Cohomology of heavy/light moduli spaces + Quasimaps

Siddarth Kannan

[joint w Stefano Serpente + Claudia Yun]

+

[joint w Terry Song, in progress]

The space $\overline{\mathcal{M}}_{g,\min}$

The space $\overline{\mathcal{M}}_{g,m|n}$

moduli space of $\left(C,\ (p_i)_{i=1}^m,\ (q_i)_{i=1}^n\right)$

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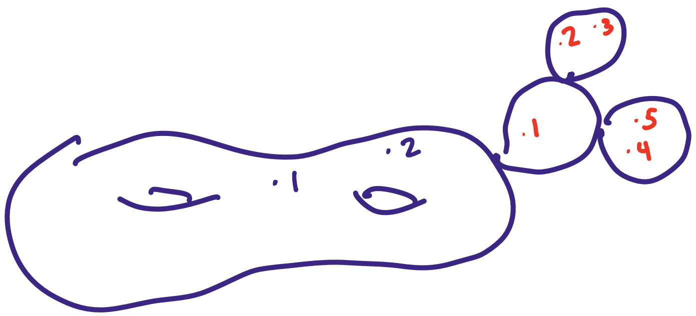

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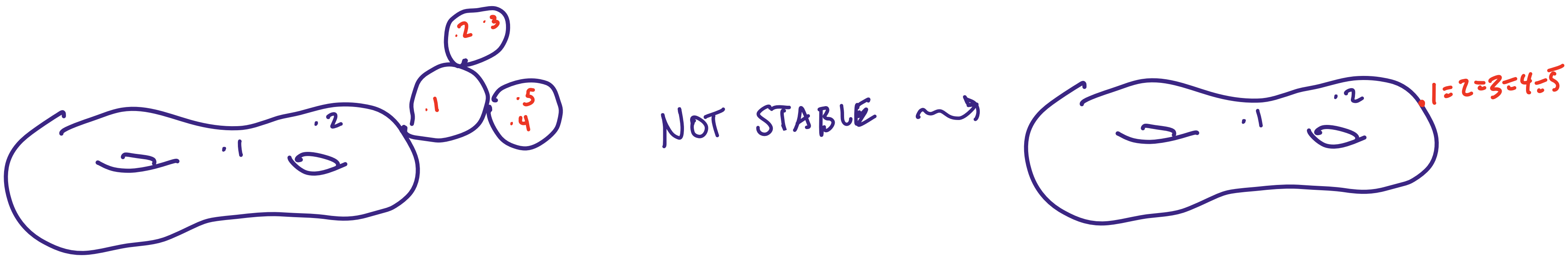

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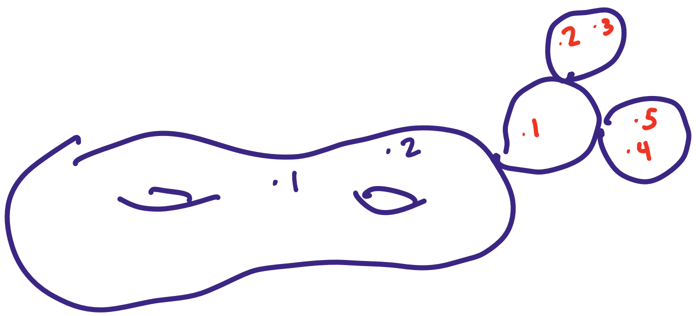

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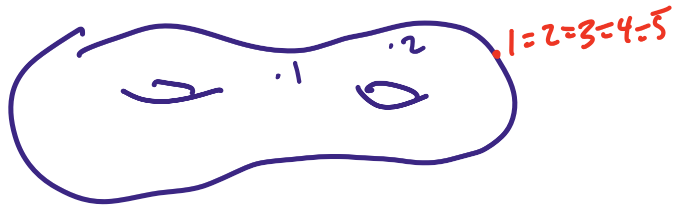

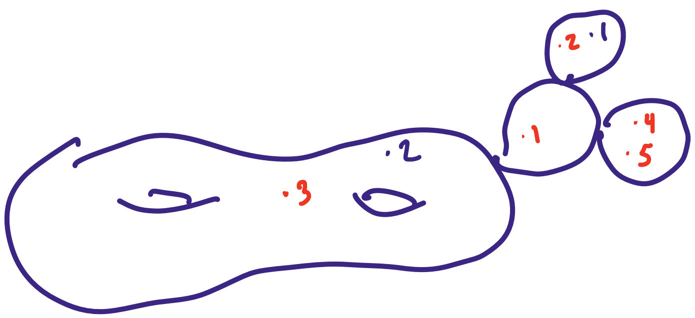

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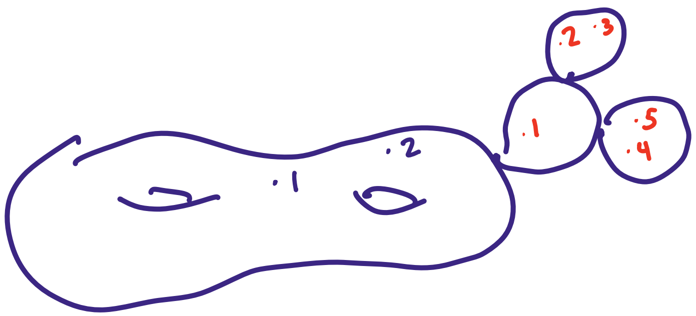

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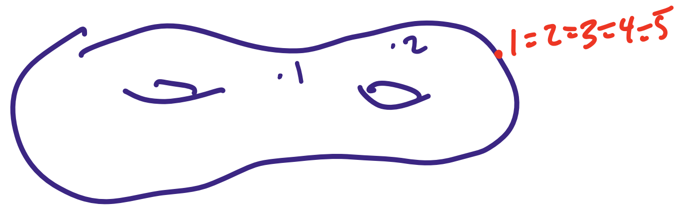

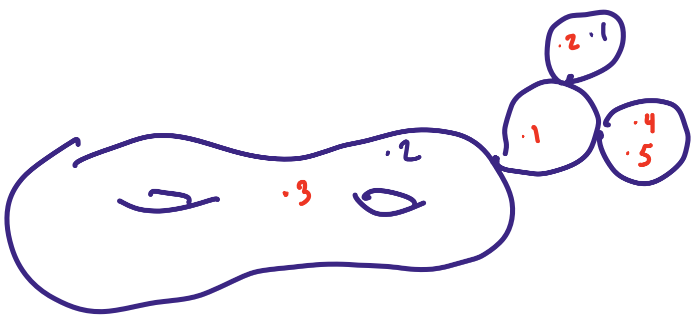

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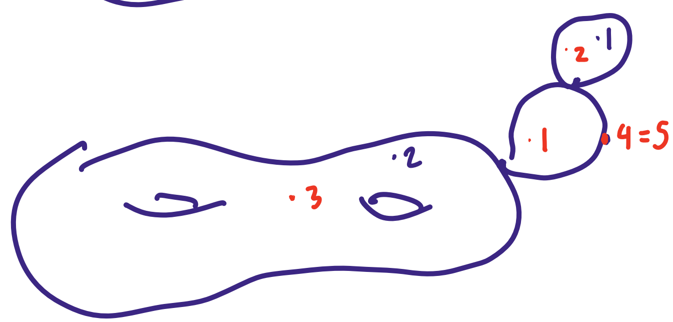

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  ☆ $\overline{\mathcal{M}}_{g,0|n}$ of particular interest ☆

- $g=0 \Rightarrow$ Kapranov blowup construction of $\overline{\mathcal{M}}_{0,n}$ / Losev-Manin moduli space

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2) Are there applications?

$$\{ H^k(\overline{\mathcal{M}}_{g,n};\mathbb{Q}) \circlearrowleft S_n \}_{k,n}$$

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$$
\begin{tikzcd}
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\left\{ H^k_c(\underset{[\text{virtual}]}{\mathcal{M}_{g,n}};\mathbb{Q}) \circlearrowleft S_n \right\}_{k,n} & & \left\{ H^k_c(\mathcal{M}_{g,m|n}) \circlearrowleft S_m \times S_n \right\}_{m,n,k}
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[all g at once]

$\updownarrow$ symmetric functions + graph combinatorics + mixed Hodge theory

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[virtual]

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[Getzler - Kapranov, Modular operads]

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Main result

[K.-Serpente-Yun]

$$\left\{ H^k(\overline{\mathcal{M}}_{g,n};\mathbb{Q}) \circlearrowleft S_n \right\}_{k,n} \xleftrightarrow[\text{[one } g \text{ at a time]}]{} \left\{ H^k(\overline{\mathcal{M}}_{g,m|n}) \circlearrowleft S_m \times S_n \right\}_{m,n,k}$$

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+

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[one g at a time]

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+
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$\overline{\mathcal{M}}_{0,n}$ geometry
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partition combinatorics

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Formulas in terms of GENERATING SERIES for (BI) SYMMETRIC FUNCTIONS, involving PLETHYSM + HOPF ALGEBRA structure

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Wide open! Ask me

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② Compute $e(\overline{\mathcal{M}}_{g,n})$, $e(\mathcal{M}_{g,n})$ when $e(\overline{\mathcal{M}}_{g,0|n})$, $e(\mathcal{M}_{g,0|n})$ known.

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③ $$\lim_{n \to \infty} \frac{\chi(\overline{\mathcal{M}}_{1,0|n})}{\chi(\overline{\mathcal{M}}_{1,n})} = 0, \text{ exponentially fast.}$$

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③ Can virtual $H_c^\circ(\mathcal{M}_{g,m|n})$ be computed $(S_m \times S_n)$-equivariantly, using tropical techniques, as Chan-Faber-Galatius-Payne do for virtual $H_c^\circ(\mathcal{M}_{g,n})$?

Stable maps and Quasimaps!

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[roughly, allow maps to acquire basepoints & deform stability condition]

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(invertible transformation between generating series)

Wall-crossing formulas [Givental] [Fontanine-Kim]

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[interiors of weighted stable map spaces constructed by Alexeev-Guy, Mustafa-Mustata]

With Terry Song, in progress.

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Are computations easier on the Quasimap side?

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[$g=0$: recursive formula for $\left\{ H^k\left(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)\right) \circlearrowleft S_n \right\}_{n,k,d}$ due to Getzler–Pandharipande.]

END