Asymptotic stabilizability[1]

Matthew Kvalheim

kvalheim@umich.edu

June 13, 2023

[1]Funding by the Army Research Office (MURI W911NF1810327) and the Office of Naval Research (VBF N00014-16-1-2817) is gratefully acknowledged.

Outline

Obstructions to stabilization

- Generalization of Brockett's test
- The homotopy theorem
- Generalization of Coron's and Mansouri's tests

Periodic orbits can be easier to stabilize

Two fundamental problems of control theory

Consider

$$\frac{dx}{dt} = f(x, u), \tag{1}$$

where $M \ni x$ is a smooth manifold and f is smooth.

1. **Controllability problem**: Given $a, b \in M$, find $u(t)$ s.t. $x(T) = b$ if $x(0) = a$ for some $T > 0$.

$$a \rightsquigarrow b$$

2. **Feedback stabilization problem**: Given a compact subset $A \subset M$, find smooth $u(x)$ s.t. A is **asymptotically stable**[2] for the **closed-loop vector field** $F(x) = f(x, u(x))$.

---

[2]For every open $W \supset A$ there is an open $V \supset A$ s.t. all forward F-trajectories initialized in V are contained in W and converge to A.

The stabilization conjecture and Brockett's solution

Often $A = \{x_*\}$ is a point, $M = \mathbb{R}^n$ in the stabilization problem.

Stabilization conjecture (pre-1983): a reasonably strong form of controllability implies smooth stabilizability of a point.

Example: the "Heisenberg system" or "nonholonomic integrator"

$$\left.\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv \end{aligned}\right\} = f(\mathbf{x}, \mathbf{u}).$$

is controllable in every sense imaginable. But Brockett (1983) showed that no point is stabilizable, refuting the conjecture. How?

Theorem (Brockett). If a point is stabilizable, then image(f) is a neighborhood of 0. (In the example, $(0, 0, \varepsilon) \notin$ image(f).)

Other stabilizability work

- Exponential (Gupta, Jafari, Kipka, Mordukhovich 2018; Christopherson, Mordukhovich, Jafari 2022),
- global (Byrnes 2008, Baryshnikov 2023),
- time-varying (Coron 1992), and
- discontinuous (Clarke, Ledyaev, Sontag, Subbotin 1997)

variants of the stabilization problem are not considered in this talk.

Coron's and Mansouri's obstructions

Krasnosel'skiĭ and Zabreĭko (1984) obtained a necessary condition for asymptotic stability of an equilibrium of a vector field.

Using this, Coron introduced a homological obstruction sharper than Brockett's, and Mansouri generalized. Define

$$\Sigma := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \colon f(x, u) \neq 0\}.$$

Theorem (Coron 1990). If $n > 1$ and a point is stabilizable,

$$f_*(H_{n-1}(\Sigma)) = H_{n-1}(\mathbb{R}^n \setminus \{0\}) \qquad (\cong \mathbb{Z}).$$

Theorem (Mansouri 2010). If a closed codimension > 1 submanifold $A \subset \mathbb{R}^n$ with Euler characteristic $\chi(A)$ is stabilizable,

$$f_*(H_{n-1}(\Sigma)) \supset \chi(A) \cdot H_{n-1}(\mathbb{R}^n \setminus \{0\}) \qquad (\cong \chi(A) \cdot \mathbb{Z}).$$

Limitations of these results

The results of Brockett, Coron, Mansouri rely on parallelizability of $\mathbb{R}^n$ to view vector fields and control systems as $\mathbb{R}^n$-valued.

Furthermore, they apply only to the special case that A is a point or a closed submanifold of $\mathbb{R}^n$ with $\chi(A) \neq 0$.

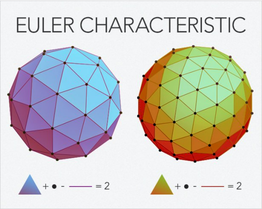
But sometimes one wants to stabilize more general subsets of more general spaces: robot gaits, safe behaviors for self-driving cars, etc.

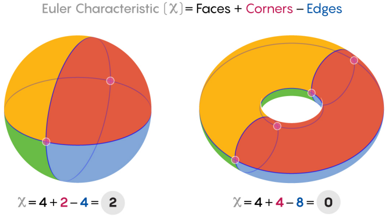
How to test for stabilizability in such general settings?

- **Generalization of Brockett's test** (MDK and Koditschek, J Geometric Mechanics, 2022).
- **Generalization of Coron's and Mansouri's tests** (MDK, SIAM J Control and Optimization, 2023).

A primer on the Euler characteristic[3]

Goes back to Francesco Maurolico (1537), Leonhard Euler (1758).





Notation: $\chi(Y) :=$ Euler characteristic of Y.

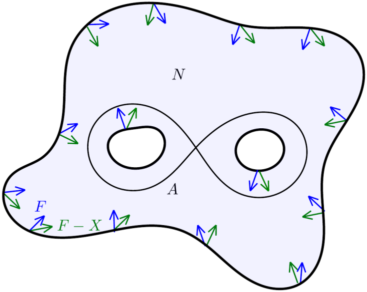
Examples: $\chi(\bullet) = 1$, $\chi(\mathbb{S}^1) = 0$, $\chi(\mathbb{S}^2) = 2$, $\chi(\text{figure 8}) = -1$

Theorem (Poincaré, Hopf): if N is a compact smooth manifold with boundary ∂N, then $\chi(N) = 0 \iff$ there exists a nowhere-zero smooth vector field on N pointing inward at ∂N.

[3] Figures from Quanta Magazine.

Generalization of Brockett's test

Theorem (MDK & DEK 2022): Let $A \subset M$ be compact & stabilizable. Then $\chi(A)$ is well-defined. If $\chi(A) \neq 0$, then for any sufficiently small vector field X, $X(x_0) = f(x_0, u_0)$ for some x_0, u_0.



Proof: Assume $\exists$ stabilizing $u(x)$ and define $F(x) := f(x, u(x))$. Lyapunov function theory $\implies$ $\exists$ compact smooth domain $N \supset A$ s.t. F points inward at ∂N and $\chi(A) = \chi(N) \neq 0$. Continuity $\implies$ $F - X$ points inward at ∂N if X is small $\implies$ $F - X$ has a zero by Poincaré-Hopf $\implies$ $\exists x_0$ s.t. $X(x_0) = F(x_0) = f(x_0, u(x_0))$.

Examples

Heisenberg system

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv \end{aligned} \tag{2}$$

Kinematic differential drive robot

$$\begin{aligned} \dot{x} &= u \cos\theta \\ \dot{y} &= u \sin\theta \\ \dot{\theta} &= v \end{aligned} \tag{3}$$

The right side of (2) $\neq X_\varepsilon := (0, 0, \varepsilon)$ for any $\varepsilon > 0$.

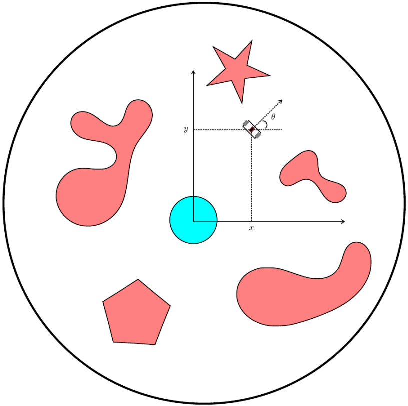
The right side of (3) $\neq X_\varepsilon := (\varepsilon \sin\theta, -\varepsilon \cos\theta, 0)$ for any $\varepsilon > 0$.

Thus, our result $\implies$ A is not stabilizable if $\chi(A) \neq 0$. E.g., if A is a stabilizable compact submanifold, A is a union of circles and tori.

Other applications: any stabilizable compact set has zero Euler characteristic for satellite orientation with ≤ 2 thrusters, for nonholonomic dynamics with ≥ 1 global constraint 1-form,...

Safety application

Our Brockett generalization implies an obstruction to a control system operating safely, i.e., ensuring trajectories initialized on the boundary of some "bad" set immediately enter some "good" set.



E.g., impossible for this differential drive robot to stably aim within ± 179 degrees of the origin while "strictly" avoiding obstacles.

Homotopy theorem & generalized Coron, Mansouri tests

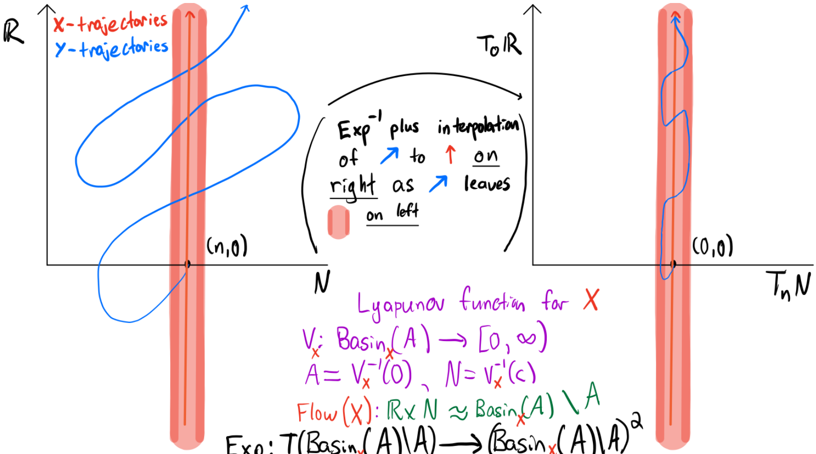
Homotopy theorem (MDK 2023). Let X, Y be smooth vector fields on a manifold M with a compact set $A \subset M$ asymptotically stable for both. There is an open set $U \supset A$ such that $X|_{U \setminus A}$, $Y|_{U \setminus A}$ are homotopic through nowhere-zero vector fields.

$\implies$ **Theorem (MDK 2023).** Let the compact set $A \subset M$ be asymptotically stable for *some* smooth vector field Y on M. If A is stabilizable for $\dot{x} = f(x, u)$, then for all small enough open $U \supset A$,

$$H_\bullet(T(U \setminus A) \setminus 0) \supset \underbrace{f_* H_\bullet(\Sigma) \supset Y_* H_\bullet(U \setminus A)}_{\text{cf. Coron, Mansouri}}.$$

These are stronger than all preceding results: there is an example (MDK 2023) for which non-stabilizability is detected by each of these theorems but not by any of the preceding theorems.

Proof of the homotopy theorem



Lyapunov function for X

$V_X : \text{Basin}_X(A) \to [0, \infty)$

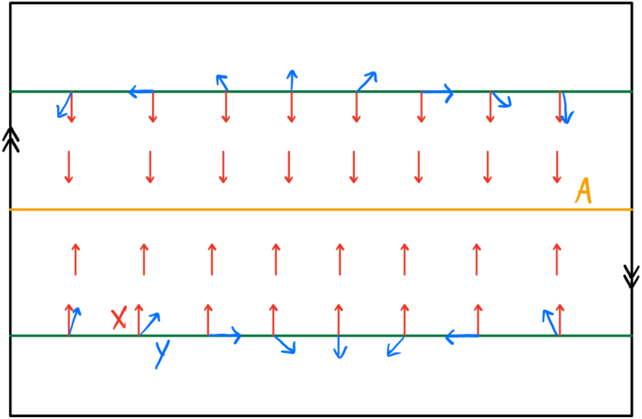
$A = V_X^{-1}(0)$, $N = V_X^{-1}(c)$

$\text{Flow}(X) : \mathbb{R} \times N \approx \text{Basin}_X(A) \setminus A$

$\text{Exp} : T(\text{Basin}_X(A) \setminus A) \longrightarrow (\text{Basin}_X(A) \setminus A)^2$

w.r.t. product of any metric on N with Euclidean metric on $\mathbb{R}$

Möbius strip example



X ≠ Y since Y ⟳ twice around ○ w.r.t. X while X ⟳ zero times w.r.t. X ⇒ A is not asymptotically stable for Y by the homotopy theorem.

Can these results detect stabilizability of periodic orbits?

If A is the image of a periodic orbit with the same orientation for X and Y, the straight-line homotopy over a sufficiently small open $U \supset A$ satisfies the homotopy theorem's conclusion regardless of whether A is attracting, repelling, or neither for X or Y.

$\implies$ **homotopy theorem gives no information on stability or stabilization of periodic orbits.** Since this is the strongest result, **preceding results also give no information.**

...Could it be that periodic orbits might be "easy" to stabilize?

Periodic orbits are sometimes easier to stabilize

Indeed:

Theorem (Bloch & MDK in prep). For a broad class of control systems including Heisenberg's and the differential-drive robot, **any periodic orbit that can be created can be stabilized**—even though *no equilibrium that can be created can be stabilized* for the mentioned examples!

Summary

Obstructions to stabilization

- Generalization of Brockett's test
- The homotopy theorem
- Generalization of Coron's and Mansouri's tests

Periodic orbits can be easier to stabilize

Questions

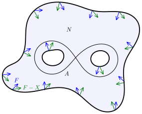
1. Is there a "universal" necessary and sufficient condition for stabilizability that can be used to actually *construct* smooth stabilizing feedbacks via numerical means?
2. Useful numerical implementation of homological tests?
3. Other open questions in conclusion of MDK 2023 (SICON).
4. Extensions, applications, analytical/numerical techniques for periodic orbit stabilization?

Thank you for your time and attention.

Asymptotic stabilizability

Generalization of Brockett's test

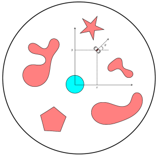
Theorem (MDK & DEK 2022): Let $A \subset M$ be compact & stabilizable. Then $\chi(A)$ is well-defined. If $\chi(A) \neq 0$, then for any sufficiently small vector field X, $X(x_0) = f(x_0, u_0)$ for some x_0, u_0.



Proof: Assume $\exists$ stabilizing $u(x)$ and define $F(x) := f(x, u(x))$. Lyapunov function theory $\implies \exists$ compact smooth domain $N \supset A$ s.t. F points inward at ∂N and $\chi(A) = \chi(N) \neq 0$. Continuity $\implies F - X$ points inward at ∂N if X is small $\implies F - X$ has a zero by Poincaré-Hopf $\implies \exists x_0$ s.t. $X(x_0) = F(x_0) = f(x_0, u(x_0))$.

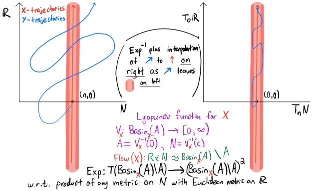
Safety application

Our Brockett generalization implies an obstruction to a control system operating safely, i.e., ensuring trajectories initialized on the boundary of some "bad" set immediately enter some "good" set.

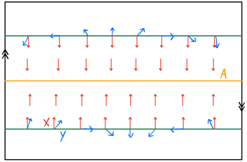


E.g., impossible for this differential drive robot to stably aim within ±179 degrees of the origin while "strictly" avoiding obstacles.

Proof of the homotopy theorem



Möbius strip example



$X \not\simeq Y$ since Y ↻ twice around ○ w.r.t. X while X ↻ zero times w.r.t. X ⇒ A is not asymptotically stable for Y by the homotopy theorem.