« "Exotic 4manifolds & KSBA surfaces" »

(from joint paper "Exotic surfaces" with Javier Reyes)

Banff TCPL 201

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13:30 - 14:20

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I A surface of general type is
- a nonsingular projective variety of dimension 2 over $\mathbb{C}$
- with a canonical class K s.t. $K^2>0$ and $K\cdot\Gamma\geq 0$ for any complex curve Γ.

Coarse moduli invariants are K^2 and $\chi = 1 - \underset{\underset{q}{\parallel}}{h^0(\Omega^1)} + \underset{\underset{p_g}{\parallel}}{h^0(\Omega^2)}$.
[by the Noether formula : $\chi_{top} = 12\chi - K^2$]

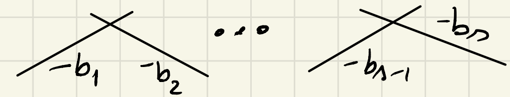
KSBA surfaces are singular surfaces which allow us to have a compactification $\overline{M}_{K^2,\chi}$ of $M_{K^2,\chi}$ = moduli space of surfaces of general type with those inv. fixed.
$\underbrace{\text{(Kollár - Shepherd-Barron}}_{1988}$ - $\underset{1994}{\text{Alexeev)}}$ $\underset{1977}{\text{(Gieseker)}}$

[This is analogue to $M_g \subset \overline{M}_g$ by Deligne-Mumford.]

II $\underline{\text{Today}}$: we will only care about singular surfaces W with
(1) Wahl singularities : $\frac{1}{n^2}(1, na-1)$ $\gcd(n,a)=1$
(2) K_W ample : $W \xhookrightarrow{mK_W} \mathbb{P}^N$.

Wahl sing. $\equiv$ c.q.s which admit a complex smoothing with $b_2 = 0$.
Milnor #

$\parallel\!|$

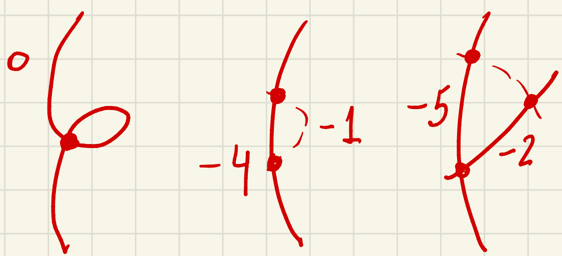
Its minimal resolution has exceptional divisor 

where $\frac{n^2}{na-1} = b_1 - \cfrac{1}{\ddots - \cfrac{1}{b_s}} = [b_1, \dots, b_s]$ s.t.

it is obtained via $[e_1, \dots, e_r] \longrightarrow [e_1+1, e_2, \dots, e_r, 2]$

$\longrightarrow [2, e_1, \dots, e_{r-1}, e_r+1)$

starting with $[4]$.



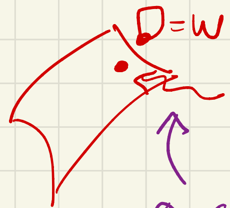
$[4]$ $\quad$ $[5,2]$

$\leftarrow [2,5,3] \leftarrow \cdots$

$\leftarrow [6,2,2] \leftarrow \cdots$

these Wahl chains appear just in too many places...

III **obs!**- KSBA surfaces with one wahl sing. typically produce divisors in $\overline{M}_{K^2,\chi}$.



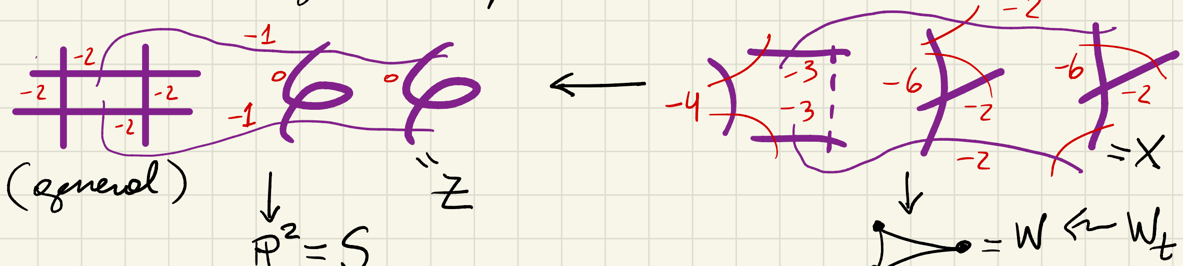
W_t = surface of general type with
$K_W^2 = K_{W_t}^2$, $\chi(\mathcal{O}_W) = \chi(\mathcal{O}_{W_t})$

$\mathbb{Q}$-Gorenstein smoothing of W

Application : Answer questions on existence of surfaces of general type with particular invariants. [Lee-Park 2007]

key : W may be rational, so hopes to construct.
Difficulty : complex smoothings.

Ex :



K_{W_t} ample
$K_{W_t}^2 = 1$
$\pi_1(W_t) = 1$
$p_g = q = 0$

Diagram in general:

$$
\begin{tikzcd}
 & & X \arrow[dl] \arrow[ddll, bend right, "\pi"'] \arrow[dd, "\phi = \text{min res. of local sing}"] \arrow[ddrr, bend left, "\text{Rational blowdown instead.}"] & & \\
 & Z \arrow[dl] & & & \\
S & & W & W_t \arrow[l, "?"'] & Y
\end{tikzcd}
$$

S with no (-1)-curves

IV **Theorem**: W KSBA surface $\Rightarrow$ RBD Y is a minimal symplectic 4-manifold. If $\pi_1(Y) = 1$ and intersection form on Y is odd then Y is an exotic $(2p_g - 1)\mathbb{P}^2_{\mathbb{C}} \# (10p_g + 9 - K^2)\overline{\mathbb{P}}^2_{\mathbb{C}}$ with $10p_g + 9 - K^2 > 0$. (so $1 \leq K^2 \leq 8 + 10p_g$)

Key property: $\pi(\mathrm{Exc}(\phi)) \subset S$ is a non-empty configuration of rational curves, and possibilities are:

1. S is rational.
2. S K3 or Enriques.
3. S $\kappa = 1$ & $b_1(S) = 0$.
4. S general type, $b_1(S) = 0$, $K_S^2 < K_W^2$.

Example: Horikawa type then 3.

$\underline{V}$ Rest : Let us assume that W is rational $\Rightarrow$ Y exotic $Bl_n(\mathbb{P}^2_{\mathbb{C}})$.
$1 \leq n \leq 8$

What's known? For $n > 1$, there is exotic $Bl_n(\mathbb{P}^2_{\mathbb{C}})$
For $n > 4$, " " " " via RBD.

Thm (Reyes-U, 2022) There are exotic $Bl_4(\mathbb{P}^2_{\mathbb{C}})$. (and "many")

$\boxed{Q}$: How to find the right config. of rational curves?

$$\left\{ \begin{matrix} \text{config. of rational} \\ \text{curves on } S \end{matrix} \right\} \overset{?}{\longleftrightarrow} \left\{ \begin{matrix} W \text{ s.t. RBD} \\ Y \text{ is exotic } Bl_n(\mathbb{P}^2) \end{matrix} \right\}$$

There is a way to organize configurations via chern classes associated to pairs (Z, A).

Point : For surfaces, $c_1^2/c_2 = K^2/\chi_{top}$ high means hard to find, hard to classify.
(e.g. obstructed in deformations, moduli smaller)

$\bar{VI}$

$$\begin{array}{ccc} Z' & \xrightarrow[\text{min log res}]{\sigma} & Z \\ \cup & & \cup \\ D = \sigma^*(A)_{red} & & A \end{array}$$

where D is a SNC divisor on Z'
(ie non-sing components & only nodes)

$\Rightarrow \Omega^1_{Z'}(\log D)$ and $c_i(\Omega^1_{Z'}(\log D)^\vee)$

$\Rightarrow \bar{c}_1^2 = c_1 \cdot c_1 = (K_{Z'} + D)^2 \qquad \bar{c}_2 = c_2 = \chi_{top}(Z') - \chi_{top}(D)$

<u>Example</u>: $Z = \mathbb{P}^2$, $A = \{L_1, \dots, L_d\}$ lines

m-point is a $p \in A$ which belongs to exactly m-lines

$t_m = \#\{m\text{-points in } A\}$

Then
$$\left.\begin{aligned} \bar{c}_1^2 &= 9 - 5d + \sum_{m \geq 2} (3m-4) t_m \\ \bar{c}_2 &= 3 - 2d + \sum_{m \geq 2} (m-1) t_m \end{aligned}\right\} \begin{aligned} &> 0 \\ &> 0 \end{aligned} \quad \text{if } t_d = t_{d-1} = 0.$$

<u>Thm</u>: $\bar{c}_1^2 / \bar{c}_2 \in [1, 3]$ combinatorially, over $\mathbb{C}$ $\bar{c}_1^2 / \bar{c}_2 \in [1, \frac{8}{3}]$

and $\bar{c}_1^2 / \bar{c}_2 = \frac{8}{3} \iff A$ is the dual Hesse arrangement

16.777.216
1 week

VII **The point**: we can relate $\bar{c}_1^2, \bar{c}_2$ to K_W^2, ℓ starting with a nodal config not points with only m-points

$$S \leftarrow (Z, A) \longleftarrow X$$

and so restrictions on $\bar{c}_1^2, \bar{c}_2$ become constraints for K_W^2, ℓ.

Thm: (via log BMY) $K_W^2 \leq 12 - \frac{2}{3}\ell - \frac{r}{3} - \frac{1}{3}K_Z^2 + \sum_{m \geq 3}(m-2)t_m^\circ$

where $t_m^\circ = \#\{\text{of excep. } \underline{\text{NOT}} \text{ in any Wahl chain}\}$.

Moreover:

→ Rana, U / Evans, Smith: we have length $\leq 4K_W^2 + 1$ if W not rational. (open the rational case)

→ Figueroa, Rana, U: $\sum_{i=1}^{\ell}(\text{length} - 1) \leq 2(K_W^2 - K_S^2) - K_S \cdot \pi(C)$

& more ... (coming up 2023)

→ For $n \leq 4$ it must be obstructed in deformations as

$$\text{obstr} = h^2(T_W) = \ell + h^1(T_W) + 2K_W^2 - 10$$

So "unobstructed" config. would not work.

→ K_W ample (or big & nef) is very subtle, there are necessary conditions; also to compute $\pi_1(Y) = 1$.

→ $2r = K_W^2 - K_Z^2 + \ell + \sum_{k \geq 2}(k-1)t_k - \sum_{k \geq 3}(k-2)t_k^0$

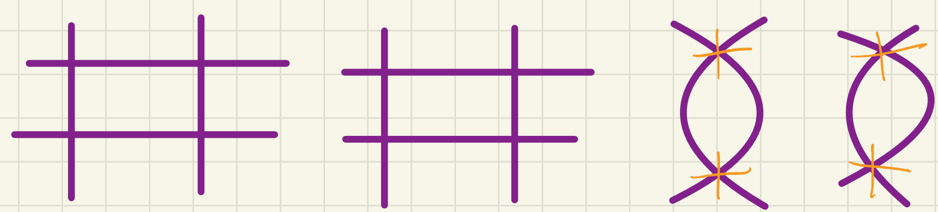
so $K_Z^2 = 0$, $t_k^0 = 0 \ \forall k \Rightarrow 2r = K_W^2 + \ell + \sum(k-1)t_k$

If ℓ small and want K_W^2 high $\Rightarrow$ r high and long wahl sing.

→ Big problem: Combinatorics to get wahl chains.

VIII What did we do for $Bl_4(\mathbb{P}^2_{\mathbb{C}})$ exotic?

(1) Consider a configuration with many special curves



+ the 8 sections
+ 4 double sections (creating multiple points at the I_2)

This is a configuration of $12+8+4=24$ $\mathbb{P}^1$ with $\bar{c}_1^2/\bar{c}_2 = 2.2\overline{16}$.

In $\mathbb{P}^2$, these are 12 lines and 2 conics.

(2) Write a computational algorithm which considers blowups at all possible subconfigurations, that verifies all the above. (obs: there is no other way) Do computer searches.
(we used 80 cores)

By the way

n	8	7	6	5	4	≤ 3
$\approx$ # of W $\ell=2$	6	433	2100	693	7	none yet

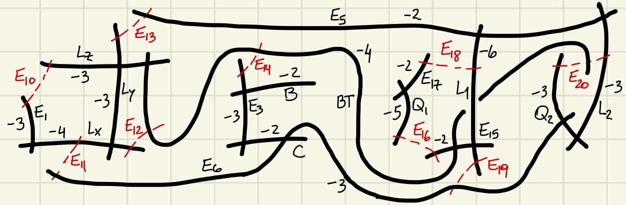
+ many with QHD sing for all $n \geq 4$.

One example for $n=4$: here $r=14$, $\ell=2$, $\bar{c}_1^2/\bar{c}_2 = \frac{7}{3} = 2.\overline{3}$

(note $K_W^2 \leq 6 + \frac{1}{3}$ since $t_3^0 = 1$)

$\bar{c}_1^2 = 14$
$\bar{c}_2 = 6$

$X \supset$



we have two wahl chains

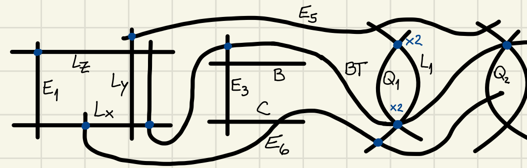
$[5,2]$

$[3,4,3,3,4,2,6,2,3,3,3,2,3,2]$

length $= 14$

$\downarrow \varphi'$

$Z \supset$



$\downarrow$

$S = \mathbb{P}^2$

$L_x,\ L_y,\ L_z$

$B = \{x+y+z=0\}$ $\quad C = \{x-y+z=0\}$

$Q_1 = \{(x+z)^2 - y(x-z) = 0\}$

$Q_2 = \{(x+z)^2 + y(x-z) = 0\}$

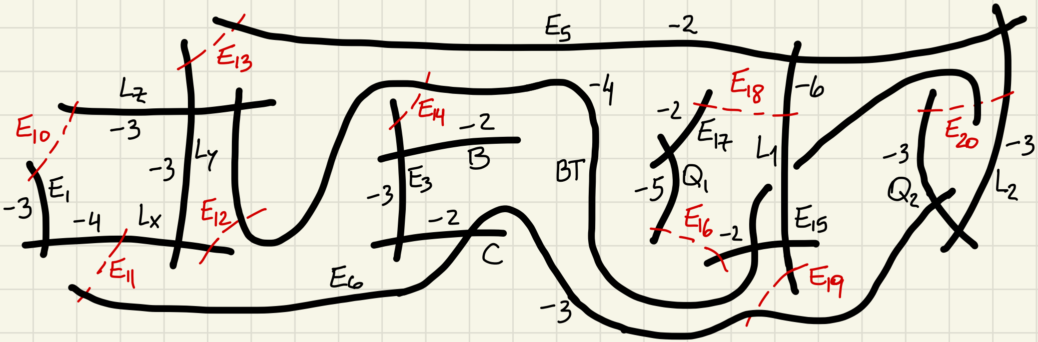
$L_1 = \{x+y-z=0\}$

$L_2 = \{x-y-z=0\}$

$BT = \{x - iy + z = 0\}$

8 lines and 2 conics with special tacnodes and simple points with complex coordinates.

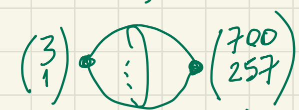
$X =$



$\longrightarrow W$

(2 sing)

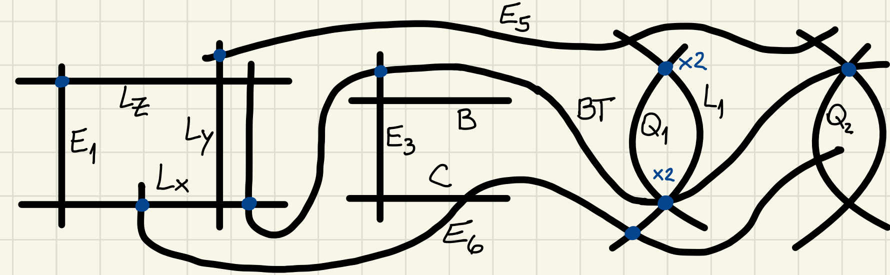
$\parallel$



$h^1(T_W) = 0 \quad h^2(T_W) = 2$

$\downarrow \varphi'$

$Z =$



$\longrightarrow \mathbb{P}^2_{\mathbb{C}}$

How to find such configurations & why do they work?

many from rational elliptic fibrations, although sections

That is even harder:

and fibers are not enough
(and geometry of sections is
hard in general)
Double, triple sections are
needed.

a right conf. satisfying
all requirements may
give nothing

(MMP give you more configurations
not even coming from eliptic fibrations)

$\boxed{\text{what is the right combinatorics for KSBA surfaces ?}}$

Computer searches:

- we have a computer program which is fed with a config. of curves (as above) and considers all subconf. & all blow-ups to find KSBA surfaces.
- we use a cluster with 80 cores.
- there are just too many conf. to try
- For $\mathbb{CP}^2 \# 5 \overline{\mathbb{CP}}^2$ we have hundreds of examples.

$\therefore$ This is a new world to be explored!