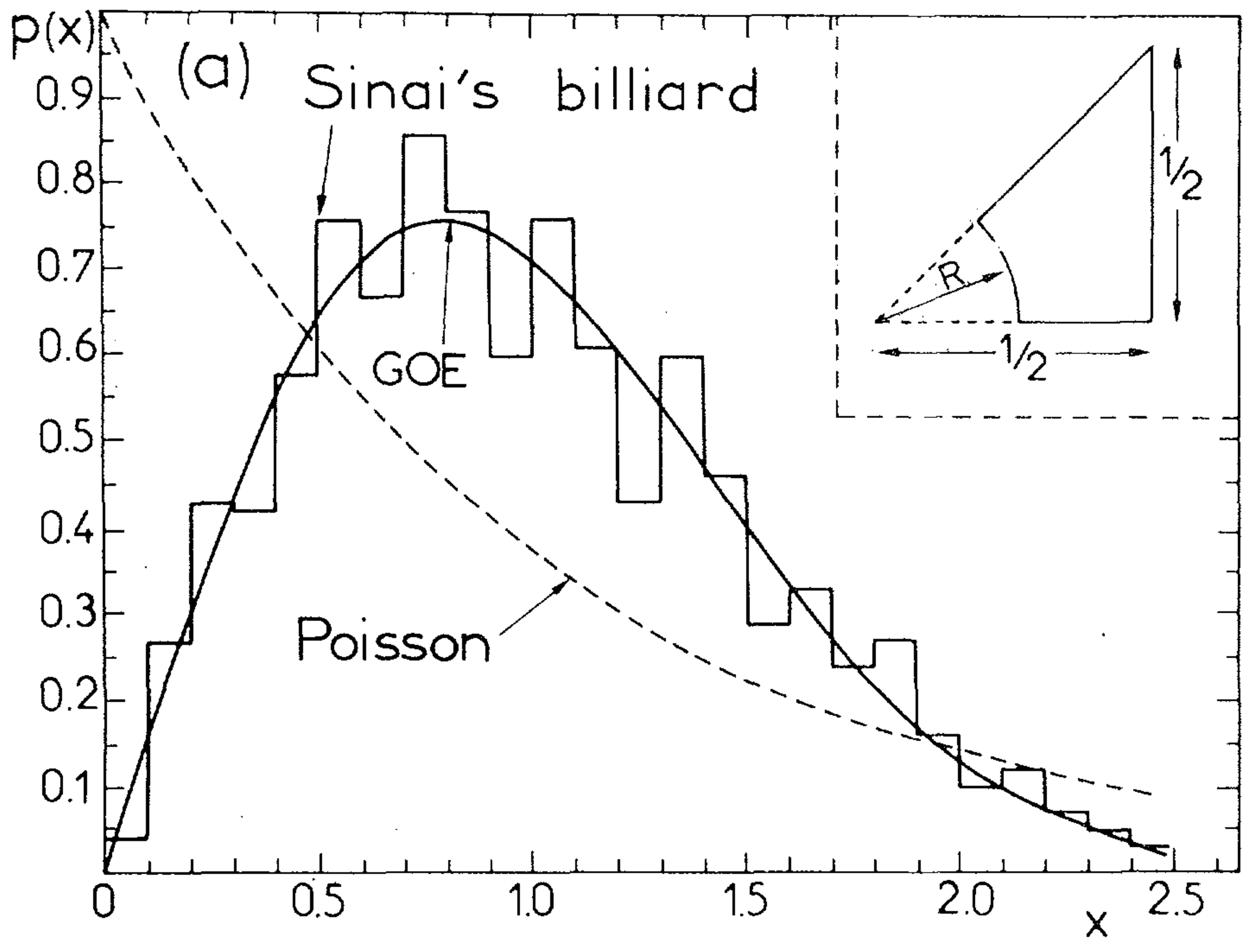
For chaotic systems, quantum energy spectra show the same fluctuation as a random matrix ensemble. [Bohigas, Giannoni, Schmit 1984]

The adjacent energy level spacings: $s_n = E_{n+1} - E_n$

- Wigner-Dyson statistics (chaotic systems)

$$P(s) = a_\beta s^\beta e^{-b_\beta s^2}$$

a_β, b_β: constants $\quad \beta = 1$ (GOE), 2 (GUE), 4 (GSE)



[Bohigas, Giannoni, Schmit 1984]

- Poisson statistics (non-chaotic systems) [Berry, Tabor 1977]

$$P(s) = e^{-s}$$

Characterization of the spectral statistics

[Oganesyan, Huse 2007] [Atas, Bogomolny, Giraud, Roux 2013]

The average $\langle \tilde{r} \rangle$ of the ratio of consecutive spacings

$$\tilde{r}_n \equiv \frac{\min(s_n, s_{n-1})}{\max(s_n, s_{n-1})} \qquad (s_n = E_{n+1} - E_n)$$

takes the following values depending on the spectral statistics:

$$\langle \tilde{r} \rangle = \begin{cases} 2\ln 2 - 1 \approx 0.38629 & \text{Poisson} \\ 4 - 2\sqrt{3} \approx 0.53590 & \text{GOE} \\ 2\frac{\sqrt{3}}{\pi} - \frac{1}{2} \approx 0.60266 & \text{GUE} \\ \frac{32}{15}\frac{\sqrt{3}}{\pi} - \frac{1}{2} \approx 0.67617 & \text{GSE} \end{cases}$$

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Krylov operator complexity

[Parker, Cao, Avdoshkin, Scaffidi, Altman 2018]

The Krylov operator complexity for a Heisenberg operator

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\mathcal{L}^n\mathcal{O} \qquad (\mathcal{L} \equiv [H, \cdot\,], \mathcal{L}^2 \equiv [H,[H, \cdot\,]], \cdots)$$

is defined as follows:

1. Introduce an inner product in the operator space: $(A|B) \equiv \mathrm{Tr}(A^\dagger B)$
2. Lanczos algorithm (next slide): $\{\mathcal{L}^n\mathcal{O}\} \to$ orthonormal basis $\{\mathcal{O}_n\}_{n=0}^{D_\mathcal{O}-1}$
3. Re-expand the Heisenberg operator as $\mathcal{O}(t) = \sum_{n=1}^{D_\mathcal{O}-1} i^n\varphi_n(t)\,\mathcal{O}_n$
4. Krylov complexity $C_\mathcal{O}(t) \equiv \sum_{n=1}^{D_\mathcal{O}-1} n|\varphi_n(t)|^2$

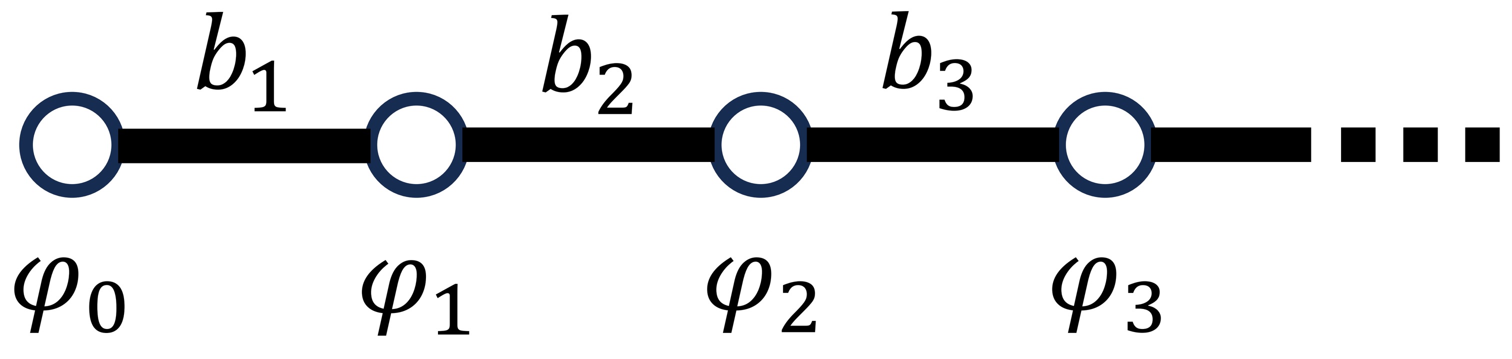
Lanczos algorithm for operators

[Parker, Cao, Avdoshkin, Scaffidi, Altman 2018]

The Lanczos algorithm (Gram-Schmidt orthogonalization for $\{\mathcal{L}^n \mathcal{O}\}$)

1. Let $b_0 \equiv 0$ and $\mathcal{O}_{-1} \equiv 0$
2. $\mathcal{O}_0 \equiv \mathcal{O}/||\mathcal{O}||$, where $||\mathcal{O}|| \equiv \sqrt{(\mathcal{O}|\mathcal{O})}$
3. For $n \geq 1$: $\mathcal{A}_n \equiv \mathcal{L}\mathcal{O}_{n-1} - b_{n-1}\mathcal{O}_{n-2}$
4. Set $b_n \equiv ||\mathcal{A}_n||$
5. If $b_n = 0$, stop; otherwise set $\mathcal{O}_n \equiv \mathcal{A}_n / b_n$ and go to step 3.

- Output: orthonormal basis $\{\mathcal{O}_n\}$ and Lanczos coefficients $\{b_n\}$
- $\varphi_n(t)$ in $\mathcal{O}(t) = \sum_n i^n \varphi_n(t)\, \mathcal{O}_n$ obeys $\dot{\varphi}_n = b_n \varphi_{n-1} - b_{n+1}\varphi_{n+1}$ $\quad (\varphi_n(0) = \delta_{n0}||\mathcal{O}||)$



Krylov state complexity

[Balasubramanian, Caputa, Magan, Wu 2022]

The Krylov state complexity (spread complexity) for a Schrödinger state

$$|\psi(t)\rangle = e^{-iHt}|\psi\rangle = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} H^n|\psi\rangle$$

is defined as follows:

1. Lanczos algorithm (next slide): $\{H^n|\psi\rangle\} \to$ orthonormal basis $\{|K_n\rangle\}_{n=0}^{D_\psi-1}$

   (There are two kinds of Lanczos coefficients a_n, b_n in this case)

2. Re-expand the Schrödinger state as $|\psi(t)\rangle = \sum_{n=0}^{D_\psi-1} \psi_n(t)|K_n\rangle$
3. Krylov complexity $C_\psi(t) \equiv \sum_{n=1}^{D_\psi-1} n|\psi_n(t)|^2$

Lanczos algorithm for states

[Balasubramanian, Caputa, Magan, Wu 2022]

The Lanczos algorithm (Gram-Schmidt orthogonalization for $\{H^n|\psi\rangle\}$)

1. $b_0 \equiv 0$, $|K_{-1}\rangle \equiv 0$
2. $|K_0\rangle \equiv |\psi\rangle$, $a_0 \equiv \langle K_0|H|K_0\rangle$
3. For $n \geq 1$: $|\mathcal{A}_n\rangle \equiv (H - a_{n-1})|K_{n-1}\rangle - b_{n-1}|K_{n-2}\rangle$
4. Set $b_n \equiv \sqrt{\langle \mathcal{A}_n|\mathcal{A}_n\rangle}$
5. If $b_n = 0$, stop; otherwise set $|K_n\rangle \equiv |\mathcal{A}_n\rangle / b_n$ and go to step 3.

- Output: orthonormal basis $\{|K_n\rangle\}$ and Lanczos coefficients $\{a_n, b_n\}$
- $\psi_n(t)$ in $|\psi(t)\rangle = \sum_{n=0}^{D_\psi - 1} \psi_n(t)|K_n\rangle$ obeys $i\dot{\psi}_n = a_n\psi_n + b_{n+1}\psi_{n+1} + b_n\psi_{n-1}$ $(\psi_n(0) = \delta_{n0})$

Lanczos coefficients and chaoticity

For finite-dimensional systems, the Lanczos algorithm must terminate.

For operators in some spin models, [Rabinovici, Sánchez-Garrido, Shir, Sonner 2021, 2022]

- if the system is non-chaotic, the Lanczos coefficient b_n behaves erratically.
- if the system is chaotic, the Lanczos coefficient b_n behaves less erratically.

They measured the magnitude of the erratic behavior of b_n by

$$\sigma^2 \equiv \mathrm{Var}(x_i) = \langle x^2 \rangle - \langle x \rangle^2, \qquad x_i \equiv \ln\left(\frac{b_{2i-1}}{b_{2i}}\right)$$

- What about quantum systems that have classical counterparts?
- How about for quantum states?

$$\sigma_a^2 \equiv \mathrm{Var}\left(x_i^{(a)}\right), \quad x_i^{(a)} \equiv \ln\left(\frac{a_{2i-1}}{a_{2i}}\right)$$

$$\sigma_b^2 \equiv \mathrm{Var}\left(x_i^{(b)}\right), \quad x_i^{(b)} \equiv \ln\left(\frac{b_{2i-1}}{b_{2i}}\right)$$

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Krylov complexity in quantum mechanics

A quantum mechanical system with the Hamiltonian

$$H = p_1^2 + p_2^2 + V(x, y)$$

- We regularize the Hilbert space by considering only a finite number N_{max} of levels and ignoring the others.
- Using the energy eigenstates as a basis, represent the operator as a matrix:

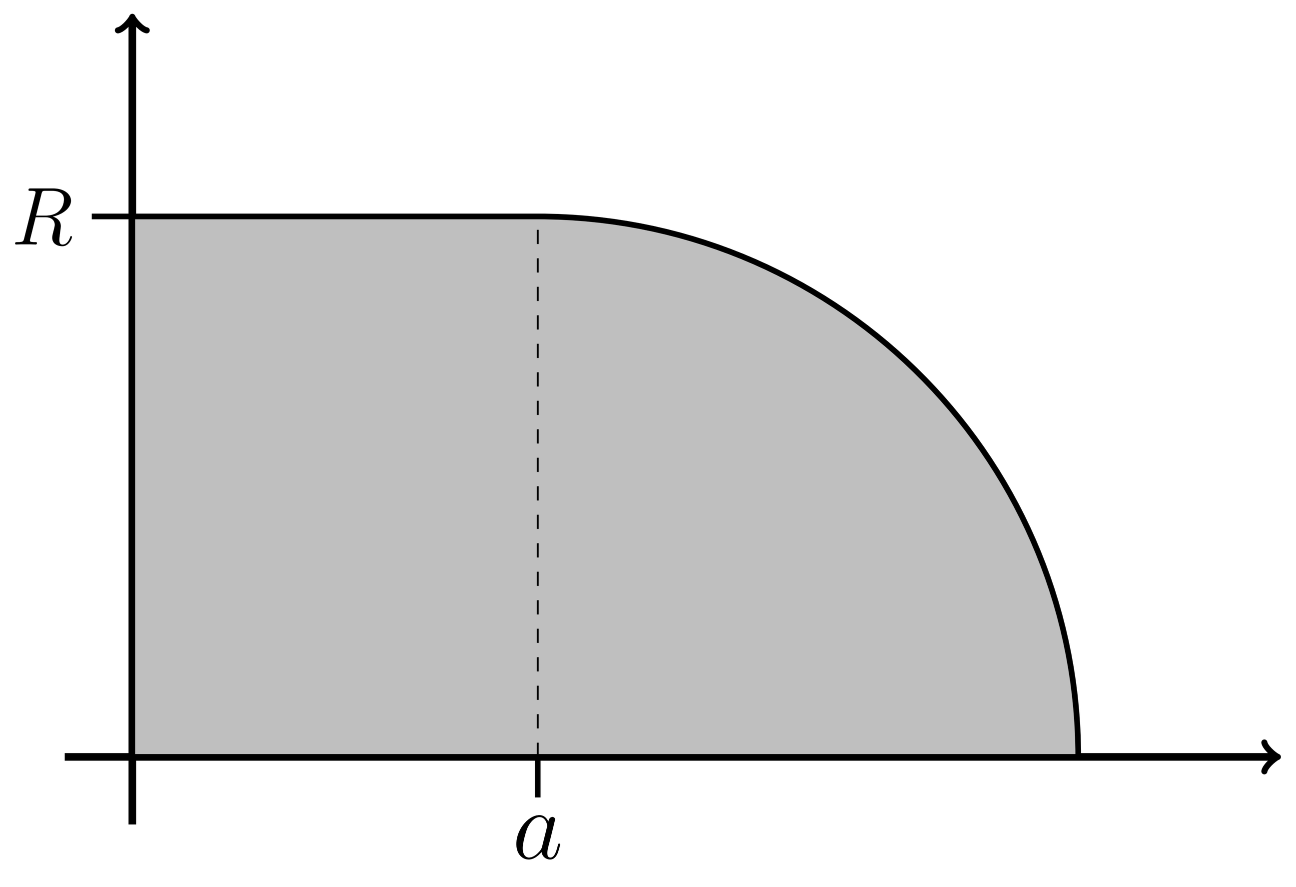
$$\mathcal{O}_{mn} \equiv \langle m|\mathcal{O}|n\rangle, \quad H|n\rangle = E_n|n\rangle \qquad m, n = 1, \cdots, N_{\mathrm{max}}$$

- Perform the Lanczos algorithm and calculate the Krylov complexity.
- For states, complexity can be calculated in the similar way.

Our system: the stadium billiard

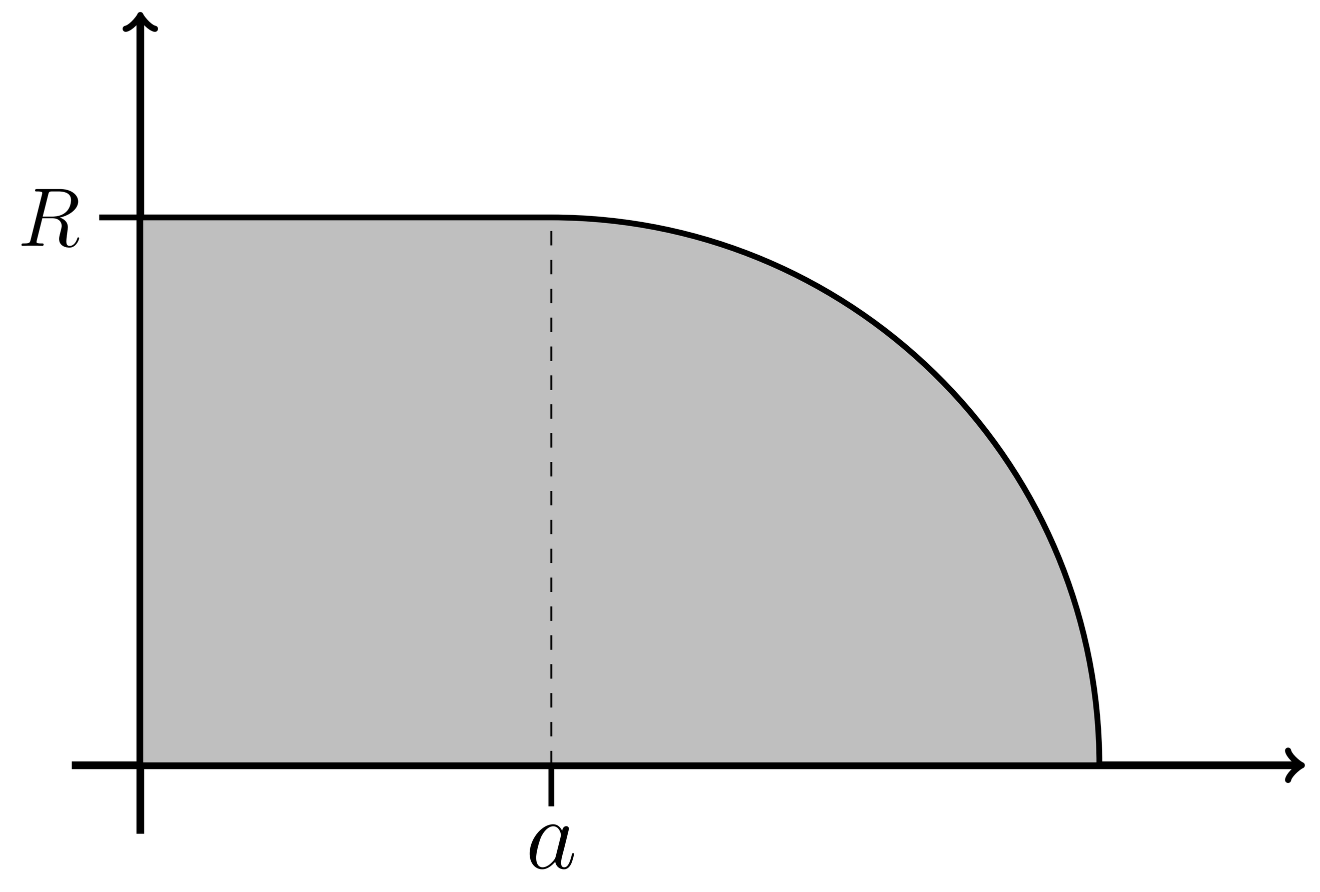
Krylov operator/state complexity in the stadium billiard

$$H = p_1^2 + p_2^2 + V(x,y)$$



- The momentum operator p_1 with $N_{\max} = 100$
- The equally-distributed state $\Psi = (\frac{1}{\sqrt{N_{\max}}}, \cdots, \frac{1}{\sqrt{N_{\max}}})$ with $N_{\max} = 500$

Classical/quantum chaos in the stadium billiard

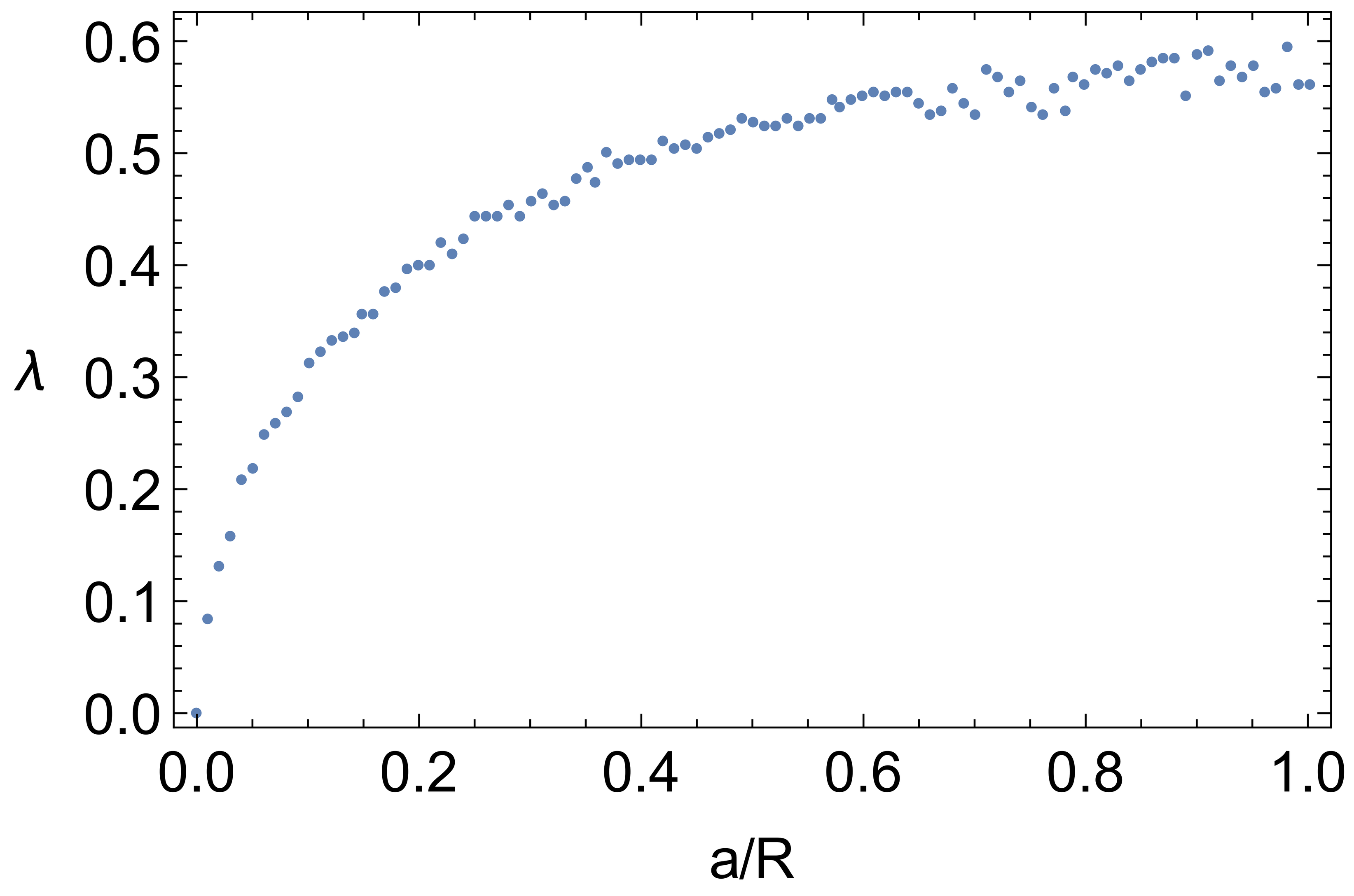


The geometry is characterized by a/R

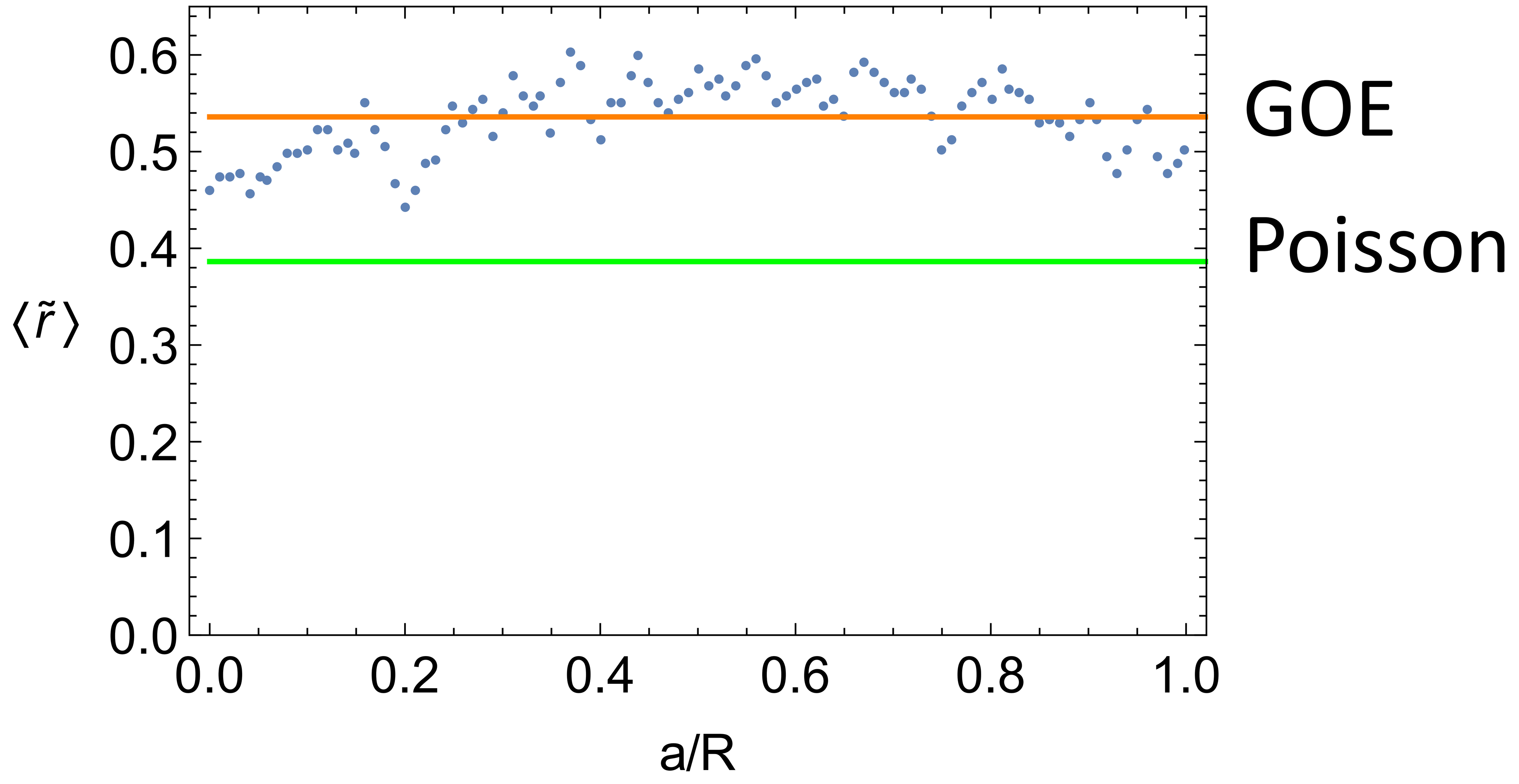
$a/R = 0$: non-chaotic

$a/R > 0$: chaotic

Classical Lyapunov exponent

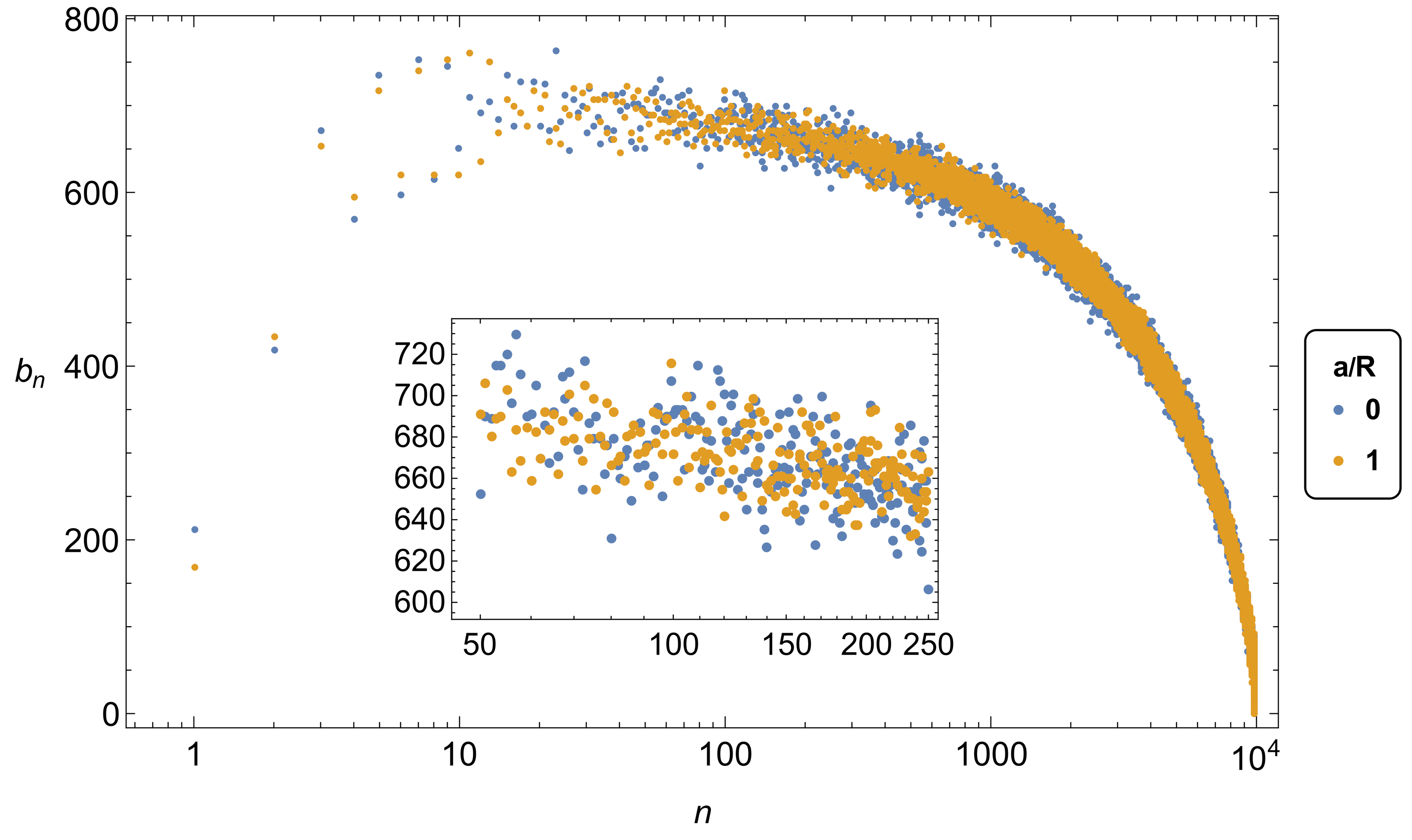


The ratio of consecutive spacings

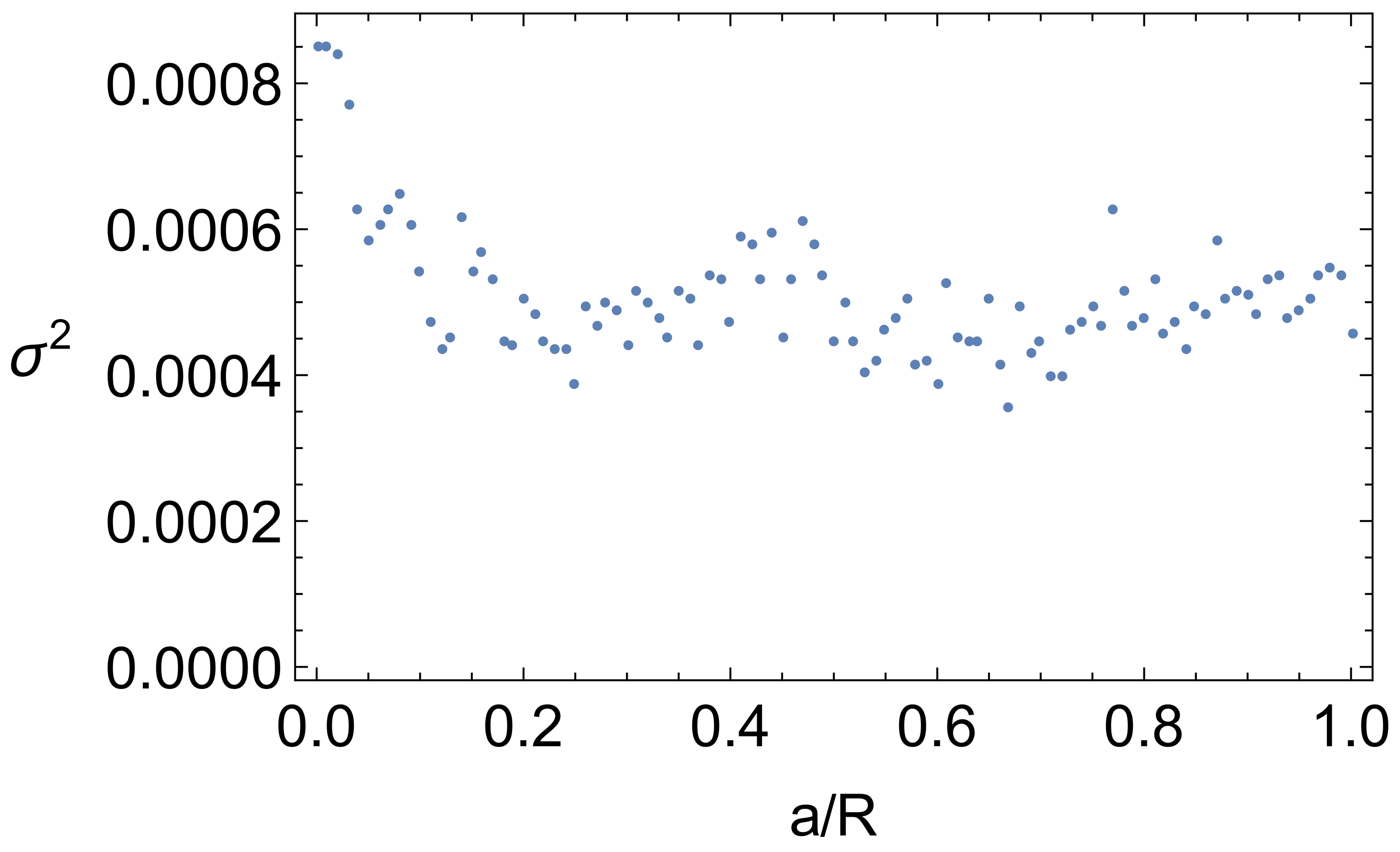


Krylov operator complexity

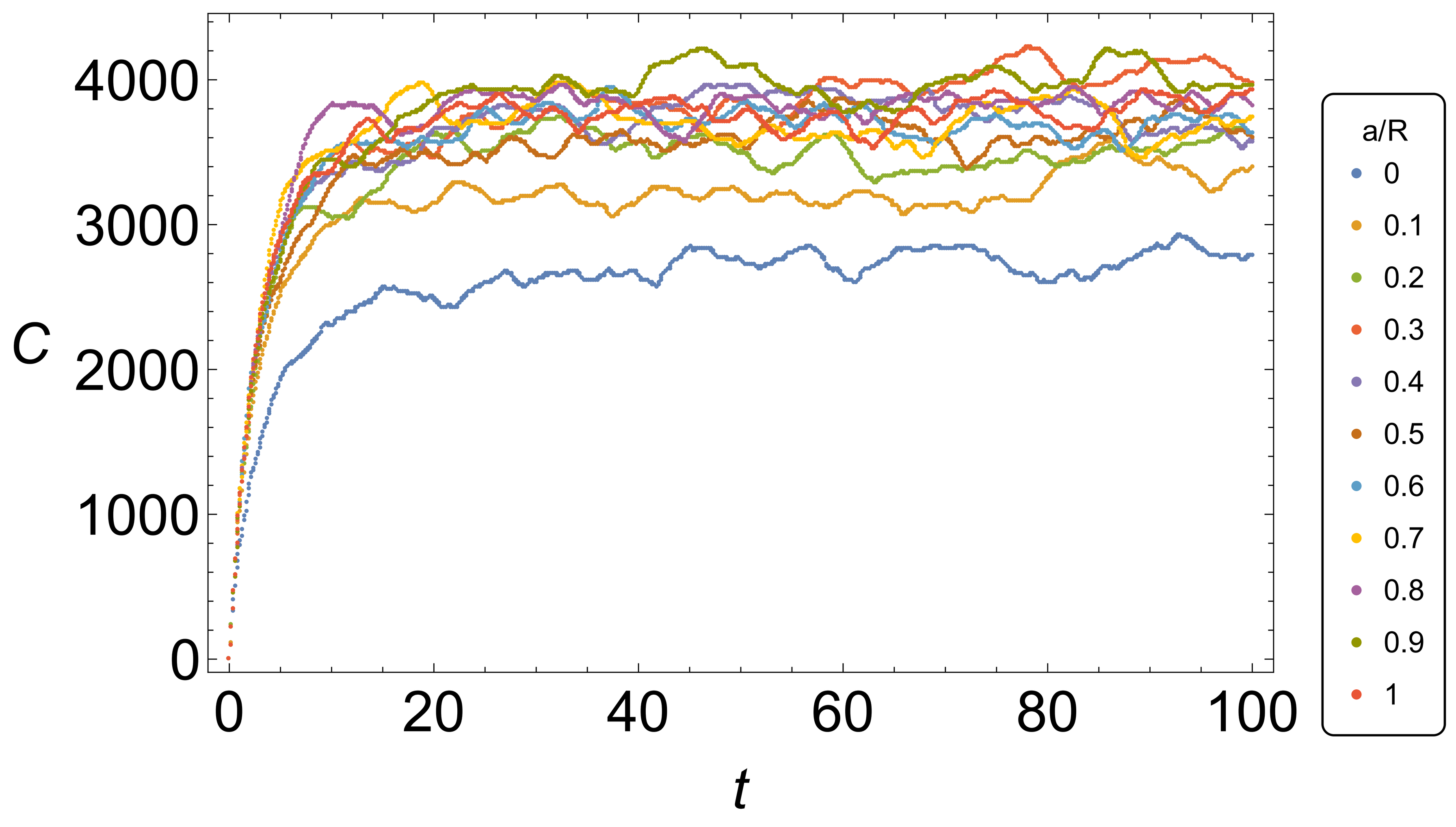
Lanczos coefficients



Variances



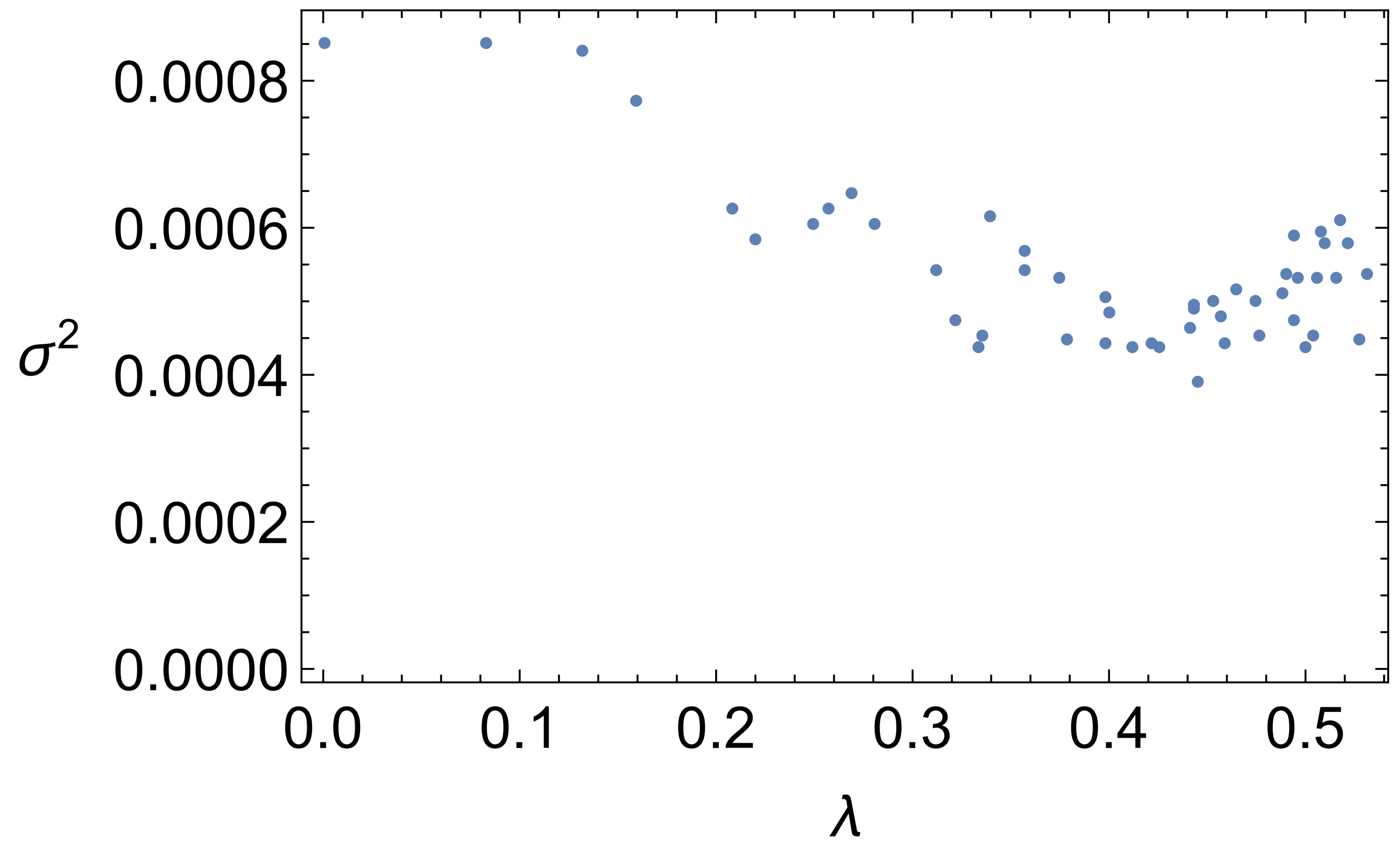
Krylov complexity

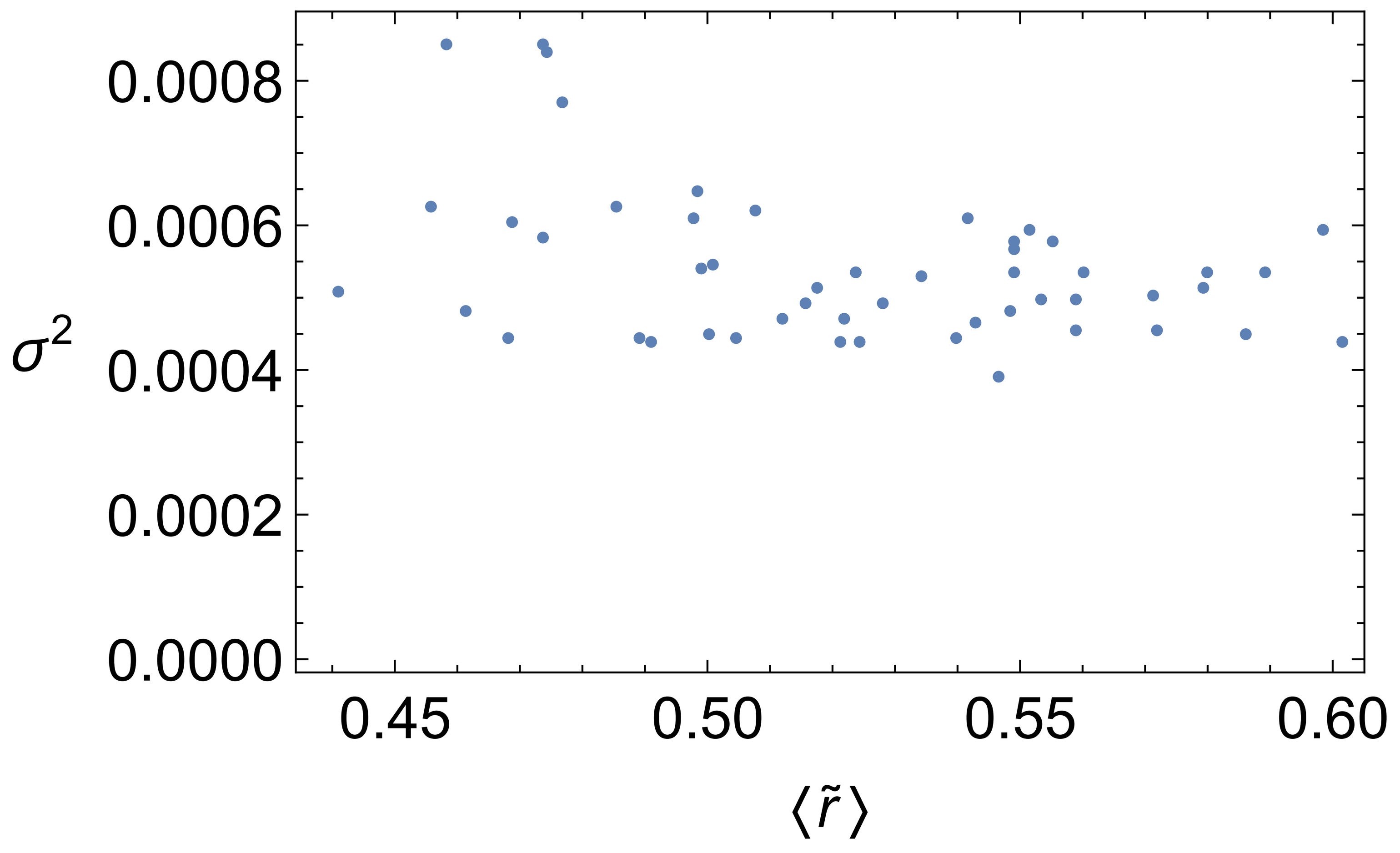


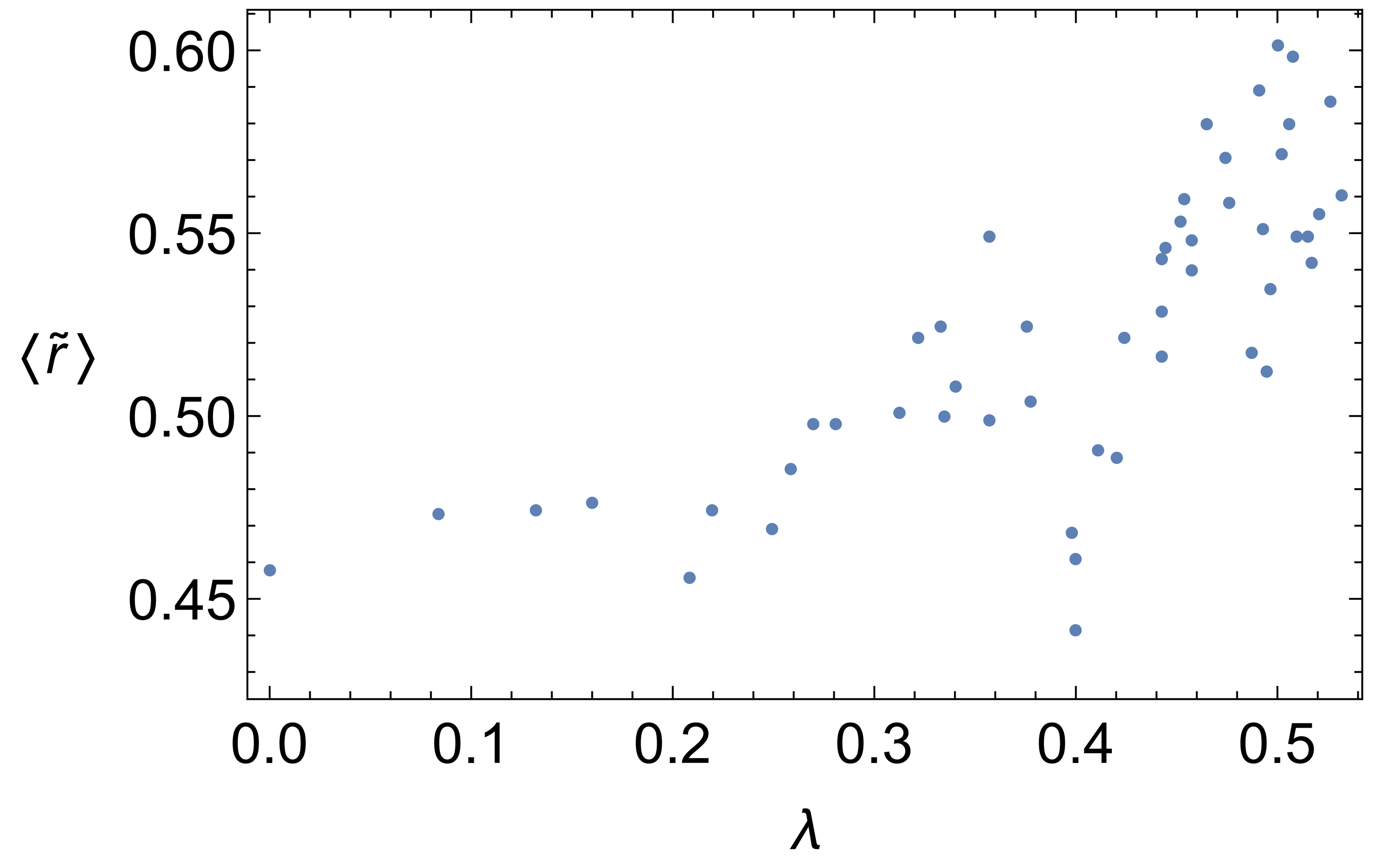
$$\sigma^2 \equiv \mathrm{Var}(x_i) = \langle x^2 \rangle - \langle x \rangle^2, \qquad x_i \equiv \ln\left(\frac{b_{2i-1}}{b_{2i}}\right)$$

- The variance becomes larger in the non-chaotic regime compared to the chaotic regime.
- The Krylov complexity does not grow exponentially.

Correlation in the stadium billiard







Correlation coefficients

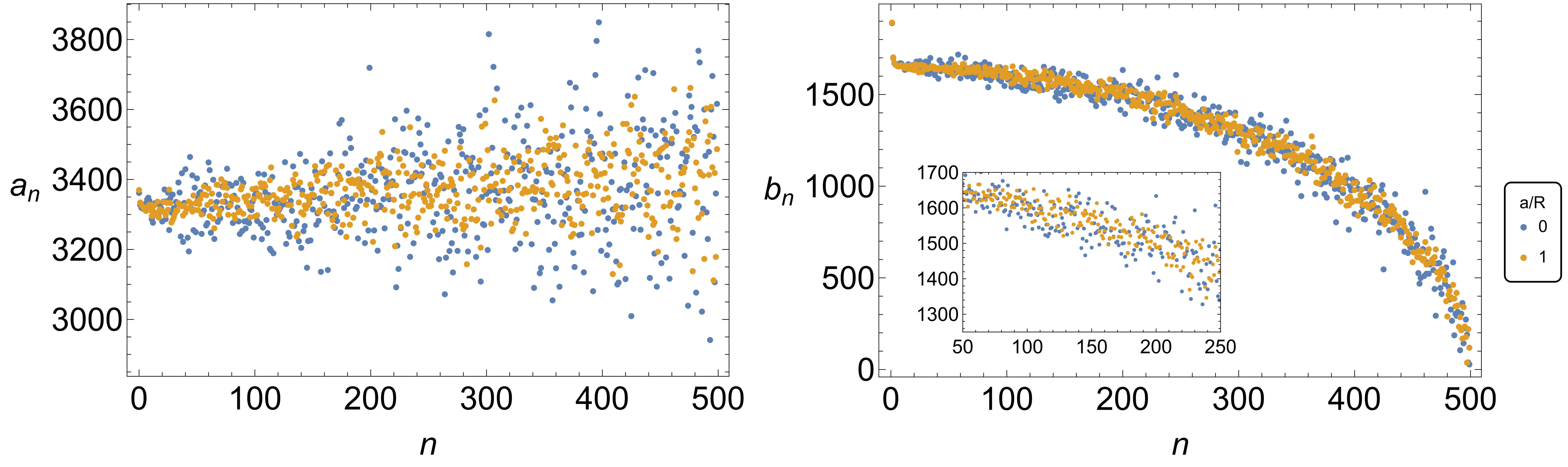
λ vs σ^2	-0.720372
$\langle \tilde{r} \rangle$ vs σ^2	-0.391709
λ vs $\langle \tilde{r} \rangle$	0.741396

- Significant correlations exist among σ^2, λ, and $\langle \tilde{r} \rangle$.
- σ^2 can be a measure of quantum chaos.

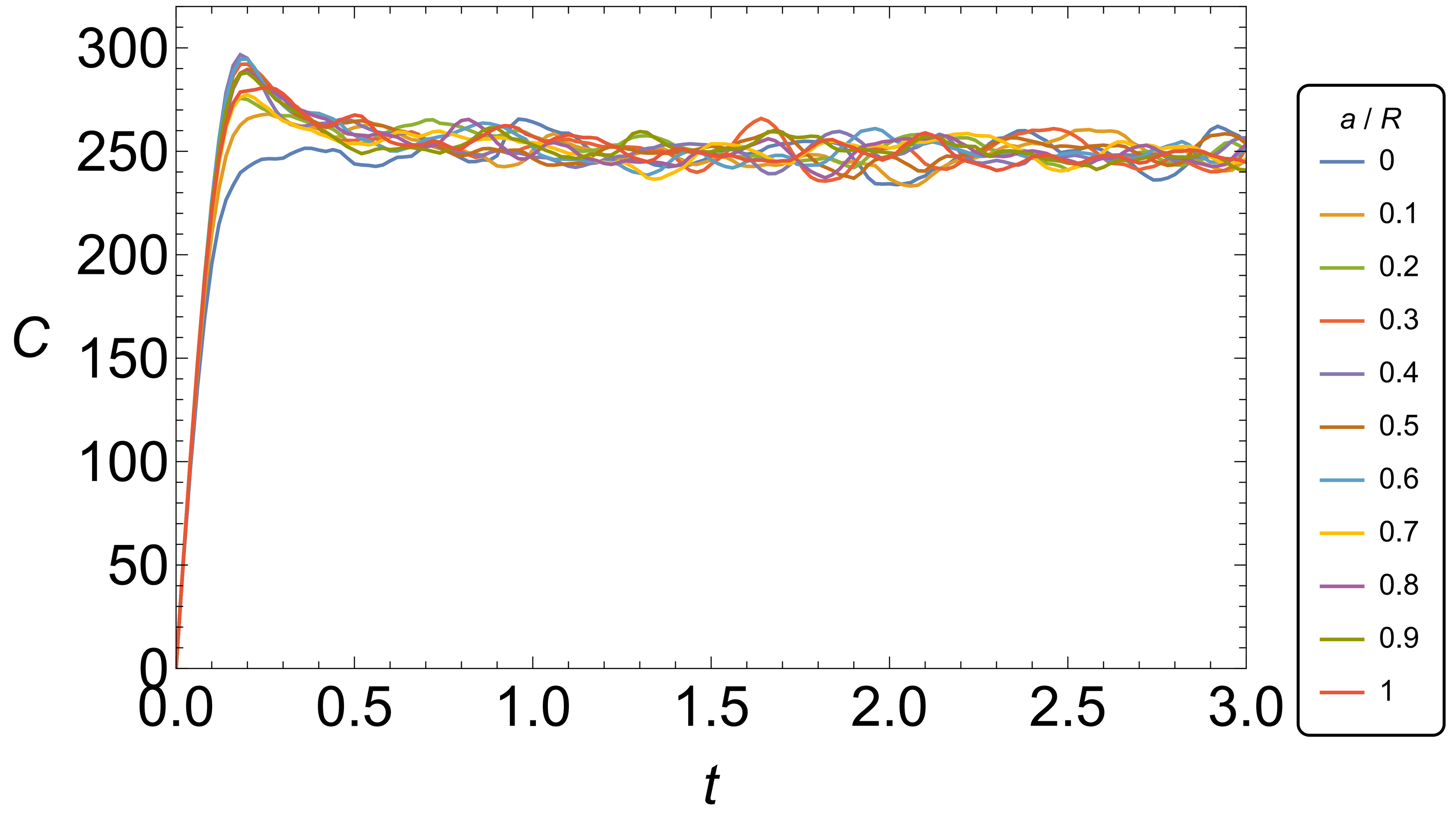
Correlation coefficients between data A and $B \equiv \dfrac{\mathrm{E}[(A-\mathrm{E}[A])(B-\mathrm{E}[B])]}{\sqrt{\mathrm{E}[(A-\mathrm{E}[A])^2]\,\mathrm{E}[(B-\mathrm{E}[B])^2]}}$

Krylov state complexity

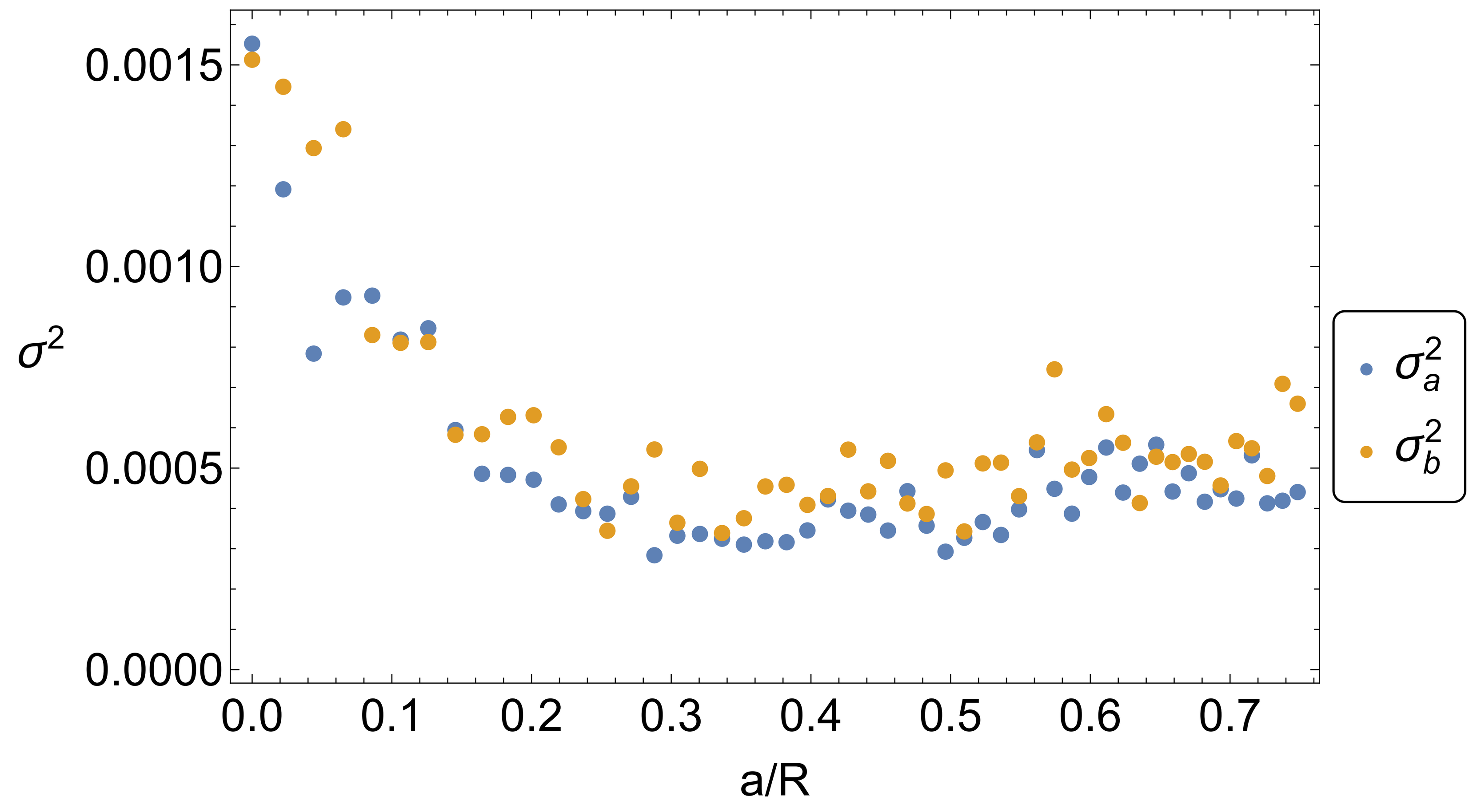
Lanczos coefficients



Krylov complexity



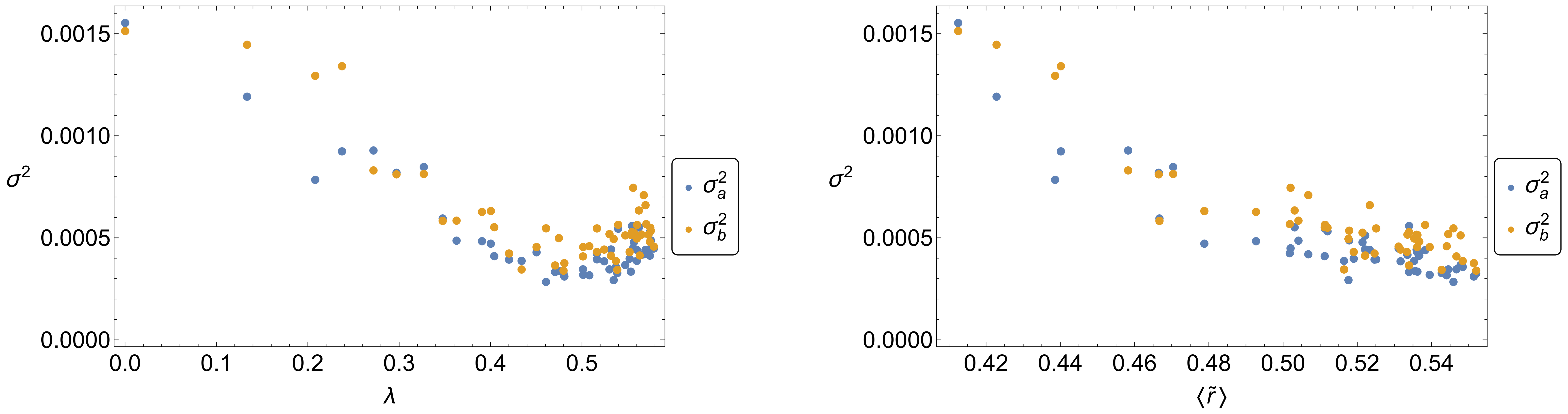
Variances



- The variance becomes larger in the non-chaotic regime compared to the chaotic regime.
- The Krylov complexity does not grow exponentially.
- The peak value of Krylov state complexity depends on a/R.

The peak behavior [Balasubramanian, Caputa, Magan, Wu 2022]
[Erdmenger, Jian, Xian 2023]

Correlation in the stadium billiard



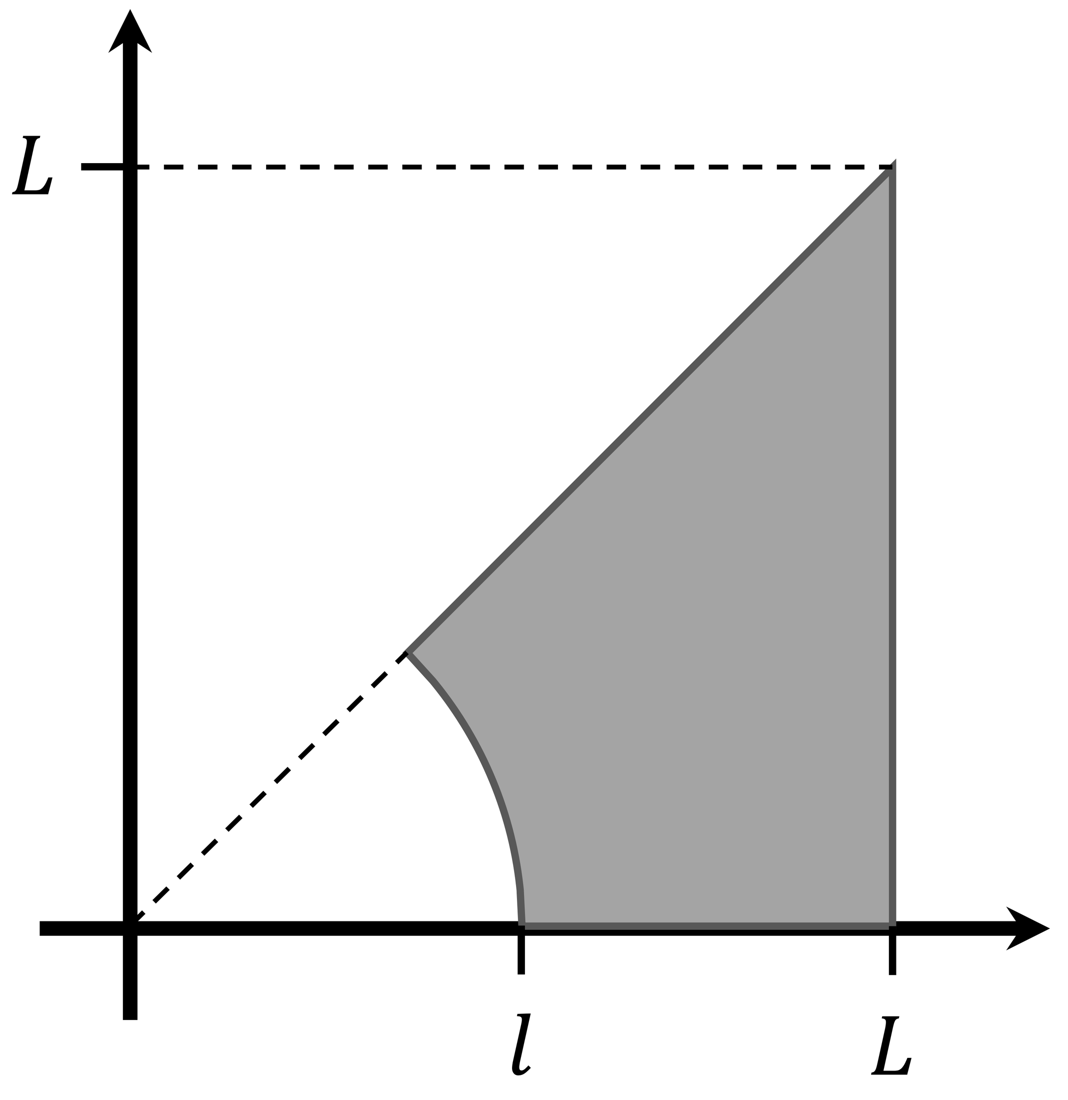
Correlation coefficients

λ vs σ_a^2	-0.832395
λ vs σ_b^2	-0.806238
$\langle\tilde{r}\rangle$ vs σ_a^2	-0.891642
$\langle\tilde{r}\rangle$ vs σ_b^2	-0.893569

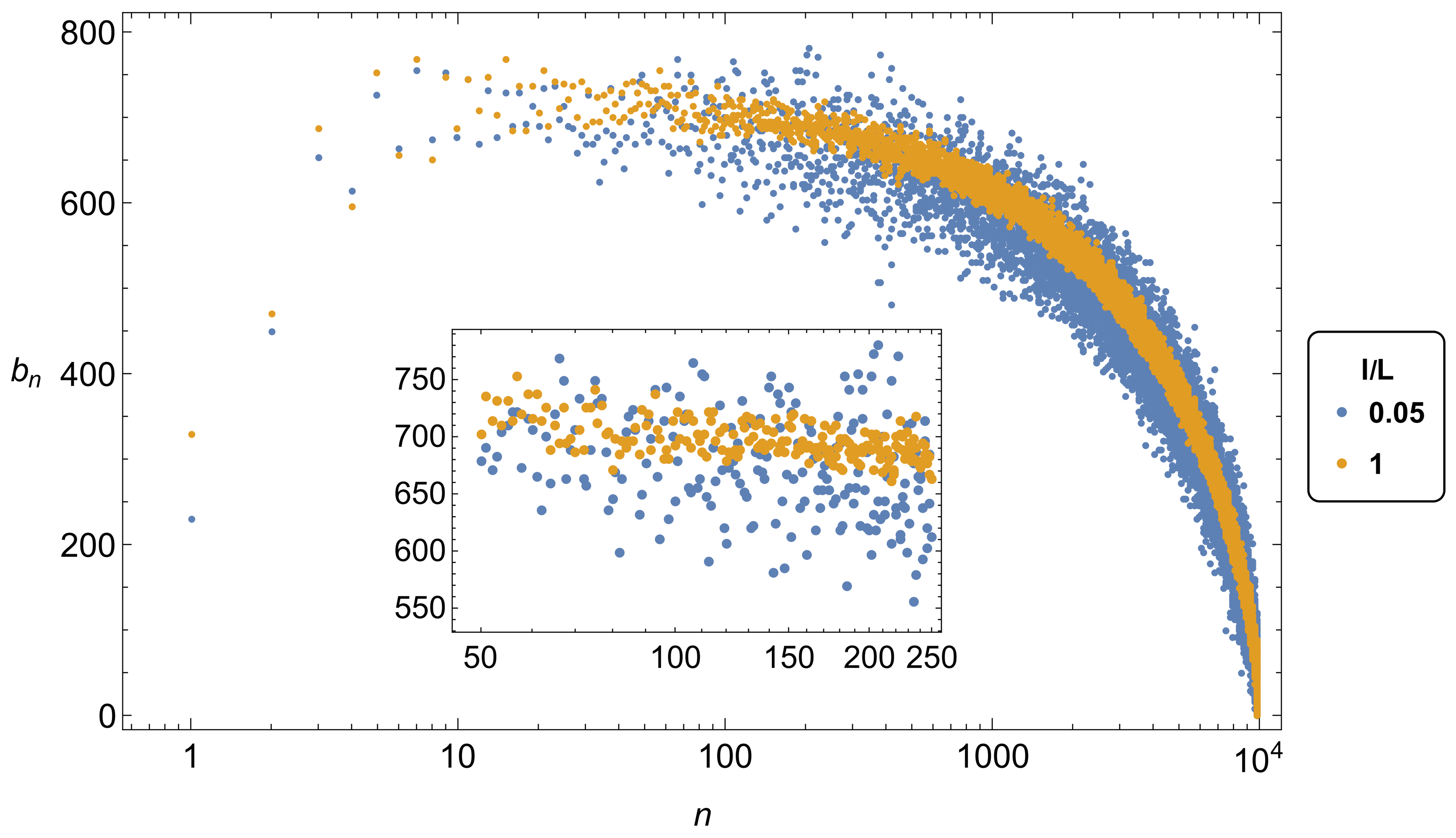
- A clear correlation exists between $\sigma_{a,b}^2$, λ, and $\langle\tilde{r}\rangle$.
- $\sigma_{a,b}^2$ can be a measure of quantum chaos.

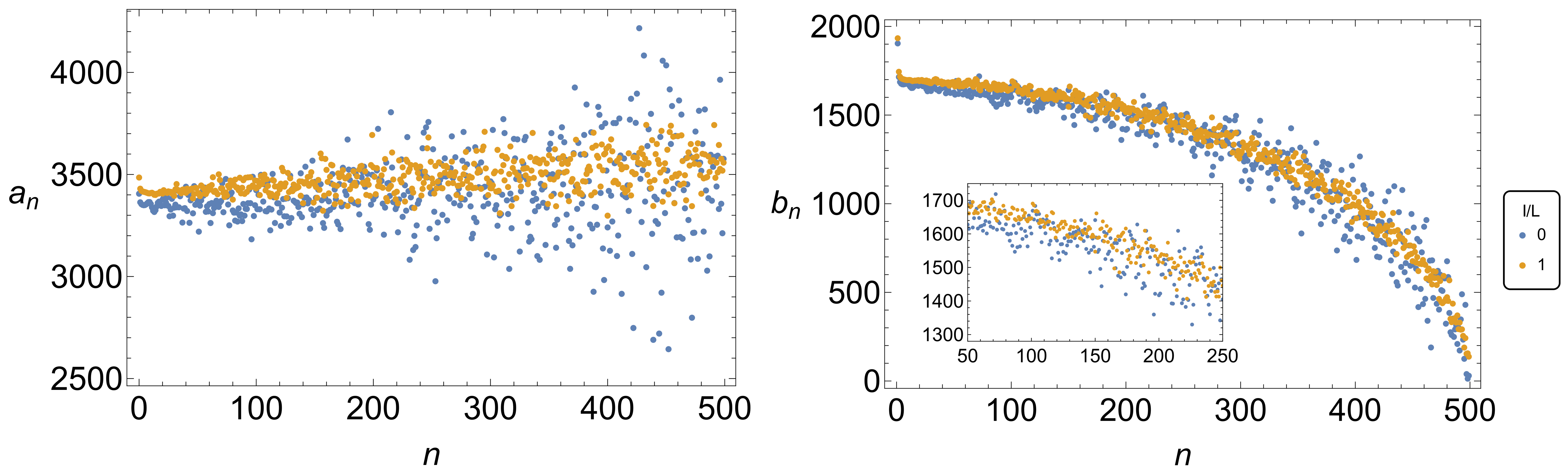
How about another billiard system?

Universality: the Sinai billiard



- Again, the variance of Lanczos coefficients becomes larger in the non-chaotic regime compared to the chaotic regime.
- The result may be universal for generic quantum mechanics.





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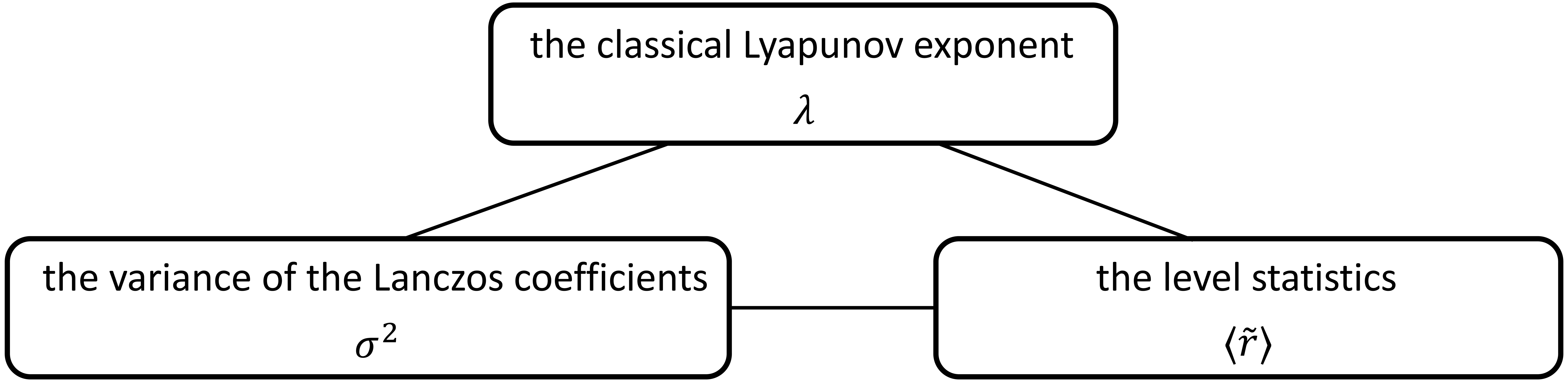
Summary

- In billiard systems, we find a significant correlation between…

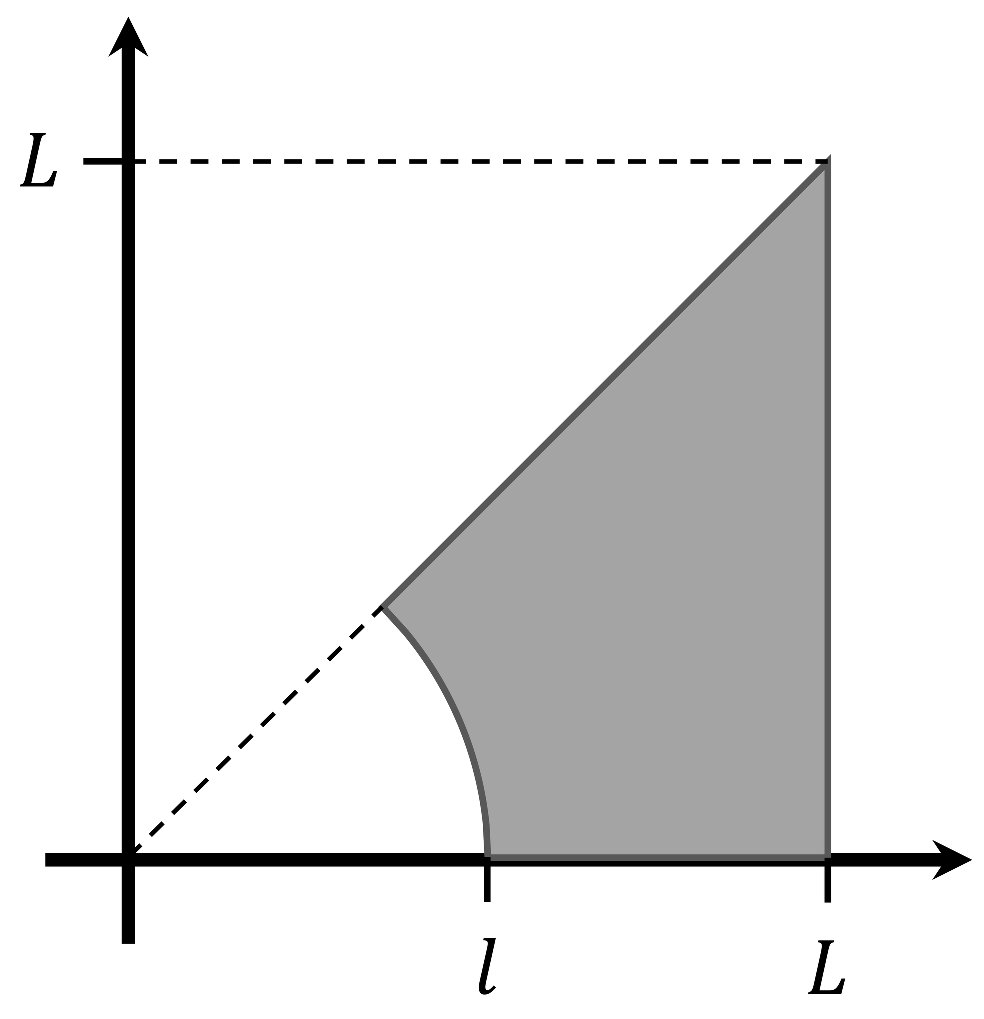


- The variance of the Lanczos coefficients can be a measure of quantum chaos.

- Other quantum mechanical systems?

- Holographic interpretation of the Krylov complexity?

Classical/quantum chaos in the Sinai billiard

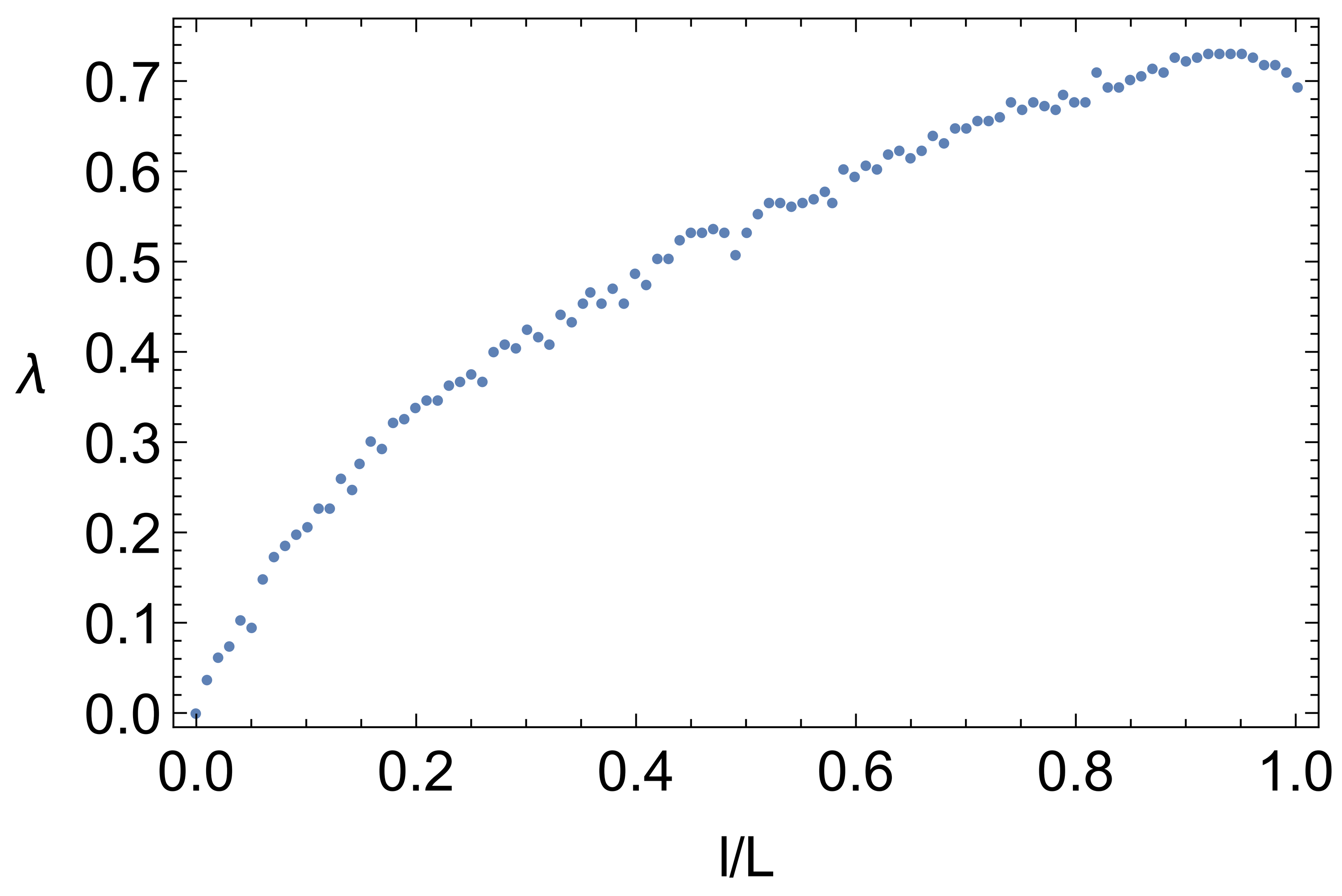


The geometry is characterized by l/L

$l/L = 0$: non-chaotic

$l/L \gtrsim 0$: chaotic

Classical Lyapunov exponent



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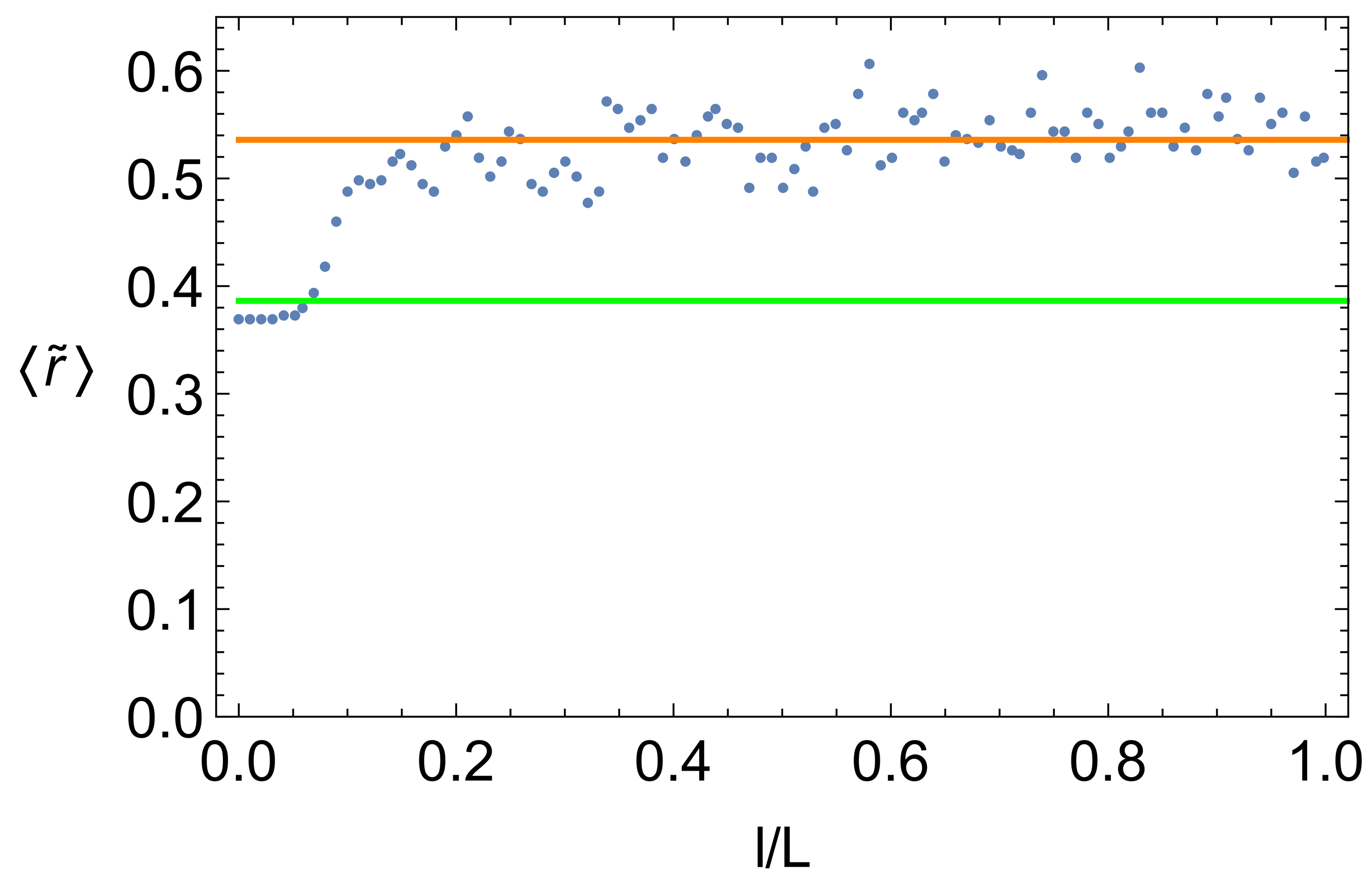
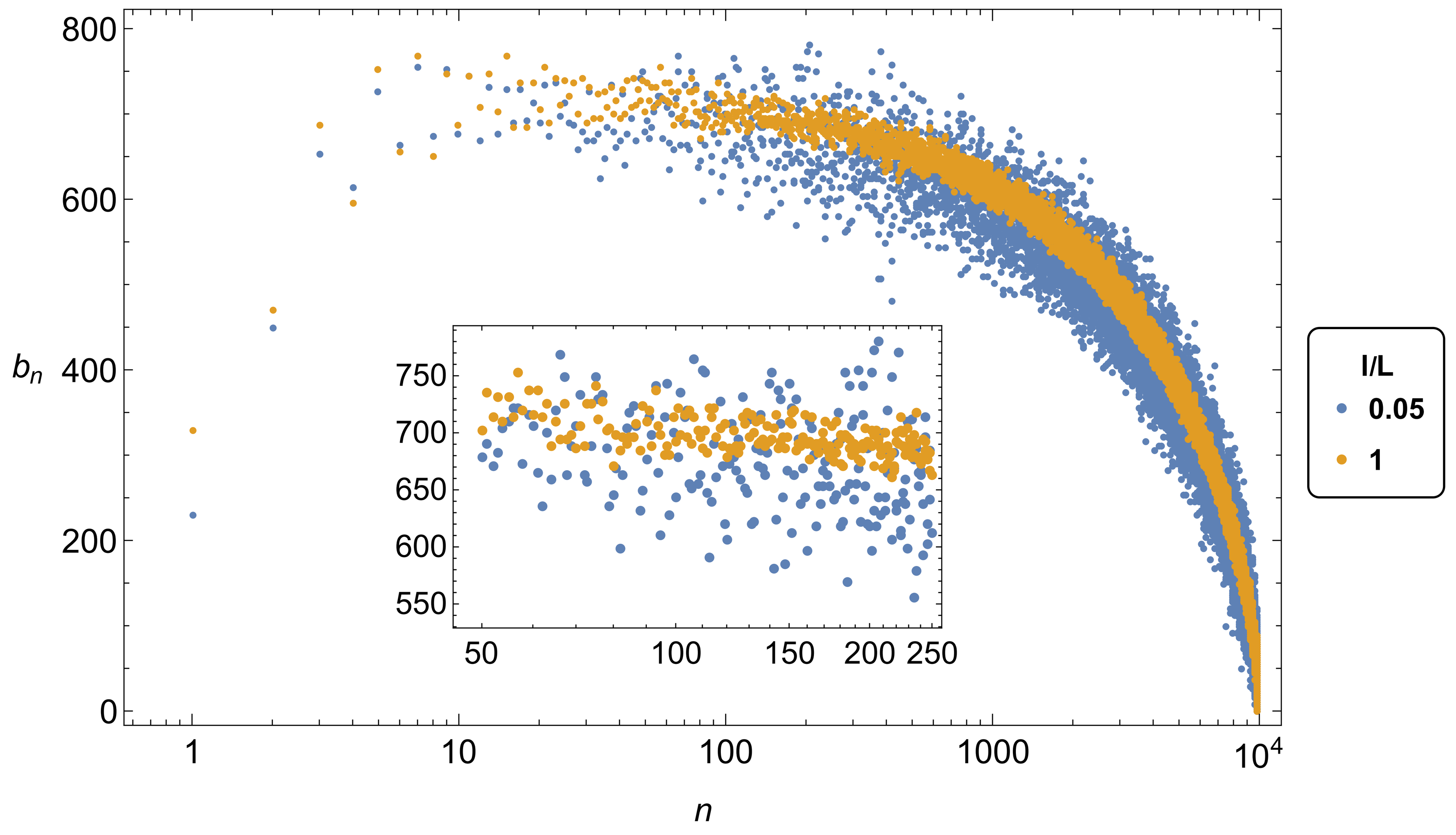
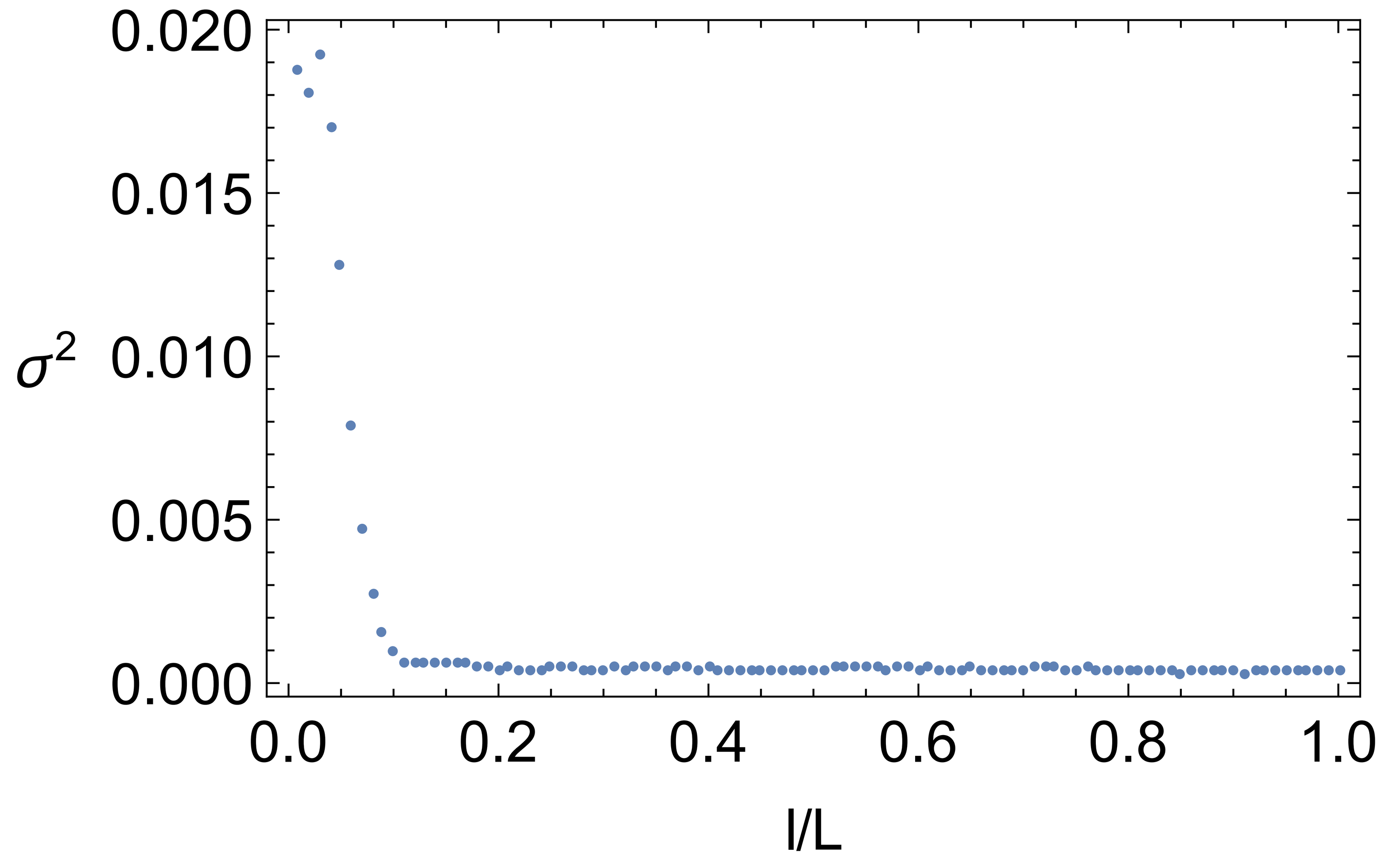


Krylov operator complexity

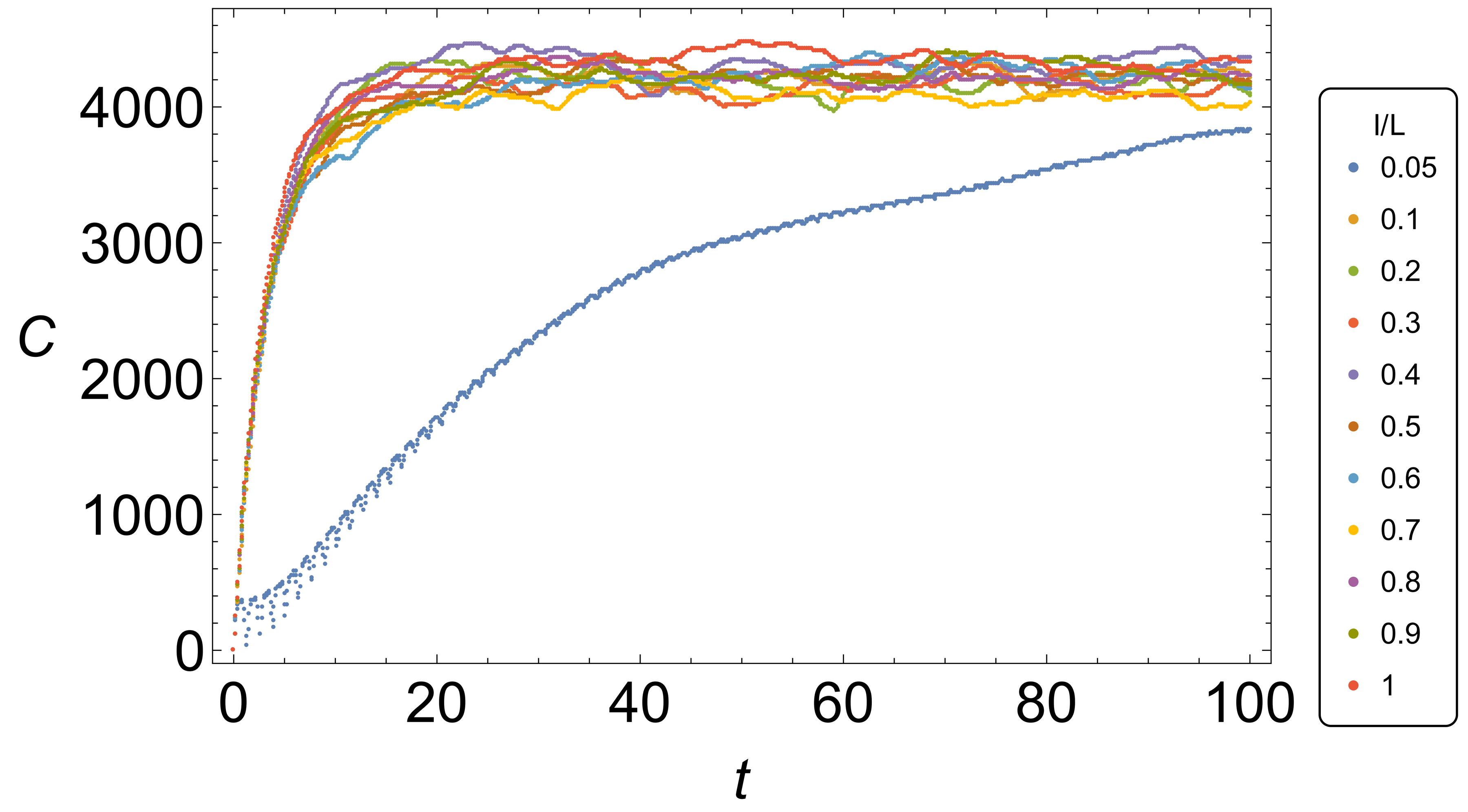
Lanczos coefficients



Variances



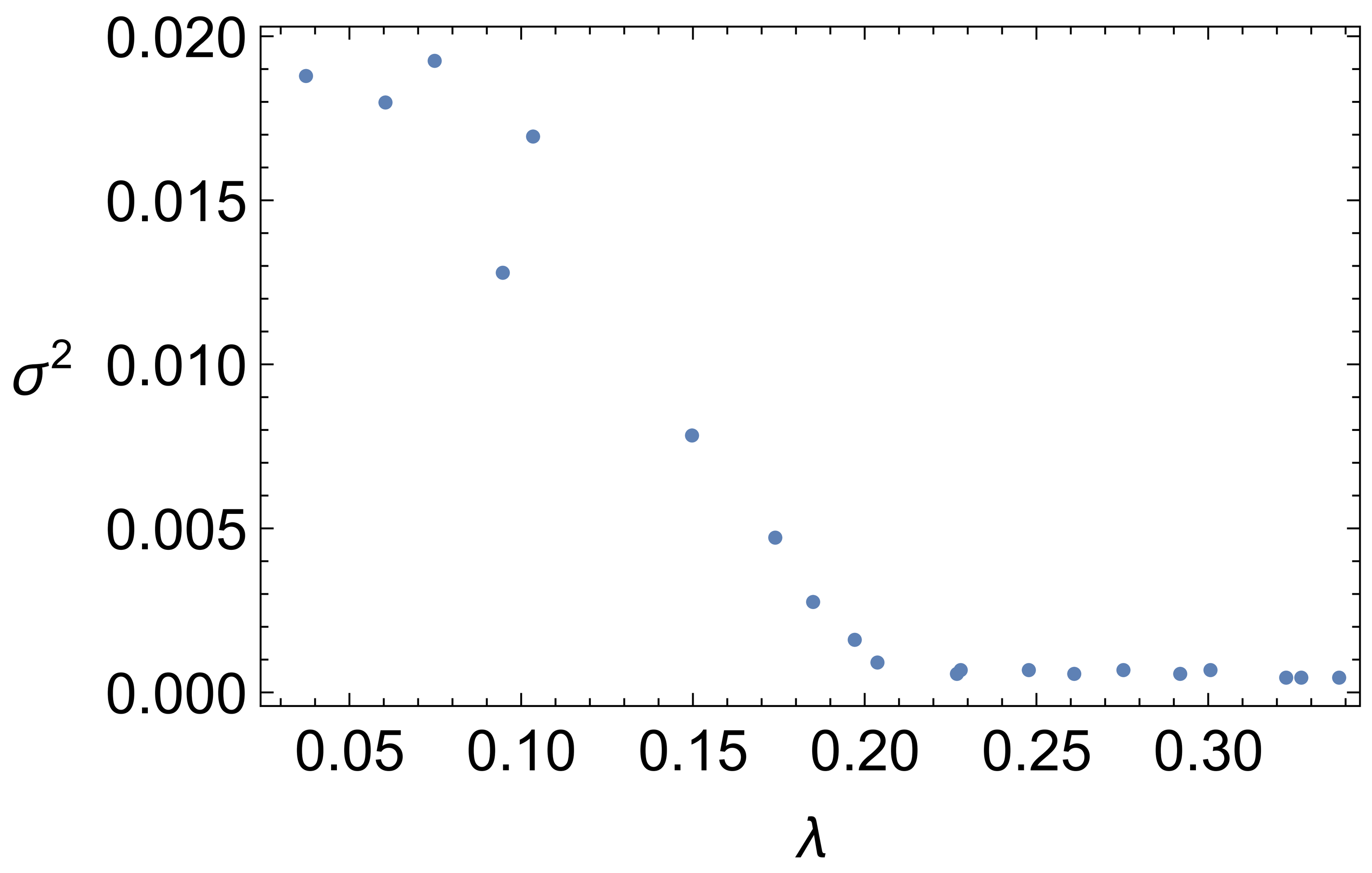
Krylov complexity

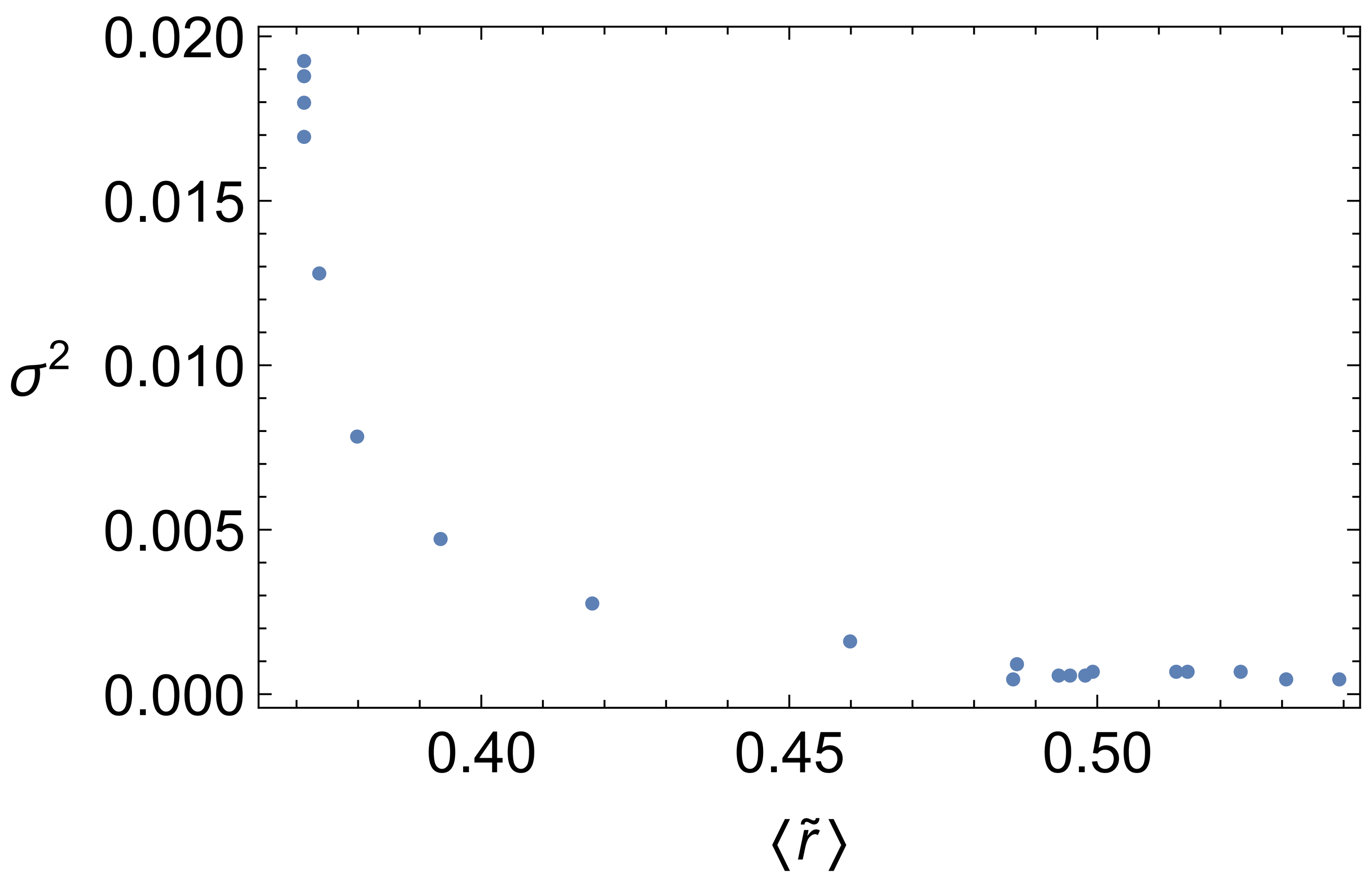


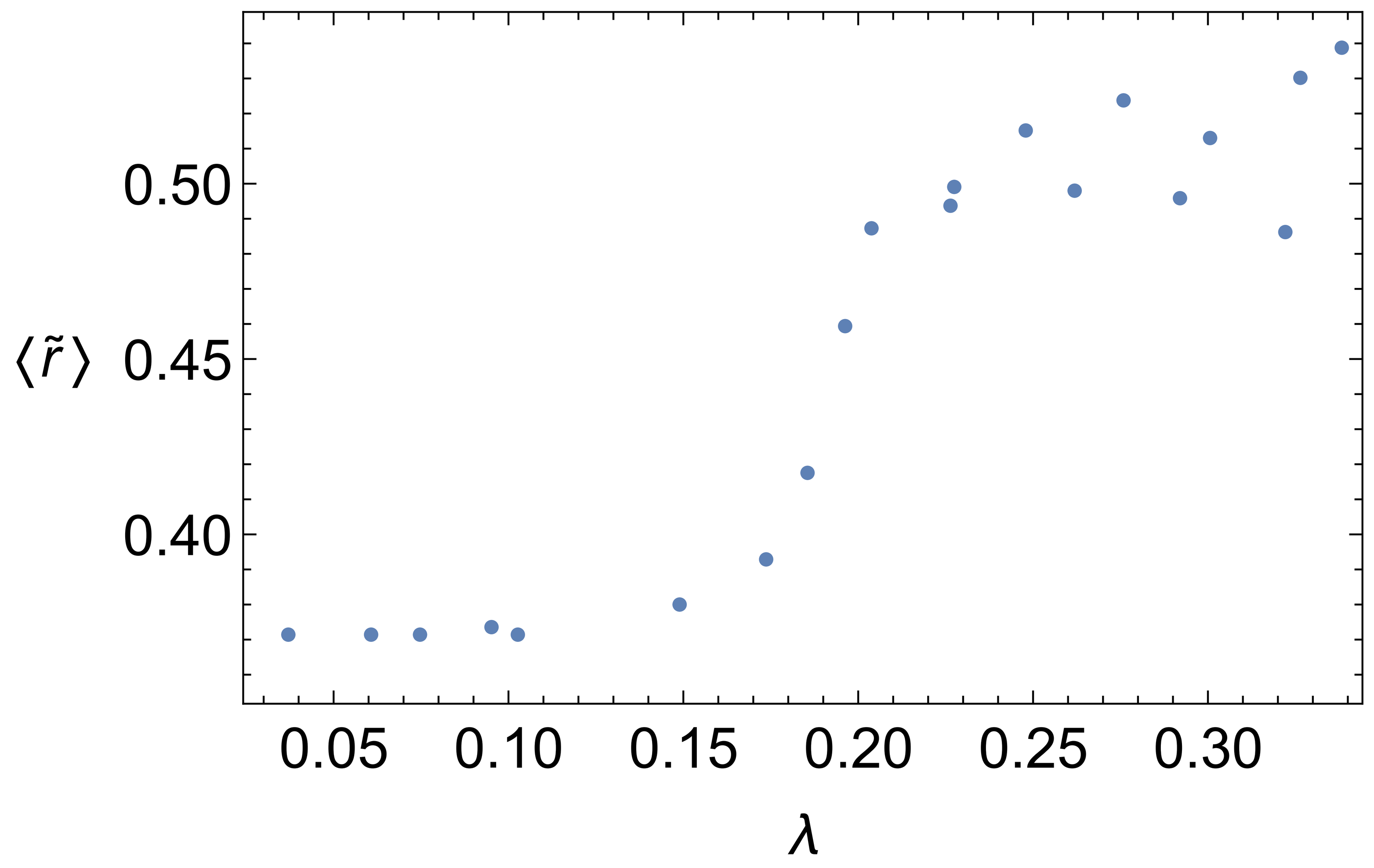
The variance becomes larger in the non-chaotic regime compared to the chaotic regime.

The Krylov complexity does not grow exponentially.

Correlation in the Sinai billiard







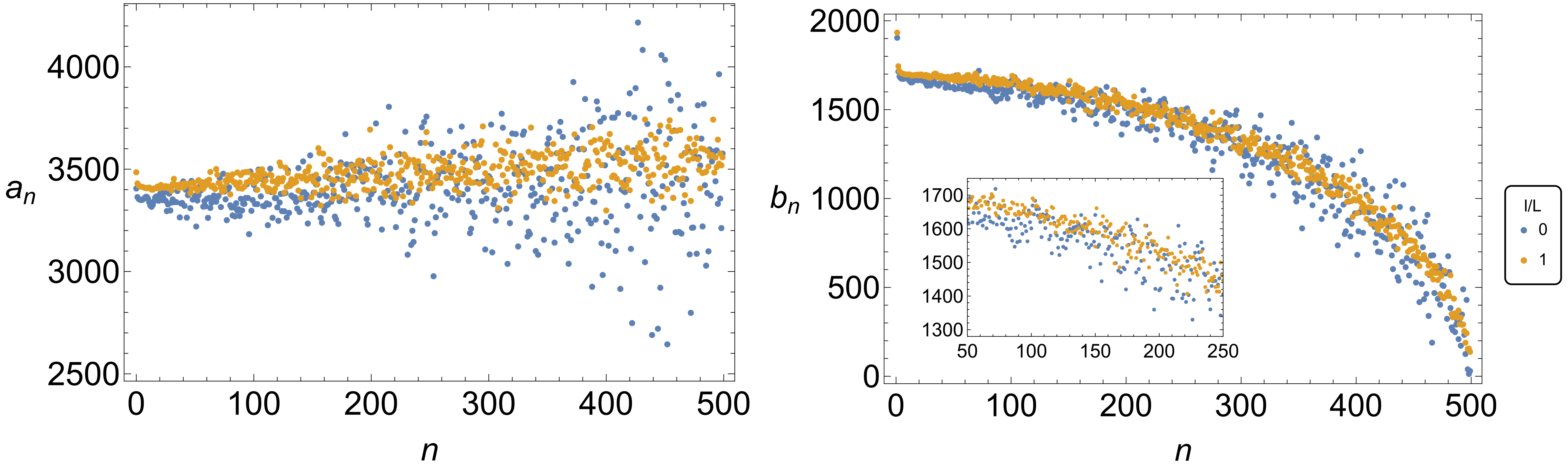
Correlation coefficients

λ vs σ^2	-0.899970
$\langle \tilde{r} \rangle$ vs σ^2	-0.872723
λ vs $\langle \tilde{r} \rangle$	0.924828

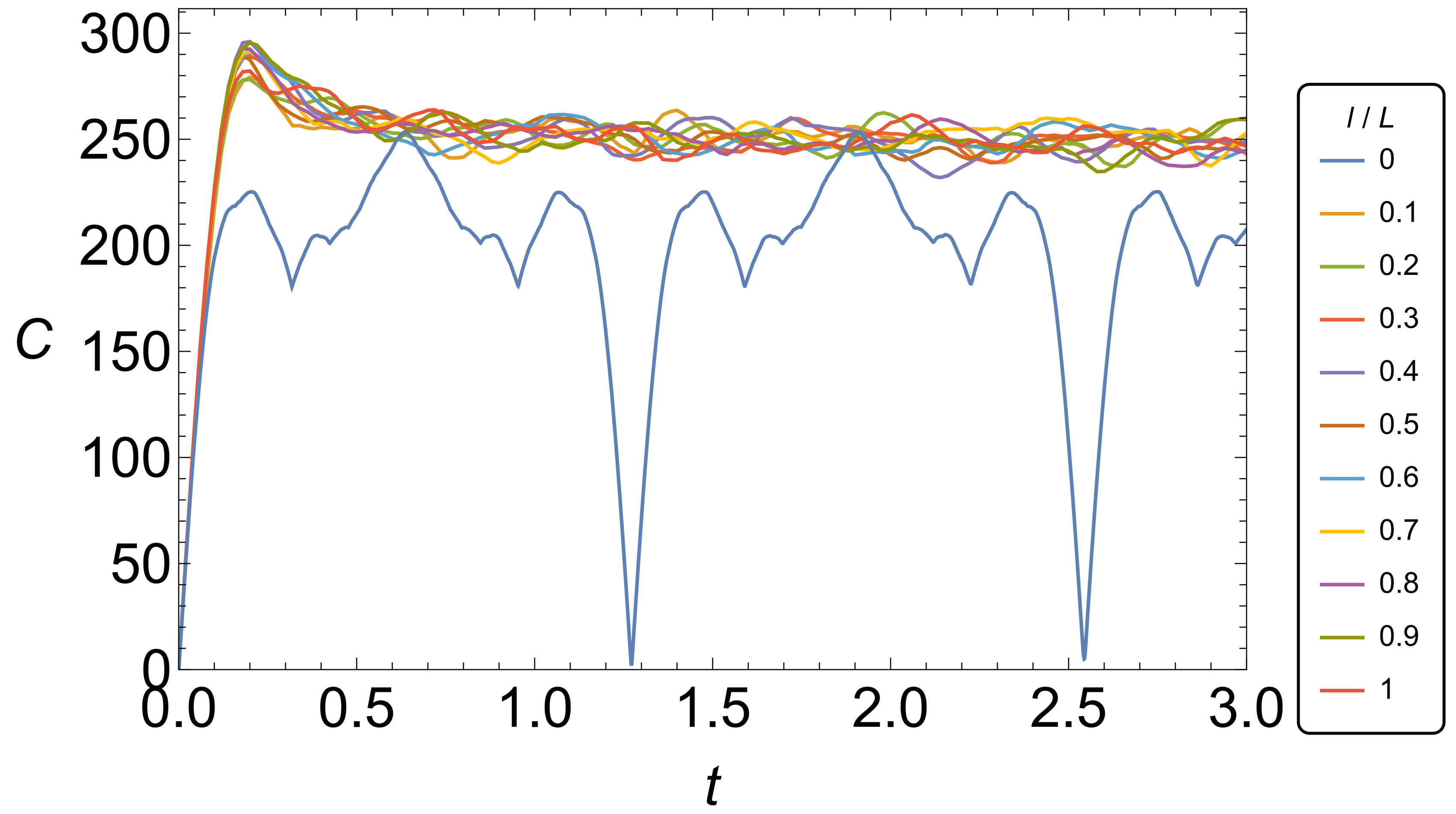
- A clear correlation exists between σ^2, λ, and $\langle \tilde{r} \rangle$.
- σ^2 can be a measure of quantum chaos.

Krylov state complexity

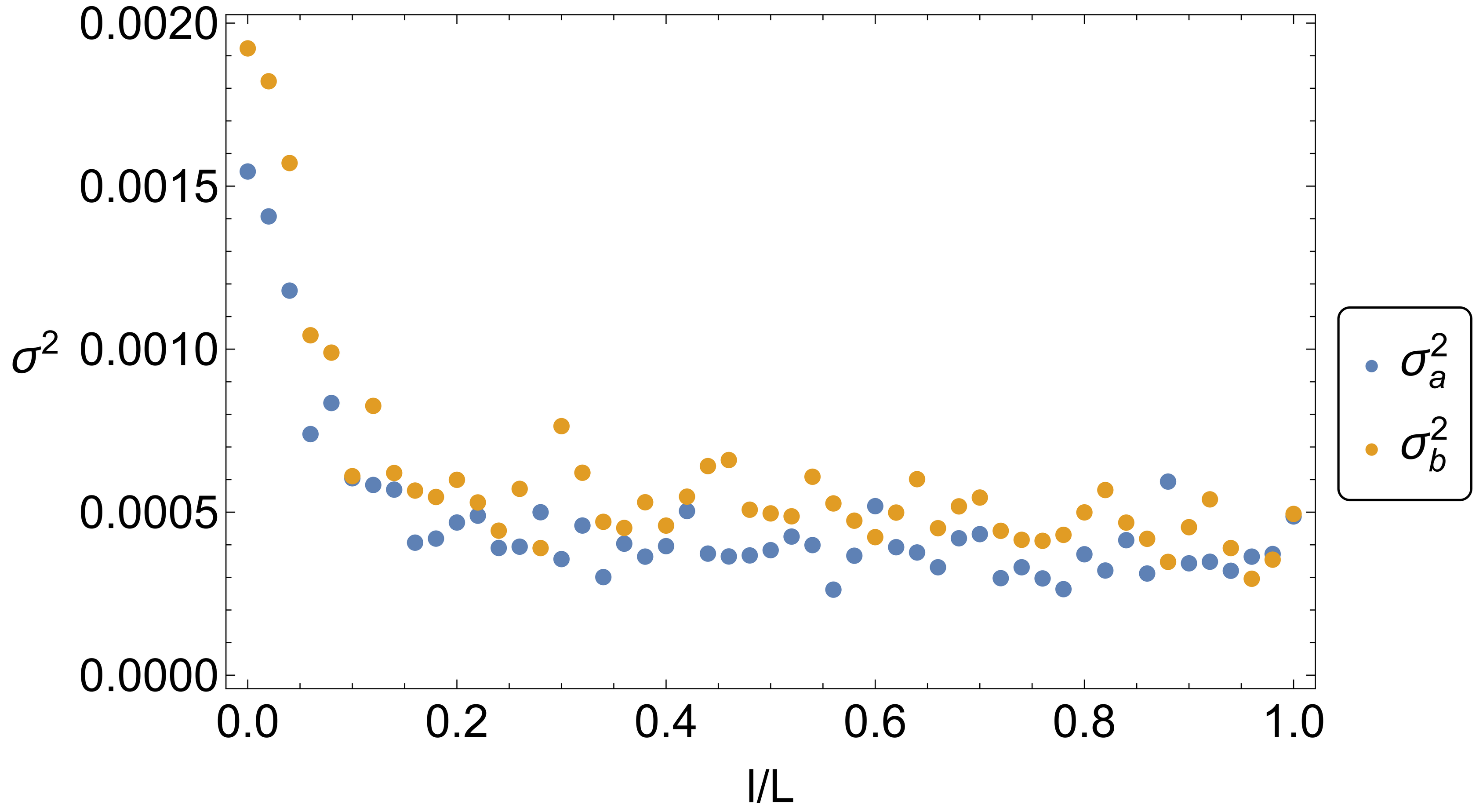
Lanczos coefficients



Krylov complexity

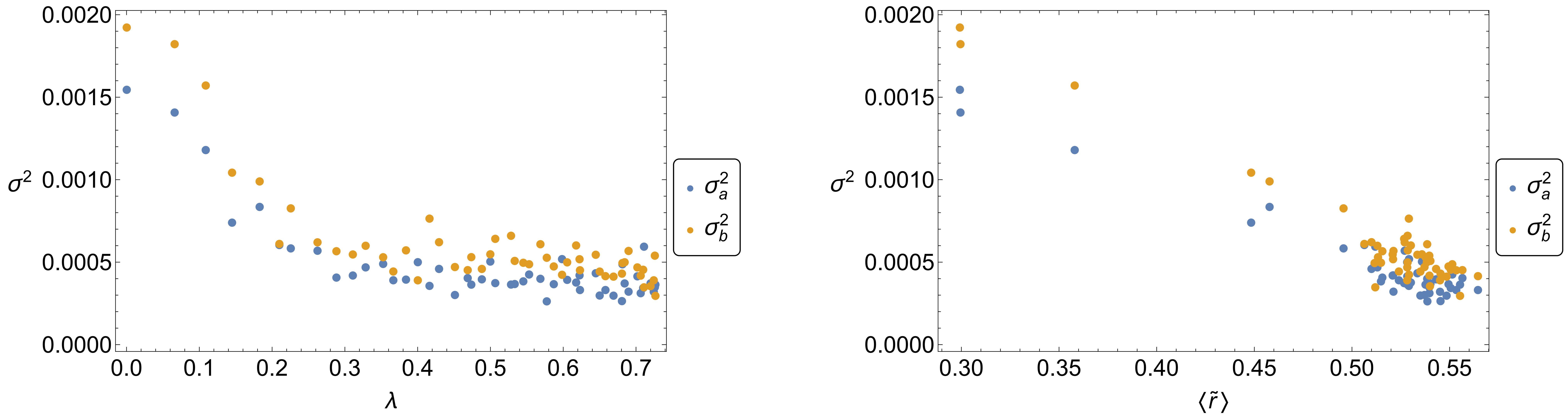


Variances



- The variance becomes larger in the non-chaotic regime compared to the chaotic regime.
- The Krylov complexity does not grow exponentially.
- The peak value of Krylov state complexity depends on a/R.

Correlation in the Sinai billiard



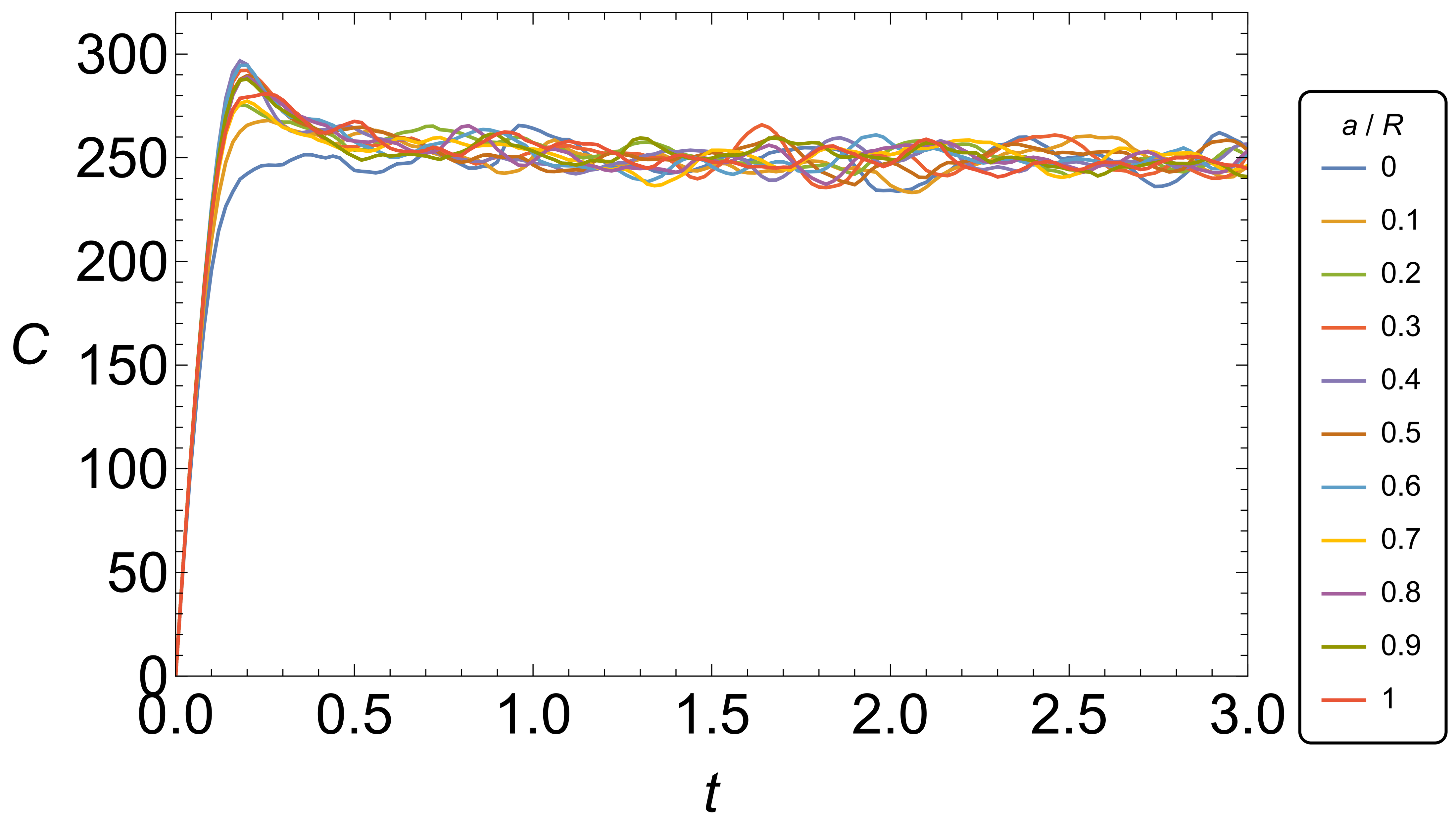
Correlation coefficients

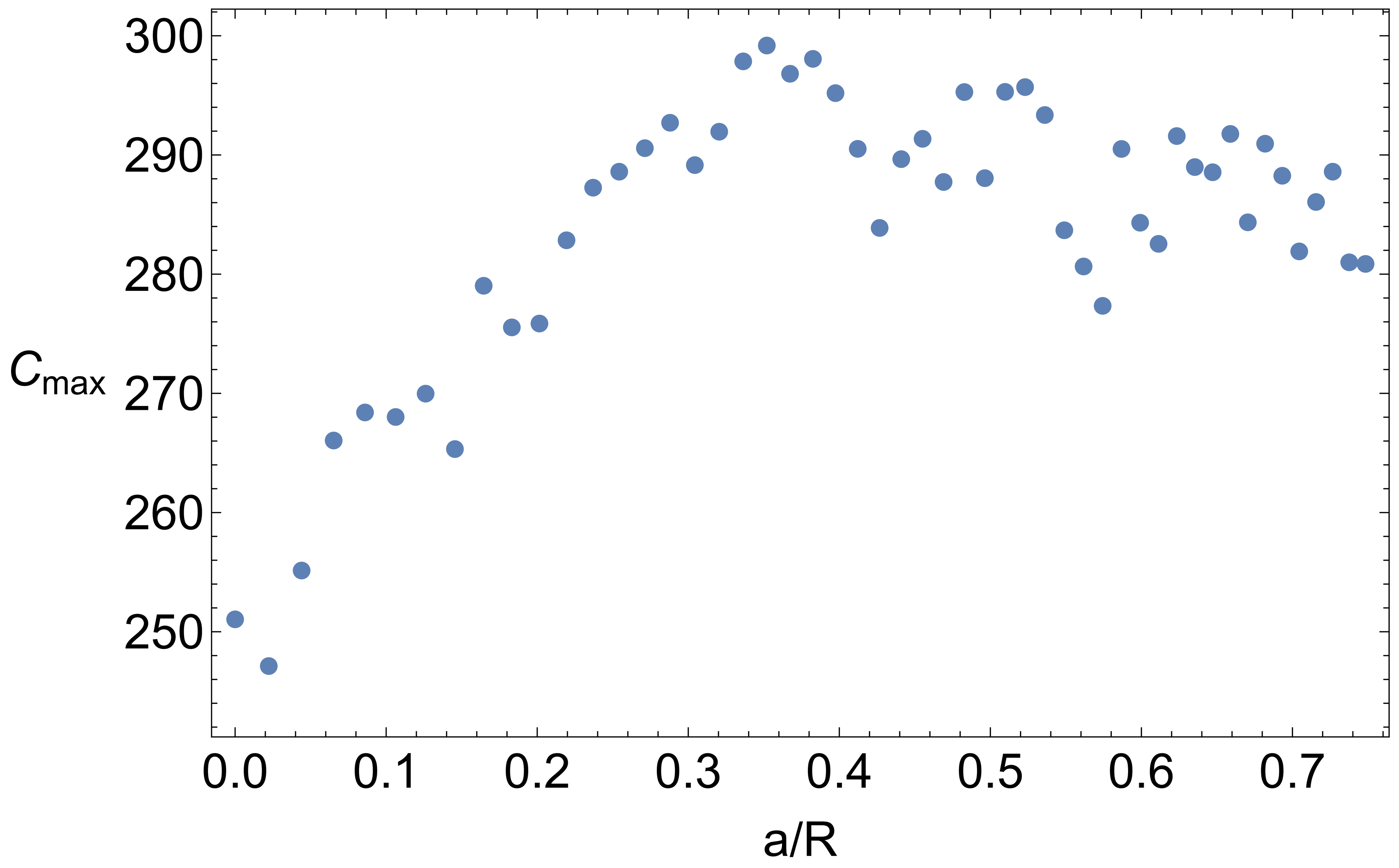
λ vs σ_a^2	-0.741803
λ vs σ_b^2	-0.757869
$\langle \tilde{r} \rangle$ vs σ_a^2	-0.965785
$\langle \tilde{r} \rangle$ vs σ_b^2	-0.962833

- A clear correlation exists between $\sigma_{a,b}^2$, λ, and $\langle \tilde{r} \rangle$.
- $\sigma_{a,b}^2$ can be a measure of quantum chaos.

The peak value of Krylov state complexity

Stadium billiard





The peak value of Krylov state complexity

Sinai billiard