Whitney Stratification of Algebraic Maps and Applications to Kinematic Singularities

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joint work with **Vidit Nanda** (**Oxford**)
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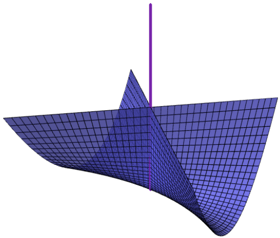

Inverse Kinematics and the Singularities of Kinematic Maps

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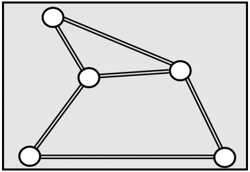

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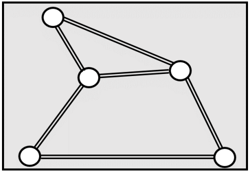

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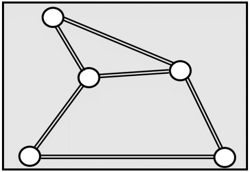

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Set $m = \dim(C)$, $n = \dim(S)$. The derivative of f at a point $p \in C$ is a linear map $T_pC \to T_f(p)S$ between tangent spaces defined by $J(p)$, the $n \times m$ Jacobian matrix of f evaluated at p.

The **singular locus of** f: $\Sigma_f := \{p \in C \mid \mathrm{rank}(J(p)) < \min(m, n)\}$.

At points $p \in \Sigma_f$ the mechanism may loose a degree of freedom causing a loss of control, but this may not happen at all points in Σ_f.

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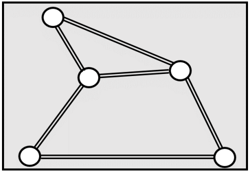

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Goal: understand the global geometry & topology of f, Σ_f, f^{-1}.

Mathematical Tools and Setting

More Precise Goal: stratify the map f, i.e. subdivide C and S into **finitely many** manifold regions so that the fibres $f^{-1}(q)$ and $f^{-1}(q')$ are topologically identical for any two points in a region of S.

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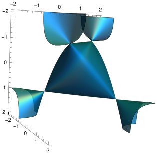
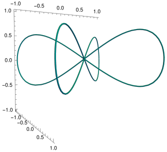


$2xyz - x^2 - y^2 - z^2 + 1 = 0$



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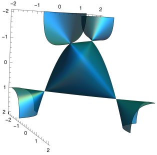

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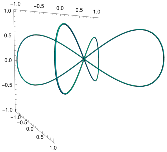

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We can see varieties as a non-linear analogue of v. spaces: we rewrite & solve poly. systems via Gröbner basis instead of linear systems via Gaussian elim.

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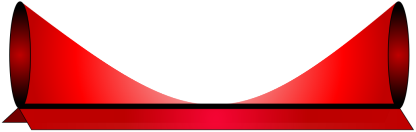

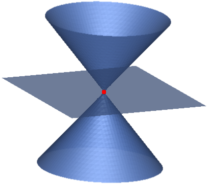

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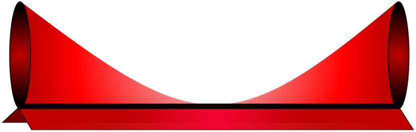

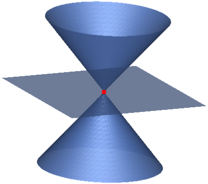

We seek to stratify these spaces by separating them into smooth manifolds which join in a nice way.

Stratifying Varieties

More precisely, we will (first) consider (complex) algebraic varieties

$$X = \mathbb{V}(I_X) = \mathbb{V}(f_1, \dots, f_r) = \{p \in \mathbb{C}^n \mid f_1(p) = \cdots = f_r(p) = 0\}.$$

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A *stratification* is a filtration, $X_\bullet$, $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$ of X s.t. $X = \cup_i X_i$ and s.t. each strata $\mathcal{M} = X_i - X_{i-1}$ is either empty or smooth, i.e. is a manifold, and has pure dimension.

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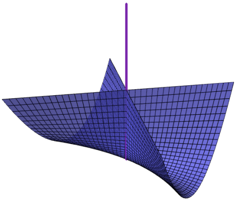
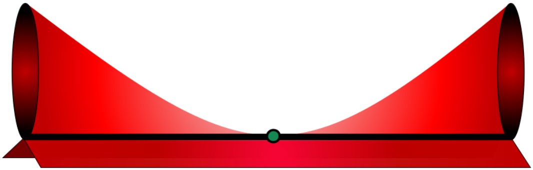
Additionally: want decomposition $X = \sqcup_i \mathcal{M}_i$ to be *equisingular*; i.e. the neighbourhood in X of any 2 points in a connected comp. of $\mathcal{M}_i$ is "similar".



Whitney Stratification

For $X_\bullet$ to be a **Whitney Stratification** these strata must satisfy
Condition B: for each pair of strata $\sigma, \tau \subset X$ and a point $y \in \tau$



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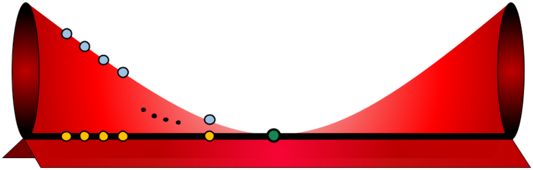

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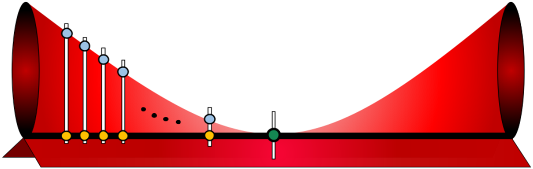

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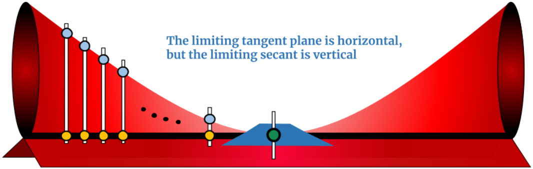
then $\ell \subset T$



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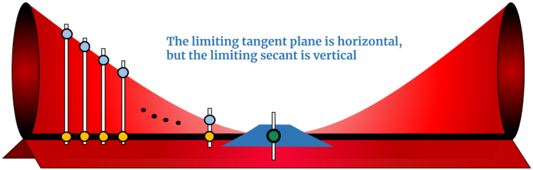

Theorem (H. Whitney, Annals of Math.,1965)

A stratification where all strata pairs satisfy Condition B exists for all algebraic varieties. Further, Condition B implies equisingularity.

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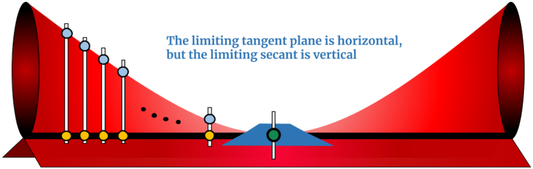

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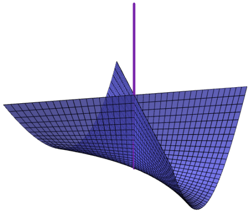
Goal: given equations defining X **efficiently compute** a Whitney stratification (compute = find equations for each X_i).

Example: Our Algorithm Applied to the Whitney Umbrella

There has long been significant interest in algorithmic computation of Whitney stratifications (e.g. Mostowski & Rannou 1991, Rannou 1998, Đinh & Jelonek 2021); previous methods have proved impractical on even the smallest examples.



```
Macaulay2, version 1.21
with packages: ConwayPolynomials, Elimination

i1 :     needsPackage "WhitneyStratifications

o1 = WhitneyStratifications

o1 : Package

i2 : R=QQ[x..z];

i3 : X=ideal(x^2*z-y^2);

o3 : Ideal of R

i4 : time W=whitneyStratify X;
     -- used 0.28765 seconds

i5 : peek W

o5 = MutableHashTable{0 => {ideal (z, y, x)}}
                      1 => {ideal (y, x)}
                                    2    2
                      2 => {ideal(x z - y )}
```

First Ingredients: Conormal Variety

Notation: write X_{reg} = set of all smooth points of a variety $X = \mathbb{V}(f_1, \ldots, f_r)$.

The **conormal variety** of X is the subvariety

$$\mathrm{Con}(X) = \overline{\{(p, \xi) \mid p \in X_{\mathrm{reg}},\ T_p X_{\mathrm{reg}} \subset \xi\}} \subset X \times (\mathbb{P}^n),$$

given by closing the set of pairs of points $p \in X_{\mathrm{reg}}$ with hyperplanes ξ containing $T_p X_{\mathrm{reg}}$; hyperplanes are represented by their normal vectors.

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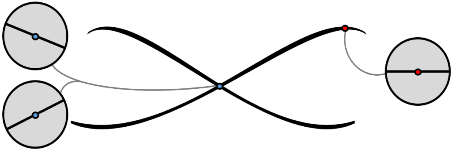
The fiber $\kappa_X^{-1}(p)$ consists of all hyperplanes containing a tangent space or limiting tangent space at p.



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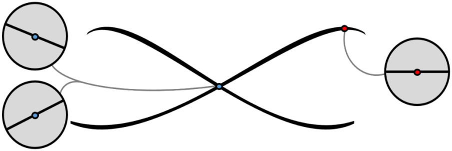

Eqs. $I_{\mathrm{Con}(X)}$ defining $\mathrm{Con}(X)$ are easily computed from the f_i via Gröbner basis.

One Last Ingredient: Associated Primes

For a polynomial ideal I we can compute a *primary decomposition*:

$$I = Q_1 \cap \cdots \cap Q_\ell$$

where Q_i is a primary ideal (if $ab \in Q_i$, either a or b^n is in Q_i).

The *associated prime* ideals $P_i = \sqrt{Q_i}$ are unique (Q_i need not be) and

$$\mathbb{V}(I) = \mathbb{V}(Q_1) \cup \cdots \cup \mathbb{V}(Q_\ell) = \mathbb{V}(P_1) \cup \cdots \cup \mathbb{V}(P_\ell),$$

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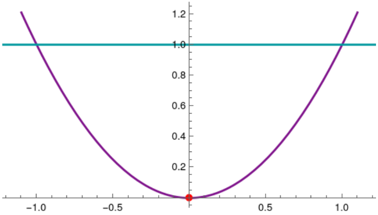
$$I = \langle x^2y^2 - x^2y - y^3 + y^2, x^3y - x^3 - xy^2 + xy\rangle = \langle y-1\rangle \cap \langle y - x^2\rangle \cap \langle x, y^3\rangle$$

The associated primes are:
$\langle y-1\rangle, \langle y-x^2\rangle, \langle x, y\rangle$.



Our Algebraic Criterion

Let X be a pure dimensional variety and $Y \subset X_{\mathrm{Sing}} = (X - X_{\mathrm{reg}})$ be a nonempty irreducible subvariety.

Goal: find all points in Y where Condition B fails w.r.t. X.

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Theorem (H. and Nanda, 2022)

Set $I_{\kappa_X^{-1}(Y)} := I_{\mathrm{Con}(X)} + I_Y$. *Let* $\{P_1, \ldots, P_s\}$ *be the associated primes of* $I_{\kappa_X^{-1}(Y)}$, *let* $\sigma \subset \{1, 2, \ldots, s\}$ *be the set of indices* i *with* $\dim \kappa_X(\mathbb{V}(P_i)) < \dim Y$ *and let*

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Note: this identifies the points of interest and is computable using Gröbner basis calculations only.

Idea of Our Algorithm

Input: An ideal I_X defining a variety X of dimension k.

Our Criterion: All points in Y_{reg} where Condition B fails with respect to X are contained in $\bigcup_{i\in\sigma}\kappa_X(\mathbb{V}(P_i))$.

Roughly speaking, our algorithm proceeds as follows:

- **Compute** $Y_1 = X_{\mathrm{Sing}}$ by finding the ideal generated by the $(n-k)\times(n-k)$-minors of the Jacobian of I_X.

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- **Compute associated primes** $\{P_1, \dots, P_s\}$ **of** $I_{\kappa_X^{-1}(Y_1)}$ and **by our criterion** Condition B fails on $Y_2 = \bigcup_{i \in \sigma} \kappa_X(\mathbb{V}(P_i)) \cup Y_{\mathrm{sing}}$ where σ collects all i with $\dim(\kappa_X^{-1}(\mathbb{V}(P_i))) < \dim Y_1$.

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- **Repeat this with** Y_2; **continue until done**.

The above leads to a procedure to construct a stratification where each peice satisfies Whitney's Condition B with respect to the top stratum.

Stratifying Maps

Recall: we wanted to globally understand kinematic maps $f : C \to S$ via stratification where the configuration and state spaces are varieties.

Exact Map Stratification Definition:

Let X, Y be algebraic varieties and $f : X \to Y$ an algebraic map. A **stratification of** f, is a Whitney stratification of X and Y so that for every strata S of X there is a strata R of Y such that the restriction $f|_S : S \to R$ is surjective, with a surjective derivative.

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Exact Map Stratification Definition:

Let X, Y be algebraic varieties and $f : X \to Y$ an algebraic map. A **stratification of** f, is a Whitney stratification of X and Y so that for every strata S of X there is a strata R of Y such that the restriction $f|_S : S \to R$ is surjective, with a surjective derivative.

Consequence: For q, q' in the same stratum N of Y the fibers $f^{-1}(q)$ and $f^{-1}(q')$ have the same topology.

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The key ingredient is the Whitney stratification algorithm presented earlier.

Stratifying Maps between Real Varieties

For planar mechanisms we can use the complex version directly via isotropic coordinates on $\mathbb{R}^2$, i.e. represent $(x, y) \in \mathbb{R}^2$ as $x + iy \in \mathbb{C}$.

This introduces unnecessary computational overhead, however, and even more so when generalized to $\mathbb{R}^3$.

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In ongoing work (funded by the AFOSR) nearing completion, which should be appearing on the arxiv in the next couple months, we extend our algorithms to real varieties.

This will allow us to compute Whitney stratification of real varieties $V \subset \mathbb{R}^n$ and of polynomial maps $f : X \to Y$, where $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, using Gröbner basis calculations only.

Summary

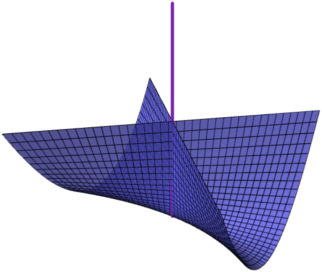
We have described an algorithm to compute a Whitney stratification an algebraic variety $X \subset \mathbb{C}^n$ and of algebraic maps $f : X \to Y$.

- Ongoing work (funded by AFOSR) gives a similar algorithm to find Whitney stratifications of **real varieties** $X \subset \mathbb{R}^n$ and to stratify algebraic maps between real varieties.
- This will allow us to stratify kinematic maps $f : C \to S$, where C and S are real varieties, (eventually) enabling a global study of their singularities using topological techniques.
- There is a **M2 package** called *WhitneyStratifications* which implements this.

Package Docs: https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2/share/doc/Macaulay2/WhitneyStratifications/html/index.html

Thank You!

Thank you for your attention!



Paper: https://doi.org/10.1007/s10208-022-09574-8

Code: https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2/share/doc/Macaulay2/WhitneyStratifications/html/index.html