Tanscendental dynamical degrees of birational maps

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Throughout this talk X will be a projective variety and f will be a dominant rational self-map of X. One wishes to study the dynamical system (X, f).

The most fundamental dynamical invariant of a dominant rational self-map $f\colon X \dashrightarrow X$ of a smooth projective variety is, arguably, the (first) *dynamical degree* $\lambda(f)$.

Definition

The dynamical degree is defined as the limit $\lim_{n\to\infty}(f^{n*}H \cdot H^{\dim X - 1})^{1/n}$ for an ample divisor H on X.

Its value does not depend on the choice of H, and it is also invariant under birational conjugacy: if $h\colon X' \dashrightarrow X$ is a birational map, then $f' := h \circ f \circ h^{-1}\colon X' \dashrightarrow X'$ is a dominant rational map with $\lambda(f') = \lambda(f)$.

Computing the dynamical degree(s) is often the first step in understanding the dynamical system (X, f).

In general, one has higher dynamical degrees (one for each i up to the dimension of X). These dynamical degrees were introduced by Friedland. We note that they were originally defined with a limsup instead of a limit, but Dinh-Sibonny showed that they exist.

This dynamical degree is of fundmamental importance in complex dynamics, but we note that it is also related to questions in arithmetic dynamics via work of Kawaguchi and Silverman. When one works over a global field, the (first) dynamical degree serves as an upper bound for the asymptotics of the growth of heights along orbits; the question of when equality holds is part of the *Kawaguchi–Silverman conjecture*.

We note that the sequence

$$d_n := (f^{n*}H \cdot H^{\dim X - 1})$$

is submultiplicative; i.e., $d_n d_m \leq d_{n+m}$ and so the limit of $d_n^{1/n}$ always exists.

But let's look at a more intuitive way of viewing the dynamical degree in the case when

$$X = \mathbb{P}^d.$$

A rational self-map $f : \mathbb{P}^d \to \mathbb{P}^d$ is given (on a dense open set) by

$$f = [P_0 : P_1 : \cdots : P_d],$$

where $P_0, \ldots, P_d \in \mathbb{C}[X_0, \ldots, X_d]$ are homogeneous polynomials of the same degree m with $P_0, \ldots, P_d$ having no common non-constant polynomial factor.

Then m is the *degree* of f, and the dynamical degree of f is

$$\lim_{n\to\infty} (\deg(f^n))^{1/n}.$$

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Now we can see intuitively why we expect submultiplicativity of the sequence of degrees: If

$$f^m = [P_0 : \cdots : P_d]$$

and

$$f^n = [Q_0 : \cdots : Q_d]$$

then

$$f^{n+m} = [P_0(Q_0, \ldots, Q_d) : \cdots : P_d(Q_0, \ldots, Q_d)],$$

so if f^m has degree a and f^n has degree b, the the polynomials $P_i(Q_0, \ldots, P_d)$ have degree ab. So

$$\deg(f^{n+m}) \leq \deg(f^n)\deg(f^m).$$

Example.

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be the map $f([X : Y]) = [X^2 : Y^2]$. Then f has degree 2 and in general

$$f^n([X : Y]) = [X^{2^n} : Y^{2^n}].$$

Thus f^n has degree 2^n and so its dynamical degree is the limit of $(2^n)^{1/n} \to 2$.

We don't get in general $\deg(f^n) = \deg(f)^n$, because there can be common factors in the polynomials. For example, consider the map

$$f : \mathbb{P}^2 \to \mathbb{P}^2$$

given by $f([X : Y : Z]) = [YZ : XZ : XY]$. Then f has degree 2 but $f^2([X : Y : Z]) = [X^2YZ : XY^2Z : XYZ^2] = [X : Y : Z]$, which has degree 1. In particular, f^n has degree 2 if n is odd and has degree 1 if n is even and so the dynamical degree of this f is 1.

In general the dynamical degree doesn't have to be an integer. Here's an example. Let $X = \mathbb{A}^2$ and let $f : X \to X$ be the map $f(u, v) = (uv, u)$. Then $f^2(u, v) = (u^2v, uv)$, $f^3(u, v) = (u^3v^2, u^2v)$, and in general

$$f^n(u, v) = (u^{F_{n+1}}v^{F_n}, u^{F_n}v^{F_{n-1}}),$$

where $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \ldots$ are the Fibonacci numbers.

So f^n has degree $F_{n+1} + F_n = F_{n+2}$ and since $F_n^{1/n} \to \rho$, the Golden ratio, we see that the dyamical degree of f, $\lambda(f)$, is ρ.

Let's do one more example! Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be the map $f(u,v) = (uv, v)$. Then $f^n(u,v) = (uv^n, v)$ and one can see that the degree of f^n is $n+1$ and so the dynamical degree is the limit of $(n+1)^{1/n} \to 1$.

The fact that the dynamical degree of this map is 1 says that the dynamical system is in some sense tamer than the preceding example.

The dynamical degree is often difficult to compute. There are a few cases where things have been worked out.

- (Sibony) If f is *algebraically stable*; i.e., $f^{n*} = f^{*n}$ for the induced pullbacks of divisors on X, then $\lambda(f)$ is equal to the spectral radius of the $\mathbb{Z}$-linear operator $f^*\colon \mathbf{NS}_{\mathbb{R}}(X) \to \mathbf{NS}_{\mathbb{R}}(X)$ on the real Neron-Severi group $\mathbf{NS}_{\mathbb{R}}(X)$, and this implies that $\lambda(f)$ is an algebraic integer.
- (Diller-Favre) for birational maps of $\mathbb{P}^2$ one can achieve algebraic stability after birational conjugation, so the dynamical degrees are algebraic integers.
- (Favre-Jonsson) For dominant endomorphisms of $\mathbb{A}^2$ the dynamical degrees are always algebraic integers.
- (Bonifant-Fornæss, Urech) There are only countably many different dynamical degrees.

These results lead naturally to the question: is the dynamical degree *always* an algebraic integer, or at least an algebraic number?

The answer is ‘NO’:

Theorem

(B-Diller-Jonsson 2019) There exists a dominant rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ whose dynamical degree is a transcendental number.

In recent work this has been extended:

Theorem

(B-Diller-Jonsson-Krieger, 2024) There exists a birational map $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ whose dynamical degree is a transcendental number.

The general strategy for this new work is similar to the one employed in the non-birational case, but there are additional subtleties that arise on both the complex dynamics side and the Diophantine approximation side and on connecting these two components, which is generally straightforward in the non-birational case.

Analysis of the non-birational case

We give a family of such examples of the form $f = \tau \circ \sigma$, where

$$\sigma(y_1, y_2) = \left(-y_1\frac{1 - y_1 + y_2}{1 - y_1 - y_2}, -y_2\frac{1 + y_1 - y_2}{1 - y_1 - y_2}\right),$$

which is a birational involution and

$$\tau(y_1, y_2) = (y_1^a y_2^b, y_1^{-b} y_2^a)$$

is monomial map with the property $(a + bi)^n \notin \mathbb{R}$ for all integers $n > 0$.

Then every f of this form has the property that $\lambda(f)$ is transcendental.

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Favre, in 2003, showed that under the above conditions the map τ cannot be conjugated to an algebraically stable map, although $\lambda(\tau) = \sqrt{a^2 + b^2}$. Note that $\lambda(\sigma) = 1$.

In general, the degrees of f^n are closely related to the degrees of τ^n as follows.

Let $d_j := \deg(\tau^j)$ for $j \geq 0$. Then $\lambda = \lambda(f)$ is the unique positive solution to the equation

$$\sum_{j=1}^{\infty} d_j \lambda^{-j} = 1. \qquad (\star)$$

The idea behind this expression for $\lambda(f)$ is very difficult and relies on a careful analysis of the actions of σ and τ on the space of valuations of the function field of $\mathbb{P}^2$. What is shown by induction is that the following holds:

$$d_n = \deg(\tau^n) \quad \text{and} \quad e_n = \deg(f^n)$$

for $n \geq 1$, then

$$e_n = \sum_{j=0}^{n-1} e_j d_{n-j}$$

for $n \geq 1$, (where we set $d_0 = 1$ and $e_0 = 2$).

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This means that if we let

$$E(z) = \sum_{n\geq 0} e_n z^n$$

and

$$D(z) = \sum_{n\geq 1} d_n z^n$$

then

$$E(z)(1 - D(z)) = 2.$$

The dynamical degree of f is $1/r_E$, where $r_E \leq 1$ is the radius of convergence of f. Similarly the dynamical degree of τ is $1/r_D$, where $r_D < 1$ is the radius of convergence of D. It is well-known that $\lim_{z\to r_D^-} D(z) = \infty$, so there is a unique point $z_0 \in (0, r_D)$ with $D(z_0) = 1$. Then the radius of convergence of E is z_0 and so we see that $\lambda = 1/z_0$ and

$$\sum_{j=1}^{\infty} d_j \lambda^{-j} = 1.$$

So now the question becomes: how do we show that the solution λ to the equation $D(\lambda^{-1}) = 1$ is transcendental?

Let's recast this as a problem in diophantine approximation. We have a power series $D(z)$ that is not algebraic and which has some nice additional structure. Then one expects that its specializations at $\bar{\mathbb{Q}}$-points inside the radius of convergence should be transcendental unless there is some compelling reason that they are not.

Example.

(Siegel-Shidlovsky, Beukers) If we take the function e^x. Then e^x is a transcendental E-function and e^α is transcendental for all nonzero algebraic values α. This is a general phenomenon for (non-polynomial) E-functions, where after excluding a finite computable set of “bad” values one always has transcendence after specialization.

Example. (Philippon, Adamczewski-Faverjon) If we take an irrational Mahler series $F(z)$ over a number field then $F(\alpha)$ is transcendental outside for algebraic α inside the radius of convergence outside of a computable set of “bad” values.

So we can now see a general approach to showing that $\lambda(f)$ is transcendental: use the structure of D to show that $D(\alpha)$ is transcendental for algebraic α in the radius of convergence of D, outside of a set of "bad" values and then show that $\alpha = 1/\lambda$ cannot not bad. But now $D(\alpha) = 1$, so α cannot be algebraic!

But how are results of this type obtained in practice?

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The idea is that one takes the series $D(z)$ and shows one has "good" approximations by rational functions $\Phi_1(z), \Phi_2(z), \ldots$ (with rational coefficients) in the sense that if Φ_n has degree a_n then

$$D(z) - \Phi_n(z) = \mathrm{O}(z^{Aa_n})$$

with $A > 2$ fixed.

One then wants to argue that $\Phi_n(\alpha)$ is a good approximation of $D(\alpha)$.

Then one wants to argue that $D(\alpha)$ has to be transcendental unless, by some fluke, $\Phi_n(\alpha)$ just happens to agree with $D(\alpha)$.

To successfully apply this strategy, we need a closed form for the series

$$D(z) = d_1 z + d_2 z^2 + d_3 z^3 + \cdots ,$$

where

$$d_j = \deg(\tau^j)$$

and

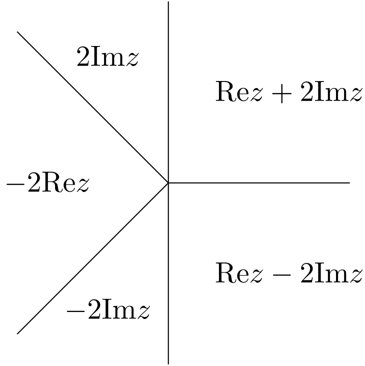
$$\tau(y_1, y_2) = (y_1^a y_2^b, y_1^{-b} y_2^a).$$

Here it is an elementary computation to show

$$d_j = \mathrm{Re}(\gamma(j)\zeta^j),$$

where $\zeta = a + bi$ and $\gamma(j) \in \{-2, \pm 2i, 1 \pm 2i\}$ is chosen to be whichever element maximizes the right side.

Intuitively $\zeta = r\exp(i\theta)$ where $r = \sqrt{a^2+b^2}$ and $\zeta^n = r^n\exp(in\theta)$. Then depending on $n\theta$ we use the following rules to compute d_n.



We are interested when the angle θ is irrational. In this case, Hasselblatt–Propp showed that the sequence $(d_j)_{j\geq 1}$ does not satisfy any finite linear recurrence relation. In particular, it is an irrational power series. In fact, it is not even algebraic!

We'll use the fact that the angle θ has reasonably good rational approximations to produce rational function approximations to the series $D(z) = \sum d_n z^n$. To then obtain transcendence we use the p-adic subspace theorem.

This approach was first employed by Corvaja and Zannier (2002) and later, famously, by Adamczewski and Bugeaud in 2007 in showing that automatic real numbers are either rational or transcendental. We have to use substantially more linear forms, however, due to the fact that we do not have a strong understanding of the angles θ that arise.

Constructing rational approximations

The easier part of the transcendence proof is to construct rational approximations to the series $D(z)$.

The idea is that if m/n is a continued fraction approximant of θ, then ζ^n is nearly a positive real number. As a result, we expect $\gamma(j+n) = \gamma(j)$ unless z^j is very close to one of the boundaries where discontinuities occur in our picture. But we expect $\gamma(j)$ to be "nearly" n-periodic in j, and so we expect that $D(z)$ is well-approximated by the rational function $D^{(n)}(z) := (1-z^n)^{-1}\sum_{j=1}^{n} D_j z^j$ obtained by assuming the $\gamma(j)$ are precisely n-periodic.

This is not quite the case, but we can show that if $C > 0$ then the number of “bad” indices $j \le Cn$ for which $\gamma(j+n) \neq \gamma(j)$ is $\le K$, where K depends only on C. Moreover, we can show that the “bad” indices are repelling in some sense, which ends up being important.

So to obtain a good approximation, we adjust $D^{(n)}(z)$ by adding on $\le K$ terms to correct for the bad j's in the range. We then obtain a series $\Phi_n(z)$ with the property that

$$D(z) - \Phi_n(z) = \mathrm{O}(z^{Cn}).$$

Showing $\Phi_n(\alpha)$ is a good approximation of $D(\alpha)$

Here is where the p-adic subspace theorem comes in. We give a quick overview of the ideas involved.

- Let K be a number field of degree $d := [K : \mathbb{Q}]$.
- Let $M(K)$ denote the set of places of K. Recall that each place $v \in M(K)$ is either finite or infinite and, in either case, determines a normalized absolute value $|\cdot|_v : K \to [0, \infty)$.

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- (finite places) If $v \in M_{\text{fin}}(K) \subset M(K)$ is finite, it corresponds to a prime ideal $\mathfrak{p}$ of the ring of integers $\mathcal{O}_K$ of K, then the order $\operatorname{ord}_{\mathfrak{p}} x$ of $x \in \mathcal{O}_K$ is the largest power $m \geq 0$ such that $x \in \mathfrak{p}^m$. If more generally $x \in K$, then one writes $x = a/b$ for some $a, b \in \mathcal{O}_K$ and $\operatorname{ord}_{\mathfrak{p}} x := \operatorname{ord}_{\mathfrak{p}} a - \operatorname{ord}_{\mathfrak{p}} b$.

  $|x|_v := 0$ if $x = 0$, and $|x|_v := \mathrm{N}(\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(x)}$ if $x \neq 0$, where $\mathrm{N}(\mathfrak{p})$ is the cardinality of the finite field $\mathcal{O}_K/\mathfrak{p}$.

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- If $v \in M_{\inf}(K) \subset M(K)$ is an infinite place, v is either *real* or *complex*. In the first case, v corresponds to a real embedding $\tau : K \to \mathbb{R}$, and we take $|x|_v = |\tau(x)|$, where $|\cdot|$ is the ordinary absolute value on $\mathbb{R}$. In the second case, v corresponds to a distinct pair $\tau, \bar{\tau} : K \to \mathbb{C}$ of complex embeddings, and we take $|x|_v = |\tau(x)|^2 = |\bar{\tau}(x)|^2$.

With these definitions we then have a product formula

$$\prod_{v \in M(K)} |c|_v = 1$$

for $c \in K^*$. (And I should add that $|c|_v = 1$ for all but finitely many places when $c \in K^*$.)

If $S \subset M(K)$ is a finite set of places containing all infinite places, then we call $\mathcal{O}_{K,S} := \{a \in K\colon |a|_v \le 1 \text{ for all } v \in M(K) \setminus S\}$ the set of S-*integers* in K.

Note that if $S = M_{\inf}(K)$, then $\mathcal{O}_{K,S} = \mathcal{O}_K$ is just the usual ring of integers. Given a vector $\mathbf{x} = (x_1, \ldots, x_m) \in \mathcal{O}_{K,S}^m$ we set

$$H_S(\mathbf{x}) = \prod_{v \in S} \max\{|x_1|_v, \ldots, |x_m|_v\}.$$

We make use of the following theorem of Evertse, which is related to the p-adic subspace theorem but is better suited to our purposes.

Theorem

Let $S \subset M(K)$ be a finite set of places of K containing all infinite places, $m \geq 2$ an integer, and $\epsilon > 0$. There is a constant $c = c(K, S, m, \epsilon) > 0$ such that if $\mathbf{x} = (x_1, \ldots, x_m) \in \mathcal{O}_{K,S}^m$ and $\sum_{k \in I} x_k \neq 0$ for every nonempty subset $I \subset \{1, 2, \ldots, m\}$, then for any $v_0 \in S$

$$|x_1 + \cdots + x_m|_{v_0} \geq c \frac{\max\{|x_1|_{v_0}, \ldots, |x_m|_{v_0}\}}{H_S(\mathbf{x})^{\epsilon} \prod_{v \in S} \prod_{k=1}^{m} |x_k|_v}.$$

How do we use Evertse's theorem?

Recall we produced a rational approximation $\Phi_n(z)$ to $D(z)$, where $\Phi_n(z)$ took a truncated series $(1-z^n)^{-1}(d_1 z + \cdots + d_n^n)$ and then corrected by adding a uniformly bounded number of monomials $c_1 z^{a_1} + \cdots + c_k z^{a_k}$ with the c_i in a fixed finite set.

We can rewrite this as

$$(1-z^n)D(z) = (d_1 z+\cdots+d_n z^n)+(1-z^n)(c_1 z^{a_1}+\cdots+c_k z^{a_k})+\mathrm{O}(z^{Cn}).$$

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Now recall that the goal is to show that if α is algebraic and inside the radius of convergence of $D(z)$ then $D(\alpha)$ is transcendental unless there is some compelling reason otherwise. The original Corvaja-Zannier strategy works as follows:

- assume towards a contradiction that $\beta := D(\alpha)$ is algebraic;
- use the p-adic subspace with forms involving β and $\Phi_n(\alpha)$ to show that this can only occur if $\beta = \Phi_n(\alpha)$ for all n;
- then show this can't occur (typically using ad hoc methods).

We return to the equation

$$(1-z^n)D(z) = (d_1 z + \cdots + d_n z^n) + (1-z^n)(c_1 z^{a_1} + \cdots + c_k z^{a_k}).$$

We take S to be the set of places that contains all infinite places and the places at which nonzero elements from $D(\alpha)$, α, the $\gamma(j)$, and the c_i are not equal to one. Then we have

$$(1-\alpha^n)D(\alpha) \approx \sum_{j=1}^{n} d_j \alpha^j + (1-\alpha)^n (c_1 \alpha^{a_1} + \cdots + c_k \alpha^{a_k}).$$

We now use technical estimates, applying Evertse's theorem and we conclude that if $D(\alpha)$ is algebraic then it is equal to $\Phi_n(\alpha)$ for all sufficiently large n.

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Finally we use an ad hoc argument to show that if α is positive and real and inside the radius of convergence of D then $D(\alpha) > \Phi_n(\alpha)$ for every n and so we conclude that $\alpha = 1/\lambda$ cannot be algebraic.

How do we go to the birational case?

As mentioned earlier, to go to the birational case, we have to go up in dimension. We can take the same idea as before, but there are additional subtleties.

Theorem

(B-Diller-Jonsson-Krieger) There exists an automorphism $\phi : \mathbb{P}^3 \to \mathbb{P}^3$ *and a matrix* $A \in \mathrm{SL}_3(\mathbb{Z})$ *such that the birational map* $f : \mathbb{P}^3 \to \mathbb{P}^3$ *given by*

$$f = \phi^{-1} \circ m_{-I} \circ \phi \circ m_A \qquad (1)$$

has transcendental dynamical degree.

Here for a 3×3 integer matrix B with $\det(B) = 1$, we define

$$m_B(x, y, z) = (x^{b_{1,1}} y^{b_{1,2}} z^{b_{1,3}}, x^{b_{2,1}} y^{b_{2,2}} z^{b_{2,3}}, x^{b_{3,1}} y^{b_{3,2}} z^{b_{3,3}}).$$

What changes?

As before, there are really two components: proving an expression for the dynamical degree that gives it as a solution λ to the equation $F(\lambda) = 1$, where $F(x)$ is some power series with coefficients in some number field; then proving transcendence of λ in a way similar to the one explained before.

As it turns out, both finding the expression and proving transcendence are considerably more difficult in this birational case and involve completely new ideas. But there is a new problem: the transcendence results require the addition of new technical conditions because the expression for the power series $F(x)$ is considerably more involved. For this reason, even seeing that there are matrices A for which all the conditions are satisfied is completely non-obvious and we require the use of deep theorems from the theory of algebraic groups to prove the existence of such an A.

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We're able to prove that there are "lots" of matrices A such that the map $f = \phi^{-1} \circ m_{-I} \circ \phi \circ m_A$ has transcendental dynamical degree, but finding a specific one is the problem of "finding hay in a haystack." Jeff Diller used Mathematica to verify that there is a specific 3×3 matrix A that works whose entries are bounded by 14, this ended up being surprisingly difficult to verify and involved showing that several integer linear recurrences have no zeros. So if one accepts computer assistance, one can find a completely explicit birational map of $\mathbb{P}^3$ that has transcendental dynamical degree.

Thanks!

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