Zero density for the Riemann Zeta function

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Overview I

Introduction

The function $\zeta(s)$ is a very important function in mathematics. Let s be a complex number with σ and t respectively it's real and imaginary parts. Then we define the zeta function as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\sigma > 1$, and for the remainder of the complex plane, it is defined as the analytic continuation of the above function.

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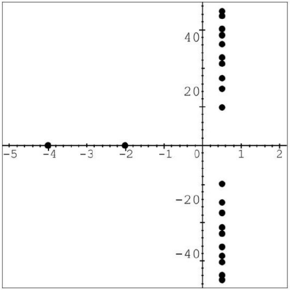
The study of zeros of the zeta function plays an important role in analytical number theory. Riemann Hypothesis(RH) is about the locations of zeros of Riemann Zeta function. According to this hypothesis $\zeta(s)$ has trivial zeros at negative even integers, that is $s = -2, -4, -6, \ldots$, and no other real zeros and non-real zeros, also called the nontrivial zeros, are lie on the critical line $R(s) = \frac{1}{2}$, and to date still is an open problem.



Motivation

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$$\pi(x) = \mathrm{Li}(x) + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right)$$

For some $c > 0$, where $\pi(x)$ is the number of primes less than or equal to x and $\mathrm{Li}(x)$ is the logarithmic integral function defined by $\mathrm{Li}(x) = \int_2^x \frac{1}{\log t} dt$. The shape and constant in the error term are determined by what we can prove about the number and location of zeros of zeta.

Zero density

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$$N(\sigma, T) = \#\{\rho = \beta + i\gamma : \zeta(\rho) = 0, 0 < \gamma < T, \sigma < \beta < 1\}. \quad (1)$$

I want to find the explicit upper bound for the number of zeros of the zeta function within this rectangular region. This type of result is commonly referred to as a **zero density** result. From Kadiri, Lumely and Ng [1], bounds on $N(\sigma, T)$ often take the shape:

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The main goal of my thesis will both improve the bound for $C_1(\sigma)$, but also, change slightly the shape of the bound.

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As T grows to infinity. In 1937 Ingham[3] showed that

$$N(\sigma, T) = O\left(T^{(2+4c)(1-\sigma)}(\log T)^5\right).$$

By assuming that $\zeta(\frac{1}{2} + it) = O(t^{c+\epsilon})$.

History

Ramaré[4] had proven an explicit version of Ingham's bound. For example, for $\sigma = 0.90$ this formula simplifies to

$$N(0.90, T) < 1293.48(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 51.50(\log T)^2.$$

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Kadiri, Lumley, and Ng[1] have presented a result that provides a tighter bound for $N(\sigma, T)$. Their result improves upon both Ramare and Ingham's estimates by following Ingham's argument but using a more general weight. If we put $\sigma = 0.90$, We can see how this improves on Ramare's estimate.

$$N(0.90, T) < 11.499(\log T)^{\frac{16}{5}} T^{\frac{4}{15}} + 3.186(\log T)^2.$$

Idea of proof

Now, I only present an outline of the proof, which are the main steps in the paper of Kadiri, Lumley, and Ng. My approach will be to follow the proof of Kadiri, Lumley and Ng but improve on it by using new results established since their publication.

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The main idea of finding an upper bound for $N(\sigma, T)$ is multiplying $\zeta(s)$ into an entire function, $P(s)$ derived as the product of $(2 - \zeta(s) \cdot M(s))$ and $M(s)$ where $M(s)$ is called a mollifier and let $f(s) = \zeta(s) \cdot M(s) - 1$. The series expansion for $f(s)$ is expressed as a below dirichlet series

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$$f(s) = \sum_{n \geq 1} \frac{\lambda(n)}{n^s}$$

$$\text{with} \begin{cases} \lambda_x(n) = 0 \text{ if } n \leq x \\ \lambda_x(n) = \sum_{\substack{d|n \\ d \leq x}} \mu(d) \text{ if } n > x \end{cases}$$

Idea of proof

We have the resulting function as below,

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\begin{aligned}
h(s) &= \zeta(s) \cdot P(s) && (2)\\
&= \zeta(s) \cdot [M(s)(2 - \zeta(s)M(s))] && (3)\\
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Since any zeros of the $\zeta(s)$ is zeros of the new function $h(s)$, so our goal becomes to find an upper bound for the number of zeros for the new function . This means the key thing is we replace the function, $\zeta(s)$ with the function $h(s)$ with more zeros that is easier to study.

Counting zeros of zeta function in a rectangle region

There exists many useful tools in complex analysis to count the zeros of the holomorphic function inside a specified rectangle region and, in my thesis, by using the classic idea of Bohr, Landau, Carlson and Titchmarsh as stated in[5] which uses the residue theorem, we can bound the number of zeros by the following integrals:

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$$N(\sigma, T) \leq \frac{1}{2\pi\left(\sigma - \sigma'\right)}\left(\int_H^T \log\left|h\left(\sigma' + it\right)\right|dt + \int_{\sigma'}^{\mu} \arg h(\tau + iT)d\tau \right.$$

$$\left. - \int_{\sigma'}^{\mu} \arg h(\tau + iH)d\tau - \int_H^T \log|h(\mu + it)|dt\right).$$

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$$\left.-\int_{\sigma'}^{\mu}\arg h(\tau+iH)d\tau-\int_H^T\log|h(\mu+it)|dt\right).$$

Thus, my goal becomes to find an upper bound for each integral. As T grows larger, the main contribution comes from the first integral and now I will give an idea of how to estimate each integral.

First Integral

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For any f non-negative and continuous, and $|h(s)| \leq 1 + |f(s)|^2$, we have

$$\int_H^T \log(|h(\sigma' + it)|)dt \leq (T - H)\log\left(1 + \frac{1}{T - H}\int_H^T |f(\sigma' + it)|^2 dt\right)$$

So the goal is to find a bound for $\int_H^T |f(\sigma + it)|^2 dt$.

Smoothing method

One idea to bound $\int_H^T |f(\sigma+it)|^2$ is we can try smoothing and unsmoothing method. By using smoothing method, we introduce the smooth weight function g as a characteristic function and multiple to our function f which becomes more easy to analyze the integral of the smoothed function rather than the original one. We have

$$\int_0^T |f(\sigma+it)|^2 dt \leq \frac{\int_{-\infty}^{\infty} |g(\sigma+it)|^2 |f(\sigma+it)|^2 dt}{\omega_2}$$

Where ω_2 is a positive function depend on g and our goal becomes to find an upper bound for $\int_{-\infty}^{\infty} |g(\sigma+it)|^2 |f(\sigma+it)|^2 dt$

Unsmoothing method

Unsmoothing seems to involve reversing or inverting the process of smoothing. So by unsmoothing method we obtain the below bound

$$\int_{-\infty}^{\infty} |g(\sigma+it)|^2 |f(\sigma+it)|^2 dt \leq \omega_1 \int_0^{\infty} x^{\beta-1} e^{-\alpha x^{\beta}} F(\sigma, xT) dx$$

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So if we have a bound for $F(\sigma, T)$ we can find the bound for $\int_{-\infty}^{\infty} |g(\sigma+it)|^2 |f(\sigma+it)|^2 dt$. Since the error term arises here is

$$\omega_1 \int_0^{\infty} x^{\beta-1} e^{-\alpha x^{\beta}} dx$$

So one way to make the estimate tighter, is choosing another general weight function. Nevertheless, it is an open problem to determine the best weights g to use in this problem.

Convexity estimate

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Since by using unsmoothing method we find a good bound for $\int_{-\infty}^{\infty} |g(\sigma + it)|^2 |f(\sigma + it)|^2 dt$ at $\sigma_1 = \frac{1}{2}$ and $\sigma_2 = 1 + \epsilon$, based on the convexity estimate we can find a bound for smoothed function at σ. If we call this integral $J(\sigma)$, by convexity estimate we obtain the below bound,

$$J(\sigma) \leq (J(\sigma_1))^{\frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1}} (J(\sigma_2))^{\frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}}$$

Where $\sigma_1 = \frac{1}{2}$ and $\sigma_2 = 1 + \epsilon$.

Boundin $F(\sigma, T)$ on the half line

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- Bound for $M(s)$ on the critical line.

$$\int_0^T M(\frac{1}{2} + it)^2 dt \leq CT \log T$$

Boundin $F(\sigma, T)$ at σ_2

Based on the expansion of f, We can write,

$$F(\sigma_2, T) = \int_0^T |f(\sigma_2 + it)|^2 \, dt = \int_0^T \left| \sum_{n \geq 1} \frac{\lambda(n)}{n^{\sigma_2 + it}} \right|^2 dt$$

The idea for finding the upper bound for the above integral is using the Montgomery and Vaughan's mean value theorem for Dirichlet polynomials.

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The idea for finding the upper bound for the above integral is using the Montgomery and Vaughan's mean value theorem for Dirichlet polynomials. Also we will use this idea to find an upper bound for the fourth integral.

Conclusion

With combining the bounds that we obtained for $J(\sigma_1)$ and $J(\sigma_2)$ by unsmoothing method, and applying the Convexity Theorem, we find an upper bound for $J(\sigma)$. Then We substitute the bound for $J(\sigma)$ in smoothing method. Finally, based on the results obtained from these methods, we provide a bound for $F(\sigma, T)$.

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$$F(\sigma, T) \leq \frac{J(\sigma)}{\omega_2}$$

Therefore we have the below bound for the first integral,

$$\int_H^T \log(|h(\sigma' + it)|)dt \leq C \frac{(\log T)^{4-2\sigma} T^{\frac{8}{3}(1-\sigma)}}{\omega_2}$$

Second and Third Integrals

The second and third integral is about finding an upper bound for

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$$\leq (\mu - \sigma') \max_{\tau \in (\sigma', \mu)} (|\arg h(\tau + iT)| + |\arg h(\tau + iH)|).$$

So the key to finding an upper bound for the difference of the integrals is to find an upper bound for $|\arg h(\tau + iT)| + |\arg h(\tau + iH)|$.

Last Integral

The last bound we need is an upper bound for $-\int_H^T \log|h(\mu+it)dt|$ or equivalently an explicit lower bound for $\int_H^T \log|h(\mu+it)|dt$. This is very similar to bounding $F(\sigma_2, T)$ that we had in the first integral.

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So the general idea for finding a bound for this integral is to use the standard mean value theorem for Dirichlet polynomials. Therefore, we have the below bound for the last integral

$$-\int_H^T \log|h_X(\mu+it)|dt \leq B\log T$$

Finally, after these steps, I'm able to compile my bounds to obtain an upper bound for the number of zeros, $N(\sigma, T)$, in a given region.


THANK YOU!

References

H. Kadiri, A. Lumley, and N. Ng, *Explicit zero density for the Riemann zeta function.*
J. Math. Anal. Appl. 465 (2018).

H.Bohr and E.Landau, *Beiträge zur Theorie der Riemannschen Zetafunktion.*
Math. Ann. 74 (1) (1913).

A. Ingham, *On the difference between consecutive primes.*
Quart. J. Mech. Appl. Math. (2) 8 (1937).

O. Ramaré, *An explicit density estimate for Dirichlet L-functions.*
Math. Comp. 85 (2016).

E. Titchmarsh, *The Theory of the Riemann Zeta-Function.*
Oxford Science Publications, 2011.

D. PATEL and A. YANG, *AN EXPLICIT SUB-WEYL BOUND FOR zeta function on the half*