PARITY BIASES IN PARTITIONS AND RESTRICTED PARTITIONS

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Outline

- Begin with a historical background on the research regarding partitions and parts of partitions.
- State required notations and definitions.
- State the main result and conjecture present in [Kim et al., 2020] .
- Sketch a proof of the main result, as given in [Banerjee et al., 2022].
- State other related theorems present in [Banerjee et al., 2022].

Preliminary Notations

- A partition λ of a non-negative integer n is the integer sequence $\{\lambda_1, \lambda_2, \ldots \lambda_\ell\}$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell$. We say that λ is a partition of n, denoted by $\lambda \vdash n$ and $\sum_{i=1}^{\ell} \lambda_i = n$.

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- The set of partitions of n is denoted by $P(n)$, and $|P(n)| = p(n)$.
- A partition $\lambda \vdash n$ can be split into λ_e and λ_o where λ_e and λ_o are the set of even parts and odd parts respectively.
- For $\lambda \vdash n$, $\ell(\lambda)$ is the length of the partition λ i.e. the number of parts of λ. $\ell(\lambda_o)$ (resp $\ell(\lambda_e)$) denotes the number of odd parts (resp. even parts) of the partition.

Parity Bias in Partitions

Parity Bias in Partitions

Parity bias is partitions, is the tendency of partitions to have more odd parts than even parts.

For example : In partitions of 5 the total number of partitions with more odd parts than even parts is 4 $\{5, 3+1+1, 2+1+1+1, 1+1+1+1+1\}$ whereas the total number of partitions of 5 with more even parts than odd parts is 1 $\{2+2+1\}$.

Historical Background

- Leibniz (1674) was the first person to ask about partitions in a letter to J. Bernoulli. He counted the number of partitions of 3, 4, 5, and 6.
- Euler was the first to introduce the concept of generating functions to solve the problem of partitioning a given integer n into a given number of parts m. In 1748 he proved a theorem which states that the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.
- Nathan Fine(1948) proved certain identities on partitions of n into odd parts with certain conditions on the largest part of the partition.
- Morris Newman (1960) gave a conjecture about the behaviour of the partition function modulo any integer, which states that for any integers m and r such that $0 \leq r \leq m-1$; the value of the partition function $p(n)$ satisfies the congruence, $p(n) \equiv r \pmod{m}$ for infinitely many non-negative integers n.

- M. Bousquet- Mélou and K. Eriksson (1997) introduced the concept of lecture hall partitions given by :

$$\mathcal{L}_n = \left\{(\lambda_1, \lambda_2, \ldots, \lambda_n) : 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \ldots \leq \frac{\lambda_n}{n}\right\}$$

  for $n \geq 1$. In their paper, they proved that the number of lecture hall partitions of length n of N equals the number of partitions of N into small odd parts: $1, 3, 5 \ldots, 2n - 1$.
- Kim, Kim, and Lovejoy (2020) proved results regarding the parity bias in partitions:
  - They showed that the number of partitions with more odd parts are greater than the number of partitions with more even parts for $n \neq 2$. This proof used q-series analysis.
  - They also conjectured that for partitions with all parts distinct, the number of partitions with more odd parts are greater than the number of partitions with more even parts for $n \geq 20$.

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More Notations

- $P_d(n)$: Set of partitions of n, with all parts distinct.
- $P_e(n)$: Set of partitions of n with more even parts than odd parts. $|P_e(n)| = p_e(n)$.
- $P_o(n)$: Set of partitions of n with more odd parts than even parts . $|P_o(n)| = p_o(n)$.
- $D_e(n)$: Set of partitions of n into distinct parts with more even parts than odd parts. $|D_e(n)| = d_e(n)$.
- $D_o(n)$: Set of partitions of n into distinct parts with more odd parts than even parts. $|D_o(n)| = d_o(n)$.
- $Q_o(n)$:Set of partitions of n with more odd parts than even parts where the smallest part is at least 2. $|Q_o(n)| = q_o(n)$.
- $Q_e(n)$; Set of partitions of n with more even parts than odd parts where the smallest part is at least 2. $|Q_e(n)| = q_e(n)$.

Main Results in [Kim et al., 2020]

Theorem 1

For all positive integers $n \neq 2$,

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Conjecture 2.1

For all positive integers $n \geq 20$,

$$d_o(n) > d_e(n).$$

Examples

Table: $p_o(7) > p_e(7)$

$\lambda \in P_o(n)$	$\lambda \in P_e(n)$
7	4+2+1
5+1+1	3+2+2
4+1+1+1	2+2+2+1
3+3+1	
3+2+1+1	
3+1+1+1+1	
2+2+1+1+1	
2+1+1+1+1+1	
1+1+1+1+1+1+1	

The above example shows that for $n = 7$, $p_o(n)$ is greater that $p_e(n)$.

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Table: $d_o(20) > d_e(20)$

$\lambda \in D_o(n)$	$\lambda \in D_e(n)$
19+1	20
17+3	18+2
17+2+1	16+4
16+3+1	14+6
15+5	14+4+2
15+4+1	12+8
15+3+2	12+6+2
14+5+1	10+8+2
13+7	10+6+4
13+6+1	10+4+3+2+1
13+5+2	8+6+4+2
13+4+3	8+6+3+2+1

Table: $d_o(20) > d_e(20)$

$\lambda \in D_o(n)$	$\lambda \in D_e(n)$
12+7+1	8+5+4+2+1
12+5+3	7+6+4+2+1
11 +9	6+5+4+3+2
11+8+1	
11+7+2	
11+6+3	
11+5+4	
11+5+3+1	
10+9+1	
10+7+3	
10+7+2+1	
10+6+3+1	

Table: $d_o(20) > d_e(20)$

$\lambda \in D_o(n)$	$\lambda \in D_e(n)$
9+8+3	
9+7+4	
9+7+3+1	
9+6+5	
9+5+3+2+1	
8+7+5	
7+5+4+3+1	

From the 3 tables showing $\lambda \in D_e(20)$ or $\lambda \in D_o(20)$, we can say that for $n = 20$, $d_o(n) > d_e(n)$.

Fundamental Principle behind the proofs in [Banerjee et al., 2022]

The fundamental idea behind the proof of theorems given in [Banerjee et al., 2022] can be discussed as follows:

Fundamental Idea

Let X and Y be the two given sets and our aim be to show that $|Y| > |X|$. We choose a subset $X_0 \subsetneq X$ and construct an injective mapping $f : X_0 \to Y$. To finish the proof it is sufficient to show that there is a subset $Y_0 \subset Y \backslash f(X_0)$ such that $|Y_0| > |X \backslash X_0|$.

Sketch of Proof of Theorem 1

- Our aim is to define a mapping f from a subset of $P_e(n)$ (denote by $G_e(n)$) to $P_o(n)$.

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- We begin our proof by defining a map from a large subset of $P_e(n)$ to $P_o(n)$ by adding 1 to all the odd parts, and some even parts, and subtracting 1 from the rest of the even parts, making sure that the numbers of 1s added or subtracted are equal, and reversing the parity.

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- This however leaves us the case of partitions $\lambda \in P_e(n)$, where $\ell(\lambda) \equiv 1 (\mod 2)$, since the number of 1s subtracted and added may not be equal, and the mapping will no longer produce a mapping to a partition of n.
- For the set of partitions such that $\ell(\lambda) \equiv 1 (\mod 2)$, we remove, the largest part. The partitions now effectively have even number of parts. We add 2 to the largest part, and apply the earlier mapping to the rest of the parts, making sure that the sum of the parts equal n.

- This however poses a problem for the set of partitions in $P_e(n)$ where $\ell_e(\lambda) - \ell_o(\lambda) = 1$, and the largest part is even, since applying the mapping defined above, we still get a partition with more even parts than odd.

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- For the set of partitions $\lambda \in P_e(n)$, where $\ell_e(\lambda) - \ell_o(\lambda) = 1$ and the largest part is even, we define a mapping $\lambda \mapsto \mu$ as

$$\mu = \{(\lambda_1 + 1), \lambda_4, \dots \lambda_\ell\} \cup \{(\lambda_2 - 2), (\lambda_3 - 2)\} \cup \{2, 1\}$$

. In the above map, we assume $\ell(\lambda) = \ell$.

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- This leaves us with the set of partitions $\lambda \in P_e(n)$ where $\ell_e(\lambda) - \ell_o(\lambda) = 1$, the largest part is even and $\lambda_3 \leq 2$.

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- This leaves us with the set of partitions $\lambda \in P_e(n)$ where $\ell_e(\lambda) - \ell_o(\lambda) = 1$, the largest part is even and $\lambda_3 \leq 2$.
- We end the proof by showing that $|P_e(n) \backslash G_e(n)| < |P_o(n) \backslash f(G_e(n))|$

Theorems in [Banerjee et al., 2022]

Theorem 2

Conjecture 2.1 is true.

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Theorem 3

For all positive integers $n > 7$, we have,

$$q_o(n) < q_e(n)$$

Parity Bias in Restricted Partitions

Parts with restrictions

In the context of this presentation, for a certain non-empty subset S of $\mathbb{Z}^+$, the notion of restriction of parts implies imposing the condition that no parts in a partition of a positive integer can belong to S.

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- For a set $S \subsetneq \mathbb{Z}^+$, we define ,

$$P_e^{\{S\}}(n) = \{\lambda \in P_e(n) : \lambda_i \notin S\}$$

$$\text{and} \qquad P_o^{\{S\}}(n) = \{\lambda \in P_o(n) : \lambda_i \notin S\}$$

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- $p_e^{\{S\}}(n) = |P_e^{\{S\}}(n)|$ and $p_o^{\{S\}}(n) = |P_o^{\{S\}}(n)|$

Theorems on Parity Bias with Restrictions

Theorem 4

For all positive integers $n \geq 1$,

$$p_o^{\{2\}}(n) > p_e^{\{2\}}(n)$$

Theorem 5

If $S = \{1, 2\}$, *then for all integers* $n > 8$, *we have,*

$$p_o^{\{S\}}(n) > p_e^{\{S\}}(n)$$

Further Research

- In [Bringmann et al., 2023], the authors have tried to answer questions about the distributions of the parts of random partitions modulo N where $N \in \mathbb{N}$.
- The authors have generalized the results given in [Kim et al., 2020] and [Banerjee et al., 2022] to give asymptotics for biases $(\mod N)$ for partitions of integers into distinct parts.

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THANK YOU