UNIVERSITY OF ALBERTA

The problem of time, relational observables, and quantum clocks

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What is the Problem of Time?

Constrained systems

Obtaining the Hamiltonian: Unconstrained Systems

Lagrangian density $\mathcal{L}(q, \dot{q})$

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Replace in $p_i \dot{q}^i - \mathcal{L}(q, \dot{q})$ and write the Hamiltonian density $\mathcal{H}(p, q) = \frac{p^2}{2m} - \frac{1}{2} k q^2$

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This p_i does not appear in $\mathcal{H}$, so

$\phi_i = p_i - f(q) = 0$ is a constraint!

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$\mathcal{H} = \frac{1}{2} p_1^2 - \frac{1}{2}\left(q^1\right)^2 - q^2 p_1 + q^1 q^2 + \lambda p_2$

Further Constraints

Constraints should be preserved during evolution

$$\dot{\phi}^i = \{\phi^i, \mathcal{H}\} = 0 \Rightarrow \begin{cases} \text{determining } \lambda^i \\ \text{new constraints } \chi^j = \{\phi^i, \mathcal{H}\} = 0 \end{cases}$$

and so on ...

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In our example

$$\dot{p}_2 = \left\{ p_2, \frac{1}{2}p_1^2 - \frac{1}{2}\left(q^1\right)^2 - q^2 p_1 + q^1 q^2 + \lambda p_2 \right\} = -p_1 + q^1$$

so we get a new constraint

$$\chi = -p_1 + q^1 = 0$$

and again

$$\dot{\chi} = 0 \Rightarrow \text{no new constraint}$$

and the full Hamiltonian

$$\mathcal{H} = \underbrace{\frac{1}{2}p_1^2 - \frac{1}{2}\left(q^1\right)^2 - q^2 p_1 + q^1 q^2}_{\mathcal{H}_0 \text{ Zero Hamiltonian}} \underbrace{+\lambda p_2 + \bar{\lambda}\left(-p_1 + q^1\right)}_{\text{Constraint}}$$

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$$\dot{p}_2 = \{ \ldots + \lambda p_2 \} = -p_1 + q^1$$

so we get

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All fields of nature (EM, weak, strong, gravity) are constrained systems!

Constraints and Gauge Transformations

First class constraint: If a constraint ϕ^i commutes with all other constraints

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For any phase space function $f(q, p)$

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Example: $\nabla \cdot \mathbf{E} = 0$ in Maxwell eqs. is actually a first class constraint:

$$\delta A_\mu = \left\{ A_\mu(x), \underbrace{\int d^4y \left(\Phi(y) \frac{\partial E^\nu(y)}{\partial y^\nu} \right)}_{\text{smearing with } \Phi} \right\} = \int d^4y \, \Phi(y) \frac{\partial}{\partial y^\nu} \{A_\mu(x), E^\nu(y)\}$$

$$= \int d^4y \, \Phi(y) \delta^\nu_\mu \frac{\partial}{\partial y^\nu} \delta(x-y)$$

$$= - \int d^4y \, \frac{\partial \Phi(y)}{\partial y^\nu} \delta^\nu_\mu \delta(x-y) = -\partial_\mu \Phi = \delta A_\mu$$

and thus under gauge transformation generated by $\nabla \cdot \mathbf{E} = 0$, we get

$$A_\mu \to A_\mu - \partial_\mu \Phi$$

Dirac Observables

(Dirac) Observable: a function $O(q,p)$ which is invariant under gauge transformations

$$\{O,\phi^i\} = \delta^{(i)} O = 0$$

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Example: **E** in EM is a Dirac observable!

$$\delta E^\mu = \left\{ E^\mu(x), \int d^4y \left(\Phi(y) \frac{\partial E^\nu(y)}{\partial y^\nu} \right) \right\} = \int d^4y \, \Phi(y) \frac{\partial}{\partial y^\nu} \{E^\mu(x), E^\nu(y)\}$$
$$= 0$$

What is the Problem of Time?

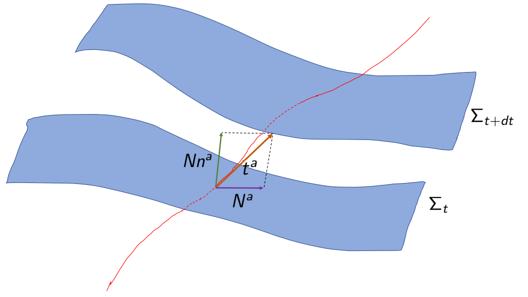
Gravity and the problem of time

GR Hamiltonian

It turns out that the vacuum GR or GR+matter is a **totally constrained** system:

$$H = \int d^3x \left(N\mathcal{H} + N^a\mathcal{D}_a + \lambda^i\mathcal{G}_i \right)$$

where



- $\mathcal{H}$: Hamiltonian constraint (1st class)
- $\mathcal{D}_a$: Diffeomorphism constraint (1st class)
- $\mathcal{G}_i$: Diffeomorphism constraint (1st class)
- N, N^a, λ^i: Lagrange multipliers
- There is **no zero Hamiltonian**
- H is nothing but a **sum of 1st class constraints**!
- Generally covariant (diffeomorphism-invariant) system
  - time reparametrization invariant

Time Evolution in GR: Pure Gauge

For any function f, time evolution in GR is

$$\dot{f} = \{f, H\} = \left\{ f, \int d^3y \left(N\mathcal{H} + N^a \mathcal{D}_a \right) \right\}$$

$$= \underbrace{\int d^3y \left(N \underbrace{\{f, \mathcal{H}\}}_{\delta^{(\mathcal{H})} f} + N^a \underbrace{\{f, \mathcal{D}_a\}}_{\delta^{(\mathcal{D})} f} \right)}_{\delta f = \text{gauge transformation!}}$$

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For a Dirac observable O, by definition

$$\dot{O} = \{O, H\} = \delta O = 0$$

Time Evolution in GR: Pure Gauge

The Problem of Time

In canonical GR (even with matter)

- All observables are constant of motion!
- There is no time evolution
- This is carried over to the quantum regime
- This is because t in GR is a pure gauge parameter: $t \to T(t)$ yields the same physics

For a Dirac observable O, by definition

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   - Measured quantity $Q(t)$, clock quantity $T(t)$; Evolution of one vs another: $Q(T)$

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   - Measured quantity $Q(t)$, clock quantity $T(t)$; Evolution of one vs another: $Q(T)$
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2. Conditional probability

$$P\left(Q = Q_0 \middle| T = T_0\right) = \frac{P\left(Q = Q_0 \cap T = T_0\right)}{P\left(T = T_0\right)} = \frac{\int_{-\infty}^{\infty} dt \mathrm{Tr}\left[\hat{\rho}\hat{\mathcal{P}}_q(t)\hat{\mathcal{P}}_{T_0}(t)\right]}{\int_{-\infty}^{\infty} dt \mathrm{Tr}\left[\hat{\rho}\hat{\mathcal{P}}_{T_0}(t)\right]}$$

$$= \frac{\int_{-\infty}^{\infty} dt \mathrm{Tr}\left[\hat{\mathcal{P}}_q(t)\hat{\mathcal{P}}_{T_0}(t)\hat{\rho}\hat{\mathcal{P}}_{T_0}(t)\right]}{\int_{-\infty}^{\infty} dt \mathrm{Tr}\left[\hat{\mathcal{P}}_{T_0}(t)\hat{\rho}\right]}$$

- $\hat{\mathcal{P}}_{T_0}(t)$ projector onto the subspace of eigenstates of $\hat{T}$ with eigenvalue T_0
- $\hat{\mathcal{P}}_q(t)$ projector onto the subspace of eigenstates of $\hat{Q}$ with eigenvalue Q_0

The Model [R. Gambini, S. Rastgoo, J. Roberts, in preparation]

- FLRW Universe

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right)$$

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- The Hamiltonian (constraint) of the system

$$\mathcal{C} = \underbrace{-\frac{6}{\gamma^2} c^2 \sqrt{|p|}}_{\text{gravity}} + \underbrace{\frac{8\pi G}{|p|^{\frac{3}{2}}} \sum_{i=1}^{2} p_{\phi_i}^2}_{\text{matter}}$$

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- Algebra

$$\{c, p\} = \frac{8\pi G\gamma}{3}, \qquad \left\{\phi_i, p_{\phi_j}\right\} = \delta_{ij}$$

Dirac Observables

EoM of the system

$$\dot{c} = \{c, N\mathcal{C}\} = -\frac{8\pi GN}{\gamma}\frac{\operatorname{sgn}(p)}{\sqrt{|p|}}\left[c^2 + \frac{4\pi\gamma^2 G}{|p|^2}\sum_i p_{\phi_i}^2\right],$$

$$\dot{p} = \{p, N\mathcal{C}\} = \frac{32\pi GN}{\gamma}c\sqrt{|p|},$$

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Remember: O is Dirac observable if $\{O, \mathcal{C}\} = 0$ so we get two Dirac observables

$$O_1 = p_{\phi_1}, \qquad\qquad O_2 = p_{\phi_2}.$$

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$$O_1 = p_{\phi_1}, \qquad\qquad O_2 = p_{\phi_2}.$$

The algebra $\left\{\phi_i, p_{\phi_j}\right\} = \delta_{ij}$, so we define the momenta conjugate to O_i as

$$\Pi_1 = -\phi_1, \qquad\qquad \Pi_2 = -\phi_2,$$

so that

$$\{O_i, \Pi_j\} = \delta_{ij}, \qquad i, j = 1, 2.$$

Dirac Observables

Now O_i look like positions and Π_i as momenta $\Longrightarrow$ new Dirac observable mimicking L_z

$$O_3 = L_3 = O_1\Pi_2 - O_2\Pi_1$$

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Now O_i look like positions and Π_i as momenta $\Longrightarrow$ new Dirac observable mimicking L_z

$$O_3 = L_3 = O_1\Pi_2 - O_2\Pi_1$$

Finally, using EoM, we can get

$$\frac{d\ln\left(|p|^{\frac{3}{2}}\right)}{d\phi_2} = \frac{3}{\gamma}\frac{cp}{p_{\phi_2}} \Rightarrow \ln\left(|p|^{\frac{3}{2}}\right) = \frac{3}{\gamma}cp\frac{\phi_2}{p_{\phi_2}} + C$$

since C is a constant, it is a Dirac observable

$$C = O_4 = \ln\left(|p|\right) - \frac{2}{\gamma}cp\frac{\phi_2}{p_{\phi_2}} = \beta\sqrt{\left(O_1^2 + O_2^2\right)}\frac{\Pi_2}{O_2}$$

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- Classical algebra

$$\{E_1(t), E_2(t)\} = \beta\sqrt{O_1^2 + O_2^2}\left(O_1\Pi_1 - O_2\Pi_2 + O_1^2t\right)$$

Quantization

T has discrete spectrum

$$\hat{T}\Psi_{m_T}(O_1, O_2) = m_T \Psi_T(O_1, O_2)$$

yields an ugly eigenstate

$$\Psi_{m_T, \sqrt{o_1^2+o_2^2}}(O_1, O_2) = \frac{1}{\sqrt{2\pi\hbar}} \sqrt{\frac{\operatorname{sgn}(O_2)}{\sqrt{o_1^2+o_2^2}}} \delta\left(\sqrt{O_1^2+O_2^2} - \sqrt{o_1^2+o_2^2}\right) \times$$
$$\exp\left[-\frac{i}{2\hbar} O_1^2 t\right] \exp\left[\pm \frac{i}{\hbar} m_T \tan^{-1}\left(\frac{O_1}{O_2}\right) \operatorname{sgn}(O_2)\right]$$

working with quantum clock!

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working with quantum clock!

- Small o_1, o_2: eigenvalues of $\hat{O}_1$ and $\hat{O}_2$

Quantization

E_2 has continuous spectrum

$$\hat{E}_2 \Psi_{e_2}(O_1, O_2) = e_2 \Psi_{e_2}(O_1, O_2)$$

yields another ugly eigenstate

$$\Psi_{e_2, o_2}(O_1, O_2) = \frac{1}{\sqrt{2\pi\beta\hbar o_2}} \frac{\delta(O_2 - o_2)}{\left(O_1^2 + O_2^2\right)^{\frac{1}{4}}} \exp\left(-\frac{i}{2\hbar}\left(O_1^2 t \mp \frac{2e_2 \tanh^{-1}\left(\frac{O_1}{\sqrt{O_1^2+O_2^2}}\right)}{\beta O_2}\right)\right)$$

Probability

Conditional probability of $E_2 \in \left[e_2^{(1)}, e_2^{(2)}\right]$ given that $T = m_T$ is expressed as

$$P\left(E_2 \in \left[e_2^{(1)}, e_2^{(2)}\right] \left| T = m_T\right.\right) = \frac{\int_{-\infty}^{\infty} dt \mathrm{Tr}\left[\hat{\mathcal{P}}_{e_2}(t)\hat{\mathcal{P}}_{m_T}(t)\hat{\rho}\hat{\mathcal{P}}_{m_T}(t)\right]}{\int_{-\infty}^{\infty} dt \mathrm{Tr}\left[\hat{\mathcal{P}}_{m_T}(t)\hat{\rho}\right]}$$

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- Projection operator for $\hat{E}_2$

$$\hat{\mathcal{P}}_{\mathrm{e}_2^{(0)}}(t) = \int_{\mathrm{e}_2^{(0)}-\Delta\mathrm{e}_2}^{\mathrm{e}_2^{(0)}+\Delta\mathrm{e}_2} d\mathrm{e}_2 \int_{-\infty}^{\infty} do_2 \left|\mathrm{e}_2, o_2, t\right\rangle \left\langle \mathrm{e}_2, o_2, t\right|$$

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- Projection operator for $\hat{T}$

$$\hat{\mathcal{P}}_{m_T^{(0)}}(t) = \int_{-\infty}^{\infty} do_1 \int_{-\infty}^{\infty} do_2 \left|m_T^{(0)}, \sqrt{o_1^2 + o_2^2}, t\right\rangle \left\langle m_T^{(0)}, \sqrt{o_1^2 + o_2^2}, t\right|$$

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- Density operator $\hat{\rho} = |\psi_\rho\rangle\langle\psi_\rho|$ with

$$|\psi_\rho\rangle = \int_{-\infty}^{\infty} dO_1 \int_{-\infty}^{\infty} dO_2 \Theta\left(O_1 - o_1^{(1)}\right)\Theta\left(o_1^{(2)} - O_1\right)\times \\ \Theta\left(O_2 - o_2^{(1)}\right)\Theta\left(o_2^{(2)} - O_2\right) N_\rho |O_1, O_2\rangle,$$

Probability: Preliminary Results

Yields

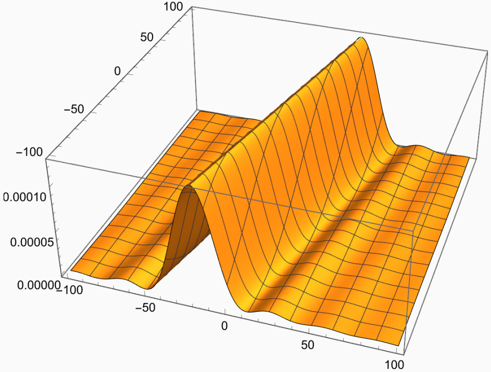
$$P\left(E_2 \in \left[e_2^{(1)}, e_2^{(2)}\right] \middle| T = m_T\right) \approx \frac{16\beta\Delta e_2 \Delta o_r}{\pi\left(\left(o_r^{(2)}\right)^2 - \left(o_r^{(1)}\right)^2\right)} \left(\frac{o_r^{(0)}\cos^2\left(o_\theta^{(0)}\right)}{e_2^{(0)} - m_T\beta o_r^{(0)}\cos^2\left(o_\theta^{(0)}\right)}\right)^2 \times$$

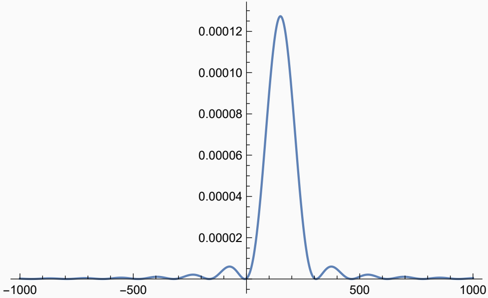
$$\sin^2\left[\frac{\Delta o_\theta}{\hbar}\left(\frac{e_2^{(0)}}{\beta o_r^{(0)}\cos^2\left(o_\theta^{(0)}\right)} - m_T\right)\right]$$

where

- $o_r^{(0)}$: central value of $O_r = O_1^2 + O_2^2$
- o_θ: central value of $O_\theta = \tan^{-1}\left(\frac{O_1}{O_2}\right)$
- $e_2^{(0)}$: central value of E_2
- ΔX : interval around X
- β: a constant including G

Probability: Preliminary Results





Summary

- Time is an illusive concept in physics: probably an emergent phenomenon
- Absolute time t in unphysical; we never have access to it, only to relation between physical quantities
- No absolute-time evolution in totally constrained systems (GR+matter)
- On quantum gravity scales, probably time does not exists, it emerges as relations between quantum observables as an approximation
- We can thus take the conditional probability interpretation and use evolving constants of motion to formulate a relational time via physical clocks
- This probability seems to agree with what we know of time
- We studied this in the context of cosmology (preliminary) and will extend the study

Conditional Probability

- Conditional probability (continuous Q, discrete T)

$$P\left(Q \in [q_1, q_2] \,\middle|\, T = T_0\right) = \frac{\int_{-\infty}^{\infty} dt \mathrm{Tr}\left[\hat{\mathcal{P}}_q(t)\hat{\mathcal{P}}_{T_0}(t)\hat{\rho}\hat{\mathcal{P}}_{T_0}(t)\right]}{\int_{-\infty}^{\infty} dt \mathrm{Tr}\left[\hat{\mathcal{P}}_{T_0}(t)\hat{\rho}\right]}$$

- Interpretation of numerator:
    - Ensemble of noninteracting systems with two quantum variables Q and T, each to be measured.
    - Each system equipped with a recording device that takes a single snapshot of Q and T at a random unknown value of the ideal time t.
    - Take a large number of such systems, initially all in the same quantum state, wait for a "long time" and concludes the experiment.
    - Recordings taken by the measurement devices are then collected and analyzed all together.
    - Computes how many times $n\left(T = T_0, Q \in [q_1, q_2]\right)$ each reading with a given value $T = T_0,\ Q \in [q_1, q_2]$ occurs
    - Take each of those values $n\left(T = T_0, Q \in [q_1, q_2]\right)$ and divides them by the number of systems in the ensemble; in the limit of infinite systems, a joint probability is given.

Ashtekar formulation

- Tetrad formulation of gravity action

$$S = \underbrace{\underbrace{\int \text{bulk integral}}_{\text{Hilbert-Palatini}} + \frac{1}{\gamma} \underbrace{\int \text{boundary term}}_{\text{Nieh-Yan term}}}_{\text{written in terms of tetrads } g_{ab} = \eta_{IJ} e^I_a e^J_b}$$

with γ = Barbero-Immirzi parameter

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$$A^i_a = \underbrace{\Gamma^i_a}_{\text{spin connection}} + \gamma \underbrace{K^i_a}_{\text{extrinsic curvature}}$$

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  - Momenta: inverse triads E^a_i, where spatial metric is $q_{ab} = \eta_{ij} E^i_a E^j_b$

Cosmology

- FLRW Universe

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  with $a(t)$ =scale factor

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The Hamiltonian (constraint) of the system

$$\mathcal{C} = -\frac{6}{\gamma^2}c^2\sqrt{|p|}$$